Derived moduli spaces of pseudo-holomorphic curves

John Pardon

April 2025

Disclaimer: This an excerpt of unfinished work-in-progress. Its release in such incomplete form is unconventional, but is made due to popular request. Comments and corrections are always welcome.

Contents

Preface

This text is intended as a logical starting point for the theory of moduli spaces of pseudo-holomorphic curves, as founded by Gromov [35] and subsequently developed by Floer, Hofer, Eliashberg, Fukaya, Kontsevich, Seidel, Abouzaid, and many others. A distinguishing feature of our treament of this subject is its generality: we formulate the main foundations of the theory in a way which is logically sufficient for applications. In fact, most of what we do is applicable well beyond the setting of pseudo-holomorphic curves, to any non-linear elliptic Fredholm problem. Despite their analytic nature, the main results of this work rely fundamentally on the framework of ∞-categories.

We assume minimal prerequisites and thus include a substantial amount of advanced graduate level background and exercises thereon, so that our treatment may qualify as self-contained. As a result, the interesting material is spread a bit thin. The reader is therefore advised not to read this text linearly, but rather to seek out their specific topics of interest, and to refer to the other parts of the text as they are cross-referenced.

We have sought to formulate statements and proofs which are as simple and down-to-earth as possible (though we cannot claim always to have met this ideal). Most of the real work has been in finding the 'right' formalism, after which the proofs fall into place with little resistance. This work is largely hidden from the view of the consumer, and so the main results may appear deceptively trivial. Although this makes our text appear less impressive superficially, we believe it is ultimately good for the subject.

Introduction and summary of results

Recall that a map $u : C \to X$ from a Riemann surface C to an almost complex manifold X (i.e. a manifold equipped with an endomorphism $J : TX \to TX$ squaring to $-\mathbf{1}$) is called *pseudo-holomorphic* when its differential $du : TC \to TX$ is $\mathbb{C}$-linear. The equation $J \circ du = du \circ j$ asserting pseudo-holomorphicity of u is a non-linear elliptic Fredholm partial differential equation. Though we focus our attention on this particular equation and its variants, the vast majority of the framework we develop applies equally to any other non-linear elliptic Fredholm partial differential equation.

The primary objects of study in this text are the *moduli spaces* of solutions to the pseudo-holomorphic map equation. Our main goal is to *describe precisely what sort of mathematical objects these moduli spaces are.*

Derived geometry

To explain the answer to our main question (what sort of mathematical objects are moduli spaces of holomorphic curves), it is helpful to begin in the linear setting. Fix a linear elliptic operator $L : E \to F$ acting on sections of vector bundles E and F over a compact manifold M. The 'space of solutions' to $Lu = 0$ is most immediately the finite-dimensional vector space $\ker L \in \mathsf{Vect}_{\mathbb{R}}$. However, for many purposes, it is better to instead consider the two-term complex $[L] = [C^\infty(M, E) \xrightarrow{L} C^\infty(M, F)]$ regarded as an object of the ∞-category $\mathsf{K}^{\geq 0}(\mathsf{Vect}_{\mathbb{R}})$ of complexes of vector spaces supported in non-negative cohomological degrees (in which it is isomorphic to $[\ker L \xrightarrow{0} \operatorname{coker} L]$). For example, if L_t is a family of operators parameterized by a smooth manifold T, then $\ker L_t$ is not generally a smooth vector bundle on T, while $[L_t]$ is, locally on the parameter space T, equivalent to a two-term complex of smooth vector bundles. It is very reasonable to regard the two-term complex $[L]$ as equally deserving of the descriptor 'space of solutions to $Lu = 0$'. Indeed, while $\ker L$ is the fiber product $C^\infty(M, E) \times_{C^\infty(M,F)} 0$ in the category $\mathsf{Vect}_{\mathbb{R}}$, the two-term complex $[L]$ is the same fiber product taken in the ∞-category $\mathsf{K}^{\geq 0}(\mathsf{Vect}_{\mathbb{R}})$.

Now let us move to the non-linear setting. The moduli space $\mathcal{M}$ of solutions to a non-linear elliptic partial differential equation on a compact manifold may be identified locally with the zero set $f^{-1}(0)$ of a smooth map $f : \mathbb{R}^n \to \mathbb{R}^m$. This is a classical fact going back to Kuranishi [63] and Atiyah–Hitchin-Singer [11], and such charts are often called *Kuranishi charts*. It is desirable to regard $\mathcal{M}$ not just as a topological space, but to also remember its Kuranishi charts and the relations among them; this generalizes the passage from $\ker L$ to $[L]$ in the linear setting. One very direct way to do this is to simply equip $\mathcal{M}$ with an *atlas* of Kuranishi charts, as first appeared in work of Fukaya–Ono [32] and developed further by Fukaya–Oh–Ohta–Ono [30, 31] and others. It is natural to ask whether $\mathcal{M}$ is naturally an object of a non-linear analogue of the ∞-category $\mathsf{K}^{\geq 0}(\mathsf{Vect}_{\mathbb{R}})$.

	linear	non-linear
category	$\mathsf{Vect}_{\mathbb{R}}$	smooth manifolds Sm
∞-category	$\mathsf{K}^{\geq 0}(\mathsf{Vect}_{\mathbb{R}})$	derived smooth manifolds $\mathcal{D}\mathsf{Sm}$

The relevant non-linear analogue of the ∞-category $\mathsf{K}^{\geq 0}(\mathsf{Vect}_{\mathbb{R}})$ is the ∞-category of *derived smooth manifolds*, which we denote by $\mathcal{D}\mathsf{Sm}$ and study in (2.9).

The ∞-category of derived smooth manifolds was introduced by Spivak [102], and it may also be called the ∞-category of locally finitely presented C^∞-schemes. It can be regarded as a special case of the rather general theory of *derived geometry* introduced by Lurie [72] and Toën–Vezzosi [105, 106]. We will adopt the perspective that the ∞-category of derived smooth manifolds $\mathcal{D}\mathsf{Sm}$ obtained from the category Sm by *formally adjoining finite ∞-limits modulo preserving finite transverse ∞-limits*. It can be shown that a derived fiber product (i.e. fiber product in $\mathcal{D}\mathsf{Sm}$) of smooth manifolds *remembers its fiber product presentation locally, modulo transverse fiber products of smooth manifolds*. The connection between multiplicities of non-transverse intersections and derived geometry was suggested long ago by the Serre intersection formula [99, V.C.1], and this has been a key motivation for the development of derived geometry since its inception. It is a deep fact that the passing from categories to ∞-categories in a differentiable context records non-transverse intersections in the necessary way.

We may now state a refined version of our goal: we seek to *construct moduli spaces of pseudo-holomorphic curves as derived smooth manifolds*.

Representable functors

To explain the construction of moduli spaces of pseudo-holomorphic curves as derived smooth manifolds, we must begin by recalling Grothendieck's technique of constructing moduli spaces by representing functors.

To specify an object $\mathcal{M}$ of a category C, it is equivalent to specify the functor $\operatorname{Hom}_{\mathsf{C}}(-,\mathcal{M}) : \mathsf{C}^{\mathsf{op}} \to \mathsf{Set}$ associating to every object $Z \in \mathsf{C}$ the set of maps $Z \to \mathcal{M}$. More precisely, given a functor $F : \mathsf{C}^{\mathsf{op}} \to \mathsf{Set}$, an object $\mathcal{M} \in \mathsf{C}$ together with an element $\xi \in F(\mathcal{M})$ is said to *represent* F when the map $\operatorname{Hom}_{\mathsf{C}}(Z,\mathcal{M}) \to F(Z)$ given by $f \mapsto f^*\xi$ is an isomorphism for every $Z \in \mathsf{C}$ (that is, pulling back ξ defines an isomorphism of functors $\operatorname{Hom}_{\mathsf{C}}(-,\mathcal{M}) \to F(-)$). When such a representing pair $(\mathcal{M},\xi)$ exists, we say that F is *representable*. It is straightforward to check that any two representing pairs $(\mathcal{M},\xi)$ and $(\mathcal{M}',\xi')$ are uniquely isomorphic. The property of a pair $(\mathcal{M},\xi)$ representing a particular functor is often also called satisfying a particular *universal property*. Bored experts may at this point take note that so far this discussion does not require any version of the Yoneda Lemma (nor is equivalent to it in any way).

The vague idea that a moduli space $\mathcal{M}$ 'parameterizes all objects of some type $\mathcal{O}$' naturally lends itself to a precise formulation in terms of representable functors. Indeed, consider the *moduli functor* F sending a space Z to the set of all families of objects of type $\mathcal{O}$ over Z (the terms 'space' and 'family' are placeholders for whatever the relevant sort of mathematical items may be). To represent F now means to find a space $\mathcal{M}$ and a family $\mathcal{U} \to \mathcal{M}$ of objects of type $\mathcal{O}$ which is 'universal' in the sense that every family of objects of type $\mathcal{O}$ over a space Z is the pullback of $\mathcal{U} \to \mathcal{M}$ under a unique map $Z \to \mathcal{M}$. As remarked above, such a pair $(\mathcal{M}, \mathcal{U} \to \mathcal{M})$ is unique up to unique isomorphism if it exists, and in this case $\mathcal{M}$ is called the 'moduli space' and $\mathcal{U} \to \mathcal{M}$ the 'universal family'. (One important caveat about this discussion is that it often needs a higher categorical context, that is we should replace the category of sets Set with the 2-category of groupoids Grpd or the ∞-category of spaces Spc.) The formalism of moduli functors may seem trivial and tautological at first glance, and it is perhaps for this reason that moduli spaces were studied for quite some time before the introduction of moduli functors.

Despite the apparent triviality of the formalism of moduli functors, it turns out to be extraordinarily useful from a technical standpoint, for a few different reasons.

First of all, the moduli functor $\mathrm{Hom}(-, \mathcal{M})$ is usually much easier to describe than the moduli space $\mathcal{M}$ itself. Indeed, the moduli functor simply consists of sets and maps between them, while the moduli space is an object of some category (e.g. smooth manifolds) which may be rather complicated to describe directly (e.g. a set, a topology on that set, and a collection of charts with smooth transition functions). The notion of a 'family of objects of type $\mathcal{O}$ parameterized by Z' is usually quite transparent, while turning 'the set of all objects of type $\mathcal{O}$' into an object of some category (e.g. describing a topology on this set) is virtually guaranteed to be quite a bit more complicated. For this reason, the moduli functor is often unquestionably canonical, while the same cannot be said for (an explicit construction of) the moduli space. Crucially, representability is a *property* (rather than *extra structure*), so whatever arbitrary choices may go into proving that a functor is representable necessarily do not affect the resulting representing object, which is automatically (and trivially) identified with the result of any other construction of a representing object.

Second, if C is a category of 'geometric objects', then one can regard the category of functors ('presheaves') $\mathsf{P}(\mathsf{C}) = \mathsf{Fun}(\mathsf{C}^{\mathsf{op}}, \mathsf{Set})$ itself as a category of geometric objects containing C (it is here that we need the Yoneda Lemma, which in particular says that $\mathsf{C} \subseteq \mathsf{P}(\mathsf{C})$). This makes it possible to reason geometrically with moduli functors, similarly to how we might reason with moduli spaces, without proving (or perhaps before we prove) they are representable. In fact, many moduli functors are simply *not* representable by objects of our 'original' geometric category C, but instead satisfy weaker conditions which nevertheless makes them reasonable geometric objects (e.g. the moduli functor of closed Riemann surfaces is not a smooth manifold, rather a smooth orbifold). A presheaf is called a *sheaf* when its value on $Z \in \mathsf{C}$ amounts to ways of specifying 'local data' on Z; sheaves form a full subcategory $\mathsf{Shv}(\mathsf{C}) \subseteq \mathsf{P}(\mathsf{C})$ of presheaves, and when regarded as geometric objects they are also called *stacks* (or C-*stacks* to indicate which category C we are working with). Moduli functors will (at least for us) always be sheaves (a 'family of objects of type $\mathcal{O}$ parameterized by Z' should evidently be local data on Z), and thus are also called *moduli stacks*. Crucially, representability is a *local property* of a stack.

The fact that representability is a local property is of decisive importance, particularly so for our application to constructing moduli spaces of pseudo-holomorphic curves. Let us explain why. As we have already noted above, the local structure of moduli spaces $\mathcal{M}$ of pseudo-holomorphic curves (or, more generally, solutions to any non-linear elliptic Fredholm problem) has been well understood since [63, 11]: we have $\mathcal{M} = f^{-1}(0)$ (locally) for smooth maps $f : \mathbb{R}^n \to \mathbb{R}^m$. However, such local charts and the data relating them are non-unique (this is inevitable given the higher homotopical nature of the ∞-category of derived smooth manifolds), and this is the root cause of the worst technical complications in the theory of moduli spaces of pseudo-holomorphic curves. The fact that representability is a local property gives a decisive solution to this problem: concretely, it tells us that the data relating local charts exists and is unique for formal reasons (provided we construct these local charts to represent a canonical moduli functor), and so its construction becomes a triviality. *The use of moduli functors thus resolves one of the main difficulties in the subject.*

Remark. There is a close analogy between the theory of distributions in analysis and the theory of presheaves in category theory. In both cases, we begin with a class of 'nice objects' (smooth functions and objects of C, respectively), and we introduce a class of 'generalized objects' (distributions = generalized functions, and presheaves on C = generalized objects of C, respectively) which are characterized by how they (formally) 'pair' with our original class of nice objects (via integration $\varphi \mapsto \int f \cdot \varphi$ and Hom functor $c \mapsto \operatorname{Hom}(c, F)$, respectively). Moreover, in both settings, it is common to produce a nice object with a certain property (say, solving a differential equation or representing a certain functor) by first arguing that a generalized object with the property exists and then arguing that this object is in fact nice. A rich theory of generalized objects may thus be useful even if all the objects in which we are ultimately interested turn out to be nice.

Moduli stacks of pseudo-holomorphic maps

We have now settled on a concrete two-part strategy for constructing a given moduli space of holomorphic curves: we should *define the relevant moduli functor*, and we should *show that it is representable*. We now explain the first of these steps, namely how we define moduli functors of pseudo-holomorphic maps.

0.0.1 Meta-Definition (elaborated in (5.3))**.** A *pseudo-holomorphic moduli problem* over a base B is a pair of submersions $W \to C \to B$ where $C \to B$ is of relative dimension two, together with vertical almost complex structures on $C \to B$ and $W \to B$ (that is, complex structures on the vector bundles $T_{W/B}$ and $T_{C/B}$) which are respected by the map $W \to C$. A *solution* to such a problem is a section $C \to W$ whose relative differential $T_{C/B} \to T_{W/B}$ is complex linear. The *moduli functor* $\underline{\mathrm{Hol}}_B(C, W) \to B$ assigns to any $Z \to B$ the set of solutions of the pullback problem $W \times_B Z \to C \times_B Z \to Z$.

More formally, the moduli functor $\underline{\mathrm{Hol}}_B(C, W) \to B$ is defined as an appropriate fiber product of functors $\underline{\mathrm{Sec}}_B(C, W) \to B$ which assign to any $Z \to B$ the set of (all) sections of

$W \times_B Z \to C \times_B Z$.

$$\begin{tikzcd} \underline{\mathrm{Hol}}_B(C,W) \arrow[r] \arrow[d] & \underline{\mathrm{Sec}}_B(C,W) \arrow[d, "u\mapsto(u,(du)^{0,1})"] \\ \underline{\mathrm{Sec}}_B(C,W) \arrow[r, "u\mapsto(u,0)"] & \underline{\mathrm{Sec}}_B(C,H) \end{tikzcd} \tag{0.0.1.1}$$

Here H is the vector bundle $\mathrm{Hom}_{\mathbb{C}}(\overline{T_{C/B}}, T_{W/C})$ over W.

This 'meta-definition' becomes a complete definition once we specify a category (or ∞-category) with an acceptable notion of *submersions* and of *vertical differentiation* thereon. In particular, these moduli functors may be defined on the categories of topological spaces, smooth manifolds, and derived smooth manifolds. We distinguish the resulting moduli stacks $\underline{\mathrm{Hol}}$ with subscripts to indicate the (∞-)category in question: $\underline{\mathrm{Hol}}_{\mathsf{Top}}$ (a topological stack), $\underline{\mathrm{Hol}}_{\mathsf{Sm}}$ (a smooth stack), $\underline{\mathrm{Hol}}_{\mathcal{D}\mathsf{Sm}}$ (a derived smooth stack), etc.

We emphasize that the definition of the moduli stack $\underline{\mathrm{Hol}}$ is always *diagrammatic*: it assigns to any Z the set (or space) of diagrams in some (∞-)categories of some particular shape. The diagrammatic nature of the moduli stacks is a significant virtue for a number of different reasons. For one, it gives rise to tautological comparison maps between the moduli stacks over different geometric categories (topological spaces, smooth manifolds, derived smooth manifolds, etc.).

Representability

Having given the (rather tautological) definition of the relevant moduli functors of pseudo-holomorphic maps (0.0.1), we can now state our main results about their representability. We should emphasize that both these results and their proofs should apply in significantly greater generality than what we state here (to any non-linear elliptic Fredholm problem).

Now consider smooth moduli stacks. It is a classical fact that regular (that is, transverse, unobstructed) loci in moduli spaces of pseudo-holomorphic curves are smooth manifolds. Here is a precise statement of this result in the language of moduli functors:

0.0.2 Regularity Theorem (proved in (5.6.2))**.** *Let $W \to C \to B$ be a pseudo-holomorphic section problem over a smooth manifold B. Suppose $C \to B$ is proper. The open substack $\underline{\mathrm{Hol}}_B(C,W)^{\mathrm{reg}}_{\mathsf{Sm}}$ of the smooth stack $\underline{\mathrm{Hol}}_B(C,W)_{\mathsf{Sm}}$ is representable, and the comparison map $(\mathsf{Sm} \to \mathsf{Top})_!\underline{\mathrm{Hol}}_B(C,W)^{\mathrm{reg}}_{\mathsf{Sm}} \to \underline{\mathrm{Hol}}_B(C,W)^{\mathrm{reg}}_{\mathsf{Top}}$ is an isomorphism of topological spaces.*

The proof of this result consists of not much more than 'classical non-linear elliptic Fredholm analysis' (precisely, a Newton–Picard iteration scheme and quadratic estimates).

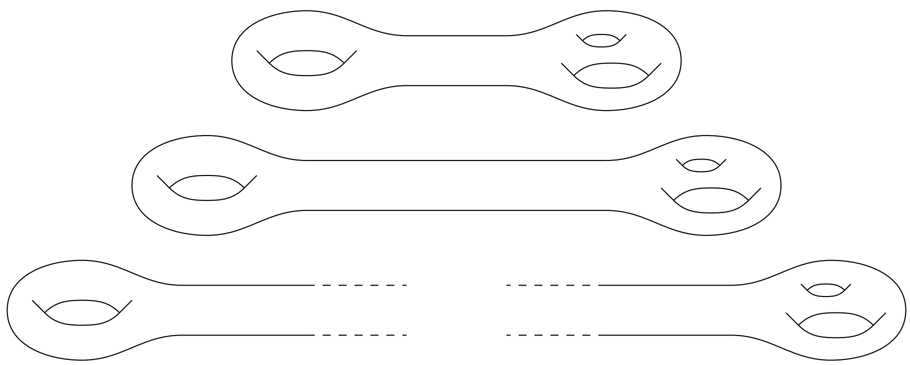
Our main representability result replaces 'smooth' with 'derived smooth' and applies to the entire moduli stack (rather than just the open regular locus inside it):

0.0.3 Derived Regularity Theorem (proved in (5.7.10))**.** *Let $W \to C \to B$ be a pseudo-holomorphic section problem over a derived smooth manifold B. Suppose $C \to B$ is proper. The derived smooth stack $\underline{\mathrm{Hol}}_B(C,W)_{\mathcal{D}\mathsf{Sm}}$ is representable, and the comparison map $(\mathcal{D}\mathsf{Sm} \to \mathsf{Top})_!\underline{\mathrm{Hol}}_B(C,W)_{\mathcal{D}\mathsf{Sm}} \to \underline{\mathrm{Hol}}_B(C,W)_{\mathsf{Top}}$ is an isomorphism.*

This result is due independently to the author [90] and Pelle Steffens [103]. It answers a conjecture of Joyce [51, §5.3] (which is a differential geometric analogue of an earlier conjecture in algebraic geometry of Kontsevich [62]).

Logarithmic structures

With only a few isolated exceptions, all non-trivial applications of pseudo-holomorphic curves involve moduli spaces of pseudo-holomorphic maps *with nodal degenerations*. A 'nodal degeneration' of curves means a family of curves locally modelled on $S^1 \times \mathbb{R}^2_{\geq 0} \xrightarrow{(\theta,x,y)\mapsto xy} \mathbb{R}_{\geq 0}$ (we will often also work in cylindrical, or log, coordinates $s = \log x$ and $s' = \log y$).



To treat such moduli spaces, we will generalize the discussion so far from smooth manifolds to (what we will call) *log smooth manifolds*, which we study in (2.7). Log smooth manifolds are relevant to the moduli theory of pseudo-holomorphic curves for three(!) separate reasons:

- The sort of asymptotically cylindrical objects we wish to consider (manifolds, differential operators thereon, and morphisms therebetween) arise naturally out of the key notion of *log smoothness* for maps between real affine toric varieties. $X_P = \operatorname{Hom}(P, \mathbb{R}_{\geq 0})$ (P a real polyhedral cone). This observation originates in the work of Melrose [79, 80, 81] on 'b-differential calculus'.
- The sort of *families* (in particular, *degenerating families*) of asymptotically cylindrical objects we wish to consider also arise naturally out of the notion of log smoothness. This was conjectured by Joyce [52, §6.2] (in the precise sense that certain natural moduli functors on log smooth manifolds are representable), who formulated the notion of log smooth manifolds (under a different name) for this purpose.
- Interesting moduli spaces of pseudo-holomorphic maps often involve not only degenerations of the source, but also degenerations of the target (for example, this is the case for the relative Gromov–Witten invariants of Jun Li [69, 70] and the Symplectic Field Theory of Eliashberg–Givental–Hofer [29]). It turns out that target expansions arise naturally from moduli functors in the log setting, and this way of encoding target expansions is, technically speaking, significantly more convenient than working with *ad hoc* explicit definitions in each individual case. This observation is due to Siebert with his proposal for logarithmic Gromov–Witten invariants, subsequently developed by Gross–

Siebert [36], Chen [15], Abramovich–Chen [2] and Abramovich–Chen–Gross–Siebert [3, 4].

Stability

Gromov compactness

Acknowledgements

The presentation in this text has benefitted in innumerable ways from many conversations over many years with many different people. With advance apologies for any omissions, let me acknowledge in particular communication with Mohammed Abouzaid, Shaoyun Bai, Dustin Clausen, Yasha Eliashberg, Tobias Ekholm, Helmut Hofer, Yusuf Barış Kartal, Daniel Litt, Sergey Nersisyan, Stefan Schwede, Paul Seidel, Mohan Swaminathan. I also thank my collaborators Sheel Ganatra and Vivek Shende from whom I learned much about how to write mathematics. A conversation with Yongbin Ruan in 2019 provided meaningful motivation for writing these notes. Finally, I would like to particularly acknowledge conversations with Pelle Steffens in Fall 2024 regarding the content herein, during which we made (yet unpursued) plans to complete the writing of these notes collaboratively.

Many references were taken from the nLab `https://ncatlab.org/`.

Any errors or omissions remain the responsibility of the author.

Chapter 1

Category theory

In any mathematical discussion, it is helpful to have available multiple different perspectives on the same situation, as it often happens that something which is opaque from one perspective turns out to be clear from another. Category theory provides such an additional perspective in a wide range of different mathematical settings. It has an uncanny ability to reveal large parts of mathematical arguments to be 'purely formal', thus clarifying where the true content really lies and *eliminating redundant arguments*. It is easy to reach the mistaken conclusion that this means all of category theory is trivial! On the contrary, its utility in crafting efficient arguments makes it indispensable in many settings.

The following observation may seem trivial, however it is in fact the main insight of category theory:

(1.0.0.1) Many mathematical objects, despite having a *rather complicated definition*, turn out to be characterized uniquely by a *simple property*.

A 'simple property' here means, more precisely, a *universal property*: being a final object in some category (any two final objects are uniquely isomorphic). Examples of (1.0.0.1) abound all throughout mathematics: classifying spaces in algebraic topology (infinite Grassmannians, Eilenberg–MacLane spaces), free groups, fiber products of schemes, moduli spaces of Riemann surfaces (in fact, virtually any moduli space at all), and all sorts of 'homotopy coherent' construction (functors of ∞-categories, ring spectra), including some of the main objects of study in this text: the ∞-category of derived smooth manifolds and derived moduli spaces of pseudo-holomorphic curves. Universal properties can be used to perform clean manipulations with the objects they characterize, which may be quite nontrivial and opaque if based instead on the definitions of these objects. While a universal property alone does not directly imply existence of the object it characterizes (a category may fail to have a final object), it can still aid in the construction of this object (for example, by providing gluing data relating different local charts).

Despite its wide ranging applications, category theory can seem rather trivial at the beginning. The first examples one sees of categorical theoretic reasoning are not particularly impressive. It is only really possible to appreciate the insight offered by category theory after seeing more complicated applications in which its power to eliminate redundant arguments becomes significant. During the process of writing other parts of this text, I have been surprised to discover that certain results whose initial proof appeared quite nontrivial could in fact be deduced by essentially formal categorical reasoning.

1.1 Categories

The reader may refer to Leinster [67] for a first introduction to category theory and to MacLane [77] for a comprehensive treatment.

Definitions and examples

⋆ **1.1.1 Definition.** A *category* C consists of the following data:

(1.1.1.1) For every pair of objects $X, Y \in \mathsf{C}$, a set $\operatorname{Hom}(X, Y)$, whose elements are called the *morphisms* $X \to Y$ in C.

(1.1.1.2) A set C, whose elements are called the *objects* of C.

(1.1.1.3) For every triple of objects $X, Y, Z \in \mathsf{C}$, a map

$$\operatorname{Hom}(X, Y) \times \operatorname{Hom}(Y, Z) \to \operatorname{Hom}(X, Z)$$

called *composition*, such that for every composable triple of morphisms a, b, c, the two compositions $(ab)c$ and $a(bc)$ are equal (composition is *associative*).

(1.1.1.4) For every object $X \in \mathsf{C}$, an element $\mathbf{1}_X \in \operatorname{Hom}(X, X)$ called the *identity morphism* such that composition with $\mathbf{1}_X$ defines the identity map $\operatorname{Hom}(X, Y) \to \operatorname{Hom}(X, Y)$ and $\operatorname{Hom}(Z, X) \to \operatorname{Hom}(Z, X)$ for all $Y, Z \in \mathsf{C}$.

The set of morphisms $\operatorname{Hom}(X, Y)$ may also be denoted $\operatorname{Hom}_{\mathsf{C}}(X, Y)$ or $\mathsf{C}(X, Y)$.

⋆ **1.1.2 Example** (Categories of sets, groups, and topological spaces)**.** The following are categories:

(1.1.2.1) Set, the category of sets: an object is a set, and a morphism is a map of sets.

(1.1.2.2) Grp, the category of groups: an object is a group, and a morphism is a group homomorphism.

(1.1.2.3) Top, the category of topological spaces: an object is a topological space, and a morphism is a continuous map.

Except not quite: a category needs a *set* of objects (1.1.1.2), and there is no 'set of all sets', 'set of all groups', or 'set of all topological spaces'. So, we should really say that we get a category of sets, groups, or topological spaces by choosing a set and, for each element of that set, a set, group, or topological space. The notation Set, Grp, Top is thus somewhat abusive, since it hides these choices. This is, in fact, an advantage, as we will see shortly that such choices are to a large extent irrelevant (see the 'principle of equivalence' (1.1.32)(1.1.33)(1.1.34) below).

1.1.3 Example. To each poset S, we can associate a category whose objects are the elements of S and in which

$$\operatorname{Hom}(s, t) = \begin{cases} * & s \leq t \\ \varnothing & \text{else} \end{cases} \tag{1.1.3.1}$$

1.1.4 Exercise (Identity morphisms are a property)**.** Show that in a category, the identity morphisms (1.1.1.4) are uniquely determined by the rest of the data (1.1.1.2)–(1.1.1.3) provided they exist.

1.1.5 Exercise (Isomorphisms and inverses)**.** A morphism $X \to Y$ in a category is called an *isomorphism* iff there exists a morphism $Y \to X$ such that the compositions $X \to Y \to X$ and $Y \to X \to Y$ are the identity morphisms $\mathbf{1}_X$ and $\mathbf{1}_Y$. Show that a given morphism $X \to Y$ as at most one such 'inverse' morphism $Y \to X$.

1.1.6 Example (Cardinal)**.** A *cardinal* is an isomorphism class of objects in the category Set.

1.1.7 Example (Groups up to finite index)**.** Consider the category whose objects are groups and whose morphisms $G \to H$ are pairs (G', f) where $G' \leq G$ is a finite index subgroup and $f : G' \to H$ is a group homomorphism, modulo the equivalence relation that $(G', f) \sim (G'', g)$ iff there exists a finite index subgroup $G''' \leq G' \cap G''$ such that $f|_{G'''} = g|_{G'''}$. In this category, all finite groups are isomorphic to the trivial group.

1.1.8 Exercise (Germs of topological spaces)**.** The category of germs of topological spaces is defined as follows. Its objects are pairs (X, x) where X is a topological space and $x \in X$ is a point. Its morphisms $(X, x) \to (Y, y)$ are pairs (U, f) where $U \subseteq X$ is an open set containing x and $f : U \to Y$ is a continuous map with $f(x) = y$, modulo the equivalence relation that $(U, f) \sim (U', f')$ iff there exists an open set $A \subseteq U \cap U'$ containing x for which $f|_A = f'|_A$. The composition of $(U, f) : (X, x) \to (Y, y)$ and $(V, g) : (Y, y) \to (Z, z)$ is given by $(f^{-1}(V), g \circ f|_{f^{-1}(V)})$. Show that a morphism $(X, x) \to (Y, y)$ in this category is an isomorphism iff can be realized as a pair (U, f) for which f is an open embedding.

1.1.9 Definition (Groupoid)**.** A *groupoid* is a category in which every morphism is an isomorphism.

1.1.10 Example (Fundamental groupoid)**.** Let X be a topological space. Its *fundamental groupoid* $\pi_1(X)$ is the category whose objects are points x of X and whose morphisms $x \to y$ are paths from x to y modulo homotopy rel endpoints, with composition being given by concatenation (which is indeed associative on homotopy classes). The automorphism group of a point $x \in X$ in $\pi_1(X)$ is the fundamental group $\pi_1(X, x)$ of X based at x.

1.1.11 Example (Core)**.** For any category C, we can consider the category $\mathsf{C}_{\simeq}$ with the same objects and whose morphisms are the isomorphisms in C; thus $\mathsf{C}_{\simeq}$ is a groupoid. For example, $\mathsf{Set}_{\simeq}$ consists of sets and bijections of sets.

1.1.12 Definition (Full subcategory)**.** For a category C, the *full subcategory* spanned by a set of objects of C is the category whose objects are this set and whose morphisms are the same as in C.

1.1.13 Example. The category of abelian groups Ab is a full subcategory of the category of groups Grp.

1.1.14 Definition (Opposite category)**.** For a category C, its *opposite* C^{op} has the same objects, but morphisms are reversed: $\mathsf{C}^{\mathsf{op}}(X, Y) = \mathsf{C}(Y, X)$.

Every notion for categories has a *dual* notion obtained by applying the original notion to the opposite; this is usually indicated linguistically with the prefix 'co-'.

⋆ **1.1.15 Definition** (Functor)**.** A *functor* $F : \mathsf{C} \to \mathsf{D}$ between categories consists of the following data:

(1.1.15.1) For every object $X \in \mathsf{C}$, an object $F(X) \in \mathsf{D}$.

(1.1.15.2) For every pair of objects $X, Y \in \mathsf{C}$, a map $F : \operatorname{Hom}(X, Y) \to \operatorname{Hom}(F(X), F(Y))$ such that $F(\mathbf{1}_X) = \mathbf{1}_{F(X)}$ and such that composing and applying F in either order define the same map

$$\operatorname{Hom}(X, Y) \times \operatorname{Hom}(Y, Z) \to \operatorname{Hom}(F(X), F(Z)).$$

1.1.16 Example (Free and forgetful functors)**.** Associating to a group or topological space its underlying set defines 'forgetful' functors $\mathsf{Grp} \to \mathsf{Set}$ and $\mathsf{Top} \to \mathsf{Set}$. Associating to a set the free group on that set or the discrete topology on that set defines functors $\mathsf{Set} \to \mathsf{Grp}$ and $\mathsf{Set} \to \mathsf{Top}$.

As in (1.1.2), there is a caveat. To define categories Set and Grp, we should first choose a set of sets and a set of groups. Then to define, say, the 'free group' functor $\mathsf{Set} \to \mathsf{Grp}$, we should choose, for each $S \in \mathsf{Set}$, a group $G \in \mathsf{Grp}$ and an identification of G with the free group generated by S. If such a group $G \in \mathsf{Grp}$ exists for every $S \in \mathsf{Set}$, we can then define the desired functor $\mathsf{Set} \to \mathsf{Grp}$. Fortunately, this sort of discussion can (and should) be systematically avoided (see the 'principle of equivalence' (1.1.32)(1.1.33)(1.1.34) below).

1.1.17 Example (Homology and homotopy groups)**.** Homology groups are a sequence of functors $H_n : \mathsf{Top} \to \mathsf{Ab}$ from topological spaces to abelian groups for integers $n \geq 0$. The homotopy groups π_n are functors $\mathsf{Top}_* \to \mathsf{Ab}$ for $n \geq 2$ and $\pi_1 : \mathsf{Top}_* \to \mathsf{Grp}$ and $\pi_0 : \mathsf{Top}_* \to \mathsf{Set}_*$, where Top_* denotes the category of pointed topological spaces and Set_* that of pointed sets (and, in both cases, pointed maps). The functors H_n and π_n are homotopy invariant, meaning they factor through the functors $\mathsf{Top} \to h\mathsf{Top}$ and $\mathsf{Top}_* \to h\mathsf{Top}_*$, where the h indicates morphisms are now homotopy classes of (pointed) maps.

1.1.18 Example (Functors on fundamental groupoids)**.** A map of topological spaces $X \to Y$ induces a functor on fundamental groupoids $\pi_1(X) \to \pi_1(Y)$. A functor $\pi_1(X) \to \mathsf{C}$ is known as a *local system* on X valued in C.

1.1.19 Example (Hom functor)**.** Sending $(X, Y) \mapsto \operatorname{Hom}(X, Y)$ is a functor $\mathsf{C}^{\mathsf{op}} \times \mathsf{C} \to \mathsf{Set}$ for any category C.

1.1.20 Definition (Fully faithful)**.** A functor F is called *fully faithful* when its constituent maps $\operatorname{Hom}(X, Y) \to \operatorname{Hom}(F(X), F(Y))$ are bijections of sets. A fully faithful functor is also called an *embedding* or an *inclusion*, and full faithfulness is often indicated with the hooked arrow $\hookrightarrow$.

1.1.21 Definition (Essential image)**.** The *essential image* of a functor $F : \mathsf{C} \to \mathsf{D}$ is the full subcategory $\mathsf{im}(F) \subseteq \mathsf{D}$ spanned by those objects $Y \in \mathsf{D}$ which are isomorphic to $F(X)$ for some $X \in \mathsf{C}$. When every object of D lies in $\mathsf{im}(F)$, we say that F is *essentially surjective.*

⋆ **1.1.22 Definition** (Natural transformation)**.** A *natural transformation* $F \to G$ between functors $F, G : \mathsf{C} \to \mathsf{D}$ consists of:

(1.1.22.1) For every object $X \in \mathsf{C}$, a morphism $F(X) \to G(X)$ such that for every morphism $X \to Y$, the two compositions $F(X) \to G(X) \to G(Y)$ and $F(X) \to F(Y) \to G(Y)$ agree.

Given categories C and D, there is a category $\mathsf{Fun}(\mathsf{C}, \mathsf{D})$ whose objects are functors $\mathsf{C} \to \mathsf{D}$ and whose morphisms are natural transformations.

1.1.23 Example (Homology of local systems)**.** Local systems on X valued in C form a category $\mathsf{Fun}(\pi_1(X), \mathsf{C})$. Homology with twisted coefficients is a sequence of functors $H_n : \mathsf{Fun}(\pi_1(X), \mathsf{Ab}) \to \mathsf{Ab}$ for $n \geq 0$.

1.1.24 Example. For every group G, there is a groupoid BG with a single object whose automorphism group is G. A functor $BG \to BH$ is a group homomorphism $\phi : G \to H$. A natural isomorphism of (functors associated to) group homomorphisms $\phi \to \phi'$ is an element $h \in H$ conjugating ϕ to ϕ', namely satisfying $h\phi(g) = \phi'(g)h$.

1.1.25 Example. If D is a groupoid, then the functor category $\mathsf{Fun}(\mathsf{C}, \mathsf{D})$ is a groupoid.

A naive notion of ‘isomorphism’ between categories is that of a compatible bijection between objects and morphisms. The following weaker notion turns out to be much more meaningful (see the ‘principle of equivalence’ (1.1.32) below):

⋆ **1.1.26 Definition** (Equivalence of categories)**.** A functor $\mathsf{C} \to \mathsf{D}$ is called an *equivalence* iff there exists a functor $\mathsf{D} \to \mathsf{C}$ such that the compositions $\mathsf{C} \to \mathsf{D} \to \mathsf{C}$ and $\mathsf{D} \to \mathsf{C} \to \mathsf{D}$ are naturally isomorphic (in $\mathsf{Fun}(\mathsf{C}, \mathsf{C})$ and $\mathsf{Fun}(\mathsf{D}, \mathsf{D})$, respectively) to the identity functors $\mathbf{1}_{\mathsf{C}}$ and $\mathbf{1}_{\mathsf{D}}$.

An equivalence of categories often expresses the fact that two different definitions of some type of mathematical object (vector spaces, smooth manifolds, etc.) are equivalent.

1.1.27 Exercise. Show that a functor is an equivalence iff it is fully faithful and essentially surjective.

1.1.28 Example. Let C be a category with set of objects S. Given any map of sets $S' \to S$, we can form a new category C' with set of objects S' and with a fully faithful functor $\mathsf{C}' \to \mathsf{C}$ acting as $S' \to S$ on objects. If $S' \to S$ is surjective, then $\mathsf{C}' \to \mathsf{C}$ is an equivalence of categories.

1.1.29 Example. Fix a field k, and let Vect_k denote the category of vector spaces and linear maps over k. Now consider a category in which an object is a vector space over k with a chosen basis. There are two reasonable notions of a morphism between two such objects:

(1.1.29.1) A linear map over k.

(1.1.29.2) A linear map over k sending basis elements to basis elements.

In the first case, the resulting category is equivalent to Vect_k, via the functor forgetting the basis. In the second case, the resulting category is equivalent to Set, via the functor remembering just the basis. This illustrates how the information in a category is carried by the morphisms, not the objects.

1.1.30 Example. Given a set S, we can regard S as a groupoid in which $\mathrm{Hom}(x,x) = *$ and $\mathrm{Hom}(x,y) = \varnothing$ for $x \neq y$. A groupoid is called *discrete* when it is equivalent to (the groupoid associated to) a set. A groupoid is discrete iff the automorphism group of every object is trivial.

1.1.31 Exercise (Posets as categories)**.** Show that a category is equivalent to the category associated to a poset (1.1.3) iff for every ordered pair of objects x, y, there is at most one morphism $x \to y$. Show that for any two such categories C and D, the groupoid $\mathsf{Fun}(\mathsf{C},\mathsf{D})_{\simeq}$ is discrete. Let Po' denote the category whose objects are categories in which there is at most one morphism for each ordered pair of objects, and whose morphisms are functors up to natural isomorphism (which, in view of the previous sentence, is unique if it exists). Let Po denote the category of posets and weakly order preserving maps ($s \leq t$ implies $f(s) \leq f(t)$). Show that the natural functor $\mathsf{Po} \to \mathsf{Po}'$ is an equivalence of categories. This equivalence justifies using the term 'poset' for an object of Po'.

The following is a fundamental principle of category theory.

⋆ **1.1.32 Remark** (Principle of equivalence)**.** Equivalence of categories (1.1.26) plays the role that isomorphism plays for most other mathematical objects one is used to dealing with. The reason for this difference is that most common mathematical objects (sets, groups, rings, modules, fields, vector spaces, topological spaces, manifolds, sheaves, schemes, cohomology theories, functors, etc.) form categories, whereas categories form a *2-category* (see (1.1.35)).

The *principle of equivalence* declares a statement involving categories to be 'meaningful' iff it is invariant under equivalence. For example, the cardinality of the set of isomorphism classes of objects in a category is invariant under equivalence, hence is a meaningful (albeit very crude) invariant to attach to a category. The cardinality of the set of objects in a category is not invariant under equivalence, hence is not a meaningful invariant of a category. A somewhat more subtle observation is that the principle of equivalence allows us to identify the notions of 'full subcategory' and 'fully faithful functor'.

Intuitively speaking, a statement about categories is invariant under equivalence provided it makes no reference to the notion of 'equality' of objects (and instead says things about morphisms between objects). Virtually any statement about categories which is invariant under equivalence is obviously so, to the extent that there is usually no need to state it explicitly. In particular, a construction involving categories will be invariant under equivalence whenever it is appropriately acted on by functors (and natural isomorphisms between them) of the categories in question (i.e. it should be 2-functorial on the 2-category of categories Cat (1.1.35)). It follows, for example, that basic constructions such as formation of functor categories respect the principle of equivalence by sending equivalences to equivalences.

The importance of the principle of equivalence stems from the fact that most 'categories' of interest, such as Set, Grp, Top (1.1.2), are, at best, only well defined up to (canonical) equivalence (1.1.33)(1.1.34), and so specializing a statement about categories to one of these is only meaningful when that statement is invariant under equivalence.

To develop the foundations of category theory in standard mathematical language does require some (minimal) breaking of the principle of equivalence. Indeed, the very definition of a category (1.1.1) involves a *set of objects*, in which there is necessarily a notion of equality. Proofs of statements about categories typically involve quantifying or inducting over sets of objects. This is unavoidable (though see Voevodsky [92]) but benign.

⋆ **1.1.33 Remark** (Small vs large categories)**.** A category in the sense of (1.1.1) is often called a *small category*, the adjective 'small' indicating that there is a *set* of objects and a *set* of morphisms between any pair of objects. As we have seen in (1.1.2), many, or perhaps most, 'categories' of interest are not small. There is thus a certain amount of dissonance between the foundations of the theory of categories in the sense of (1.1.1) and the scope of the intended applications of this theory.

A *large category* has a 'notion of object', a 'notion of morphism between objects', a 'notion of equality of between morphisms', and a 'notion of associative composition of morphisms'; one similarly has a notion of functor between large categories. We do not regard this sentence as a precise mathematical definition. Rather, the notion of a large category is a meta-mathematical framework into which typical categories of interest such as Set, Grp, Top (1.1.2) fall.

A large category is called *essentially small* when it is equivalent to a small category. Equivalently, a large category is essentially small when there is a set of objects representing every isomorphism class and the collection of morphisms between any pair of objects is a set (this latter condition is called being *locally small*). For example, the large categories Set, Grp, Top are not essentially small, although their full subcategories Set_κ, Grp_κ, Top_κ of sets, abelian groups, and topological spaces of cardinality less than a given cardinal κ are essentially small.

Given an essentially small category C, a *small model* of C is a small category C_0 together with an equivalence $\mathsf{C}_0 \to \mathsf{C}$. Small models always exist (by definition of essentially small), and they are moreover unique up to canonical equivalence (1.1.34). It follows that any result for small categories which adheres to the principle of equivalence remains valid for essentially small categories.

Applying category theory to large categories which are not essentially small requires either realizing that the underlying arguments go through without any smallness assumptions (that is to say, they are meta-mathematical) or working with appropriately chosen essentially small subcategories.

1.1.34 Remark (Uniqueness of small models)**.** We explain uniqueness of small models in the case of Top_κ, but the reasoning applies to any essentially small category.

(1.1.34.1) Given a set $\mathcal{U}$ along with, for every $s \in \mathcal{U}$, a topological space X_s, such that every topological space of cardinality $< \kappa$ is isomorphic to some X_s, we obtain a small category $\mathsf{Top}_\kappa^{\mathcal{U}}$ (whose set of objects is $\mathcal{U}$ and in which a morphism $s \to s'$ is a continuous map $X_s \to X_{s'}$). Such sets $\mathcal{U}$ exist: for example, fix a set Q of cardinality $\geq \kappa$, and let $\mathcal{U}$ consist of all subsets of Q of cardinality $< \kappa$ equipped with a topology.

(1.1.34.2) Given any two $\mathcal{U}$ and $\mathcal{U}'$ as above, a choice of function $f : \mathcal{U} \to \mathcal{U}'$ along with isomorphisms $X_s \xrightarrow{\sim} X_{f(s)}$ defines an equivalence of categories $\mathsf{Top}_\kappa^{\mathcal{U}} \to \mathsf{Top}_\kappa^{\mathcal{U}'}$; this recipe is moreover compatible with composition of functors. Such functions f exist by the axiom of choice.

(1.1.34.3) Given any two $f, g : \mathcal{U} \to \mathcal{U}'$ as above, there is a canonical natural isomorphism between the two induced functors $\mathsf{Top}_\kappa^{\mathcal{U}} \to \mathsf{Top}_\kappa^{\mathcal{U}'}$, namely that defined by the isomorphisms $X_{f(s)} \xleftarrow{\sim} X_s \xrightarrow{\sim} X_{g(s)}$. This construction is also compatible with composition.

1.1.35 Example (Categories of categories)**.** There are at least three diffferent answers to the question of what is the *category of categories*, related by functors

$$\mathsf{Cat}_{\mathsf{strict}} \to \mathsf{Cat} \to \mathsf{hCat}. \tag{1.1.35.1}$$

At one extreme is the category $\mathsf{Cat}_{\mathsf{strict}}$, whose objects are (small) categories and whose morphisms are functors. The category $\mathsf{Cat}_{\mathsf{strict}}$ does not see natural transformations between functors. Because of this, an equivalence of categories need not be an isomorphism in $\mathsf{Cat}_{\mathsf{strict}}$. Regarding categories as objects of $\mathsf{Cat}_{\mathsf{strict}}$ thus violates the principle of equivalence.

At another extreme is the category hCat, whose objects are (small) categories and whose morphisms are natural isomorphism classes of functors. It is promising to note that a functor is an isomorphism in hCat iff it is an equivalence of categories. Unfortunately, it turns out that hCat is a poor input to most other categorical constructions, notably limits and colimits.

The objects of Cat are again (small) categories, and $\mathrm{Hom}_{\mathsf{Cat}}(\mathsf{C}, \mathsf{D}) = \mathsf{Fun}(\mathsf{C}, \mathsf{D})_\simeq$. As $\mathsf{Fun}(\mathsf{C}, \mathsf{D})_\simeq$ is not a set but rather a groupoid, Cat is not a category but rather a *2-category* as we will explain in more detail (1.1.149) once we have in hand the language of 2-categories. It is this 2-category Cat which is really the true category of categories.

Properties of objects and morphisms

⋆ **1.1.36 Definition** (Monomorphism and epimorphism)**.** A morphism $X \to Y$ is called a *monomorphism* (or *monic*) iff the induced map $\mathrm{Hom}(Z, X) \to \mathrm{Hom}(Z, Y)$ is injective for all Z. Dually, $X \to Y$ is an *epimorphism* (or *epic*) when $\mathrm{Hom}(Y, Z) \to \mathrm{Hom}(X, Z)$ is injective for all Z. The arrows $\hookrightarrow$ and $\twoheadrightarrow$ are often used to indicate monomorphisms and epimorphisms, respectively.

1.1.37 Exercise. Show that a morphism of sets is monic iff it is injective, and is epic iff it is surjective. Show that a morphism of commutative rings is monic iff it is injective. Show that surjections and localizations of commutative rings are epimorphisms.

1.1.38 Exercise. Given a pair of morphisms $X \xrightarrow{f} Y \xrightarrow{g} X$ composing to the identity $\mathbf{1}_X$, we say that the morphism g is a *retraction* of f and that f is a *section* of g; we also say that the object X is a *retract* of Y. A morphism admitting a retraction (resp. section) is called a *split monomorphism* (resp. *split epimorphism*); these notions are dual. Show that a split monomorphism (resp. split epimorphism) is a monomorphism (resp. epimorphism). Show that a morphism which is both a split monomorphism and a split epimorphism is an isomorphism.

* *

1.1.39 Definition (Property of objects)**.** A *property* of objects in a category C is a set $\mathcal{P}$ of isomorphism classes in C. An object 'has $\mathcal{P}$' or 'is $\mathcal{P}$' when its isomorphism class is in $\mathcal{P}$.

1.1.40 Definition (Property closed under retracts)**.** A property of objects $\mathcal{P}$ is said to be *closed under retracts* iff every retract of an object with $\mathcal{P}$ has $\mathcal{P}$.

1.1.41 Definition (Arrow category)**.** The morphisms in a category C are themselves the objects of a category, namely the *arrow category* $(\mathsf{C} \downarrow \mathsf{C}) = \mathsf{Fun}(\Delta^1, \mathsf{C})$ where $\Delta^1 = (\bullet \to \bullet)$ denotes the category with two objects and a single non-identity morphism from one to the other. A morphism is said to be a retract of another when it is true for the corresponding objects of $\mathsf{Fun}(\Delta^1, \mathsf{C})$. A *property of morphisms* in C is a property of objects in $\mathsf{Fun}(\Delta^1, \mathsf{C})$.

1.1.42 Example. In any category, properties of morphisms include being an isomorphism, being a monomorphism, or being an epimorphism.

1.1.43 Example. Consider the category whose objects are finite subsets $S \subseteq \mathbb{Z}$ and whose morphisms are arbitrary maps $S \to T$. The property of a map $f : S \to T$ satisfying $f(s) \leq f(s')$ for $s \leq s'$ not a property of morphisms, because it is not invariant under isomorphisms in the category. This category is equivalent to the category of finite sets, a context in which asking for a morphism to be weakly increasing evidently has no meaning.

1.1.44 Definition (Property closed under composition)**.** A property of morphisms $\mathcal{P}$ is said to be *closed under composition* iff every isomorphism has $\mathcal{P}$ and the composition of any two $\mathcal{P}$-morphisms has $\mathcal{P}$.

1.1.45 Example. Isomorphisms, monomorphisms, and epimorphisms (in any category) are closed under composition and retracts.

1.1.46 Definition (2-out-of-3 property)**.** A property of morphisms $\mathcal{P}$ is said to have the *2-out-of-3 property* when any two out of f, g, $g \circ f$ having $\mathcal{P}$ implies that the third does too.

1.1.47 Example. In the category of abelian groups, the property of having finite kernel and finite cokernel satisfies the 2-out-of-3 property.

1.1.48 Definition (Preservation, reflection, and lifting of properties)**.** Let $\mathcal{P}$ be a property of objects in categories C and D, and let $F : \mathsf{C} \to \mathsf{D}$ be a functor. We say F *preserves* $\mathcal{P}$-objects when $c \in \mathcal{P}$ implies $F(c) \in \mathcal{P}$ for every object $c \in \mathsf{C}$. We say F *reflects* $\mathcal{P}$-objects when $F(c) \in \mathcal{P}$ implies $c \in \mathcal{P}$ for every object $c \in \mathsf{C}$. We say F *lifts* $\mathcal{P}$-objects when every $d \in \mathcal{P}$ is isomorphic to $F(c)$ for some $c \in \mathcal{P}$.

1.1.49 Example. Every functor preserves isomorphisms. The forgetful functor $\mathsf{Grp} \to \mathsf{Set}$ reflects isomorphisms (a group homomorphism is an isomorphism iff it is a bijection of sets). The forgetful functor $\mathsf{Top} \to \mathsf{Set}$ does not reflect isomorphisms (a continuous bijection of topological spaces need not have a continuous inverse).

1.1.50 Exercise. Let $F : \mathsf{C} \to \mathsf{D}$ be a functor for which $F(f)$ being an isomorphism implies f is a split epimorphism. Show that F reflects isomorphisms.

1.1.51 Exercise (Checking isomorphism on morphism sets)**.** Let $f : x \to y$ be a morphism in a category C. Show that f is an isomorphism iff $(f \circ -) : \mathrm{Hom}_{\mathsf{C}}(a, x) \to \mathrm{Hom}_{\mathsf{C}}(a, y)$ is an isomorphism for all $a \in \mathsf{C}$ (taking $a = y$ and lifting 1_y gives a one-sided inverse to f, then apply (1.1.50)).

Limits and colimits

⋆ **1.1.52 Definition** (Final and initial objects)**.** A *final object* in a category C is an object X such that $\mathrm{Hom}(Z, X) = *$ for every $Z \in \mathsf{C}$. Final objects are *unique up to unique isomorphism*: if X and X' are both final objects, then there is a unique isomorphism $X \to X'$; because of this, we may speak of *the* final object of C (in accordance with the principle of equivalence (1.1.32)). Dually, an object X is *initial* when $\mathrm{Hom}(X, Z) = *$ for every Z. An object which is both final and initial is called a *zero object*. A category which has a zero object is called *pointed*.

1.1.53 Example. The initial objects of Set and Top are the empty set/space $\varnothing$. The final objects of Set and Top are the one-point set/space $*$. The category Grp is pointed: the trivial group $\mathbf{1}$ is a zero object (both initial and final).

1.1.54 Exercise. Show that a functor which lifts final objects also reflects final objects.

⋆ **1.1.55 Definition** (Diagram)**.** A *diagram shape* J consists of a set of 0-cells (vertices), a set of 1-cells (arrows between vertices), and a set of 2-cells, disks with boundary of the form

$$\begin{tikzcd} & a_1 \arrow[r] & \cdots \arrow[r] & a_n \arrow[dr] & \\ x \arrow[ur] \arrow[dr] & & & & y \\ & b_1 \arrow[r] & \cdots \arrow[r] & b_m \arrow[ur] & \end{tikzcd} \tag{1.1.55.1}$$

for some integers $n, m \geq 0$. A *diagram* of shape J in a category C is a map $D : J \to \mathsf{C}$ associating to each 0-cell an object, to each 1-cell a morphism, such that for each 2-cell (1.1.55.1), composition along the two paths from x to y yields the same morphism $x \to y$. Diagrams form a category $\mathsf{Fun}(J, \mathsf{C})$ in which a morphism $D \to D'$ associates to each 0-cell $j \in J$ a morphism $D(j) \to D'(j)$ such that for each 1-cell $j \to j'$, the two compositions $D(j) \to D'(j) \to D'(j')$ and $D(j) \to D(j') \to D'(j')$ coincide.

There is an evident similarity between a diagram shape and a category, and between a diagram and a functor; in fact, this is more than just a similarity. We can regard a category as a diagram shape by taking its objects to be the 0-cells, its morphisms to be the 1-cells, and adding a triangular 2-cell

$$\begin{tikzcd} & b \arrow[dr] & \\ a \arrow[ur] \arrow[rr] & & c \end{tikzcd} \tag{1.1.55.2}$$

for each pair of morphisms $a \to b \to c$ composing to a morphism $a \to c$. In the other direction, a diagram shape determines a category whose objects are the 0-cells and whose morphisms are directed paths (formal compositions) of 1-cells, modulo the relation that the formal compositions of the two maximal paths bounding a 2-cell (1.1.55.1) are the same. A diagram $J \to \mathsf{C}$ is then exactly the same as a functor to C from the category associated to J.

We emphasize that a diagram $J \to \mathsf{C}$ consists of *specified data* for each 0-cell and 1-cell, satisfying a *property* for each 2-cell. The assertion that a given diagram 'commutes' is simply the assertion that certain evident 2-cells (usually all possible 2-cells) are present; often this assertion is implicit in writing the diagram (diagrams commute unless the contrary is explicitly specified).

1.1.56 Exercise. Consider a pullback diagram in the category of sets.

$$\begin{tikzcd} B \arrow[r] \arrow[d] & A \arrow[d] \\ Y \arrow[r] & X \end{tikzcd} \tag{1.1.56.1}$$

Show that if $Y \sqcup A \to X$ is surjective, then the diagram is also a pushout.

1.1.57 Exercise (Cancellation for fiber products)**.** Fix a diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[d] \\ D \arrow[r] & E \arrow[r] & F \end{tikzcd} \tag{1.1.57.1}$$

in which the right square (involving B, C, E, F) is a fiber square. Consider the induced maps

$$\begin{tikzcd} A \arrow[r] \arrow[rrr, bend left] & B \times_E D \arrow[r, "\sim"] & C \times_F E \times_E D \arrow[r, equal] & C \times_F D \end{tikzcd} \tag{1.1.57.2}$$

and conclude that the composite square (involving A, C, D, F) is a fiber square iff the left square (involving A, B, D, E) is a fiber square.

1.1.58 Exercise. In the situation of cancellation for fiber products (1.1.57), it is not true in general that if the left square and the composite square are pullbacks then the right square is a pullback. Show that this does hold, however, under the additional assumption that the fiber product functor $\mathsf{C}_{/E} \xrightarrow{\times_E D} \mathsf{C}_{/D}$ reflects isomorphisms. Note that this is the case for any surjection $D \twoheadrightarrow E$ in the category of sets.

1.1.59 Definition (Pullback and pushout of a morphism)**.** Let $f : X \to Y$ be a morphism. A pullback of f is a morphism $f' : X' \to Y'$ fitting into a pullback square:

$$\begin{tikzcd} X' \arrow[r] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r] & Y \end{tikzcd} \tag{1.1.59.1}$$

Dually, a pushout of f is a morphism f' fitting into a pushout square:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\ X' \arrow[r, "f'"] & Y' \end{tikzcd} \tag{1.1.59.2}$$

1.1.60 Definition (Property preserved under pullback)**.** A property of morphisms $\mathcal{P}$ is said to be *preserved under pullback* when the following implication holds:

(1.1.60.1) For every $\mathcal{P}$-morphism $X \to Y$ and every morphism $Z \to Y$, the pullback $X \times_Y Z \to Z$ exists and has $\mathcal{P}$.

More generally, we say $\mathcal{P}$ is *preserved under* $\mathcal{Q}$*-pullback* ($\mathcal{Q}$ another property of morphisms) when the implication (1.1.60.1) holds provided $Z \to Y$ has $\mathcal{Q}$.

1.1.61 Exercise. Show that isomorphisms, monomorphisms, and split epimorphisms are preserved under pullback.

1.1.62 Exercise. Suppose $\mathcal{P}$ is a property of morphisms which is preserved under pullback and closed under composition. Show that $\mathcal{P}$ is preserved under fiber product, in the sense that for $\mathcal{P}$-morphisms $X \to Y$ and $X' \to Y'$ and any morphisms $Y \to Z \leftarrow Y'$, if $Y \times_Z Y'$ exists then so does $X \times_Z X'$ and the morphism $X \times_Z X' \to Y \times_Z Y'$ has $\mathcal{P}$.

⋆ **1.1.63 Definition** (Relative diagonal)**.** For any morphism $X \to Y$ in a category, the diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & X \arrow[d] \\ X \arrow[r] & Y \end{tikzcd} \tag{1.1.63.1}$$

induces a morphism $X \to X \times_Y X$ called the *(relative) diagonal* of $X \to Y$. The *nth diagonal* is the nth iterate of this construction: the zeroth diagonal of $X \to Y$ is $X \to Y$ itself, the first diagonal is $X \to X \times_Y X$, the second diagonal is $X \to X \times_{X \times_Y X} X$, etc.

1.1.64 Exercise. Show that the diagonal of any map of sets is injective, and that the diagonal of an injective map of sets is an isomorphism. Show that in any category, the diagonal of any morphism (if it exists) is a monomorphism, and that the diagonal of any monomorphism exists and is an isomorphism.

1.1.65 Definition (Properties of the diagonal)**.** Let $\mathcal{P}$ be any property of morphisms in a category which has all fiber products. A morphism is said to have property $\mathcal{P}_\Delta$ when its relative diagonal has property $\mathcal{P}$.

1.1.66 Exercise (The diagonal of a pullback is a pullback of the diagonal)**.** Use cancellation for fiber products (1.1.57) to show that if the left square below is a fiber square, then so are the right two squares.

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & Y' \arrow[d] \\ X \arrow[r] & Y \end{tikzcd} \quad \Longrightarrow \quad \begin{tikzcd} X' \arrow[r] \arrow[d] & X' \times_{Y'} X' \arrow[r] \arrow[d] & Y' \arrow[d] \\ X \arrow[r] & X \times_Y X \arrow[r] & Y \end{tikzcd} \tag{1.1.66.1}$$

Conclude that if $\mathcal{P}$ is preserved under pullback then so is $\mathcal{P}_\Delta$.

1.1.67 Exercise (The diagonal of a composition is a composition of pullbacks of diagonals)**.** Show that if $X \to Y \to Z$ are morphisms, then $X \times_Y X \to X \times_Z X$ is a pullback of $Y \to Y \times_Z Y$. Conclude that if $\mathcal{P}$ is preserved under pullback and closed under composition then so is $\mathcal{P}_\Delta$.

1.1.68 Lemma (Cancellation)**.** *Let $\mathcal{P}$ be a property of morphisms preserved under pullback and closed under composition. If the composition $X \to Y \to Z$ has $\mathcal{P}$ and $Y \to Z$ has $\mathcal{P}_\Delta$, then $X \to Y$ has $\mathcal{P}$.*

Proof. Factor $X \to Y$ into $X \to X \times_Z Y \to Y$. The map $X \to X \times_Z Y$ is a pullback of $Y \to Y \times_Z Y$ so has $\mathcal{P}$. The map $X \times_Z Y \to Y$ is a pullback of $X \to Z$ so has $\mathcal{P}$. □

1.1.69 Exercise. If $X \to Y$ and $Y \to Z$ are maps of sets whose composition $X \to Z$ is injective, then the first map $X \to Y$ is also injective. Prove this using the abstract cancellation property (1.1.68).

1.1.70 Definition (Twisted arrow category)**.** Let C be a category, and recall the arrow category $(\mathsf{C} \downarrow \mathsf{C})$ (1.1.41), whose objects are morphisms $c \to d$ in C and whose morphisms $(c \to d) \to (c' \to d')$ are commutative squares of the following shape.

$$\begin{tikzcd} c \arrow[r] \arrow[d] & c' \arrow[d] \\ d \arrow[r] & d' \end{tikzcd} \tag{1.1.70.1}$$

The *twisted arrow category* $(\mathsf{C}^{\mathsf{op}} \downarrow \mathsf{C})$ has the same objects, but a morphism $(c \to d) \to (c' \to d')$ is a commutative square of the following shape.

$$\begin{tikzcd} c \arrow[d] & c' \arrow[l] \arrow[d] \\ d \arrow[r] & d' \end{tikzcd} \tag{1.1.70.2}$$

Note the direction of the top arrow.

1.1.71 Definition (End and coend)**.** Let $F : \mathsf{C}^{\mathsf{op}} \times \mathsf{C} \to \mathsf{D}$ be a functor. Its *end* is the limit of its pullback to the twisted arrow category $(\mathsf{C}^{\mathsf{op}} \downarrow \mathsf{C})$.

$$\lim_{c^{\mathsf{op}} \to c} F(c^{\mathsf{op}}, c) = \mathrm{Eq}\Bigg(\prod_c F(c,c) \overset{\prod_f (\mathbf{1}_{\mathsf{C}^{\mathsf{op}}} \times f)}{\underset{\prod_f (f \times \mathbf{1}_{\mathsf{C}})}{\rightrightarrows}} \prod_{f : c_1 \to c_2} F(c_1, c_2) \Bigg) \tag{1.1.71.1}$$

Dually, the *coend* of a functor $F : \mathsf{C} \times \mathsf{C}^{\mathsf{op}} \to \mathsf{D}$ is the colimit of its pullback to $(\mathsf{C} \downarrow \mathsf{C}^{\mathsf{op}})$.

$$\operatorname*{colim}_{c \to c^{\mathsf{op}}} F(c, c^{\mathsf{op}}) = \operatorname{Coeq}\left(\coprod_{f : c_1 \to c_2} F(c_1, c_2) \overset{\coprod_f (\mathbf{1}_{\mathsf{C}} \times f)}{\underset{\coprod_f (f \times \mathbf{1}_{\mathsf{C}^{\mathsf{op}}})}{\rightrightarrows}} \coprod_c F(c, c) \right) \tag{1.1.71.2}$$

1.1.72 Exercise. Let $F, G : \mathsf{C} \to \mathsf{E}$ be two functors, which together determine a functor of the following shape.

$$\mathsf{C}^{\mathsf{op}} \times \mathsf{C} \to \mathsf{Set} \tag{1.1.72.1}$$
$$(c^{\mathsf{op}}, c) \mapsto \operatorname{Hom}_{\mathsf{E}}(F(c^{\mathsf{op}}), G(c)) \tag{1.1.72.2}$$

Show that the set of natural transformations $F \to G$ is naturally identified with end of this functor.

$$\operatorname{Hom}_{\mathsf{Fun}(\mathsf{C},\mathsf{E})}(F, G) = \lim_{c^{\mathsf{op}} \to c} \operatorname{Hom}_{\mathsf{E}}(F(c^{\mathsf{op}}), G(c)) \tag{1.1.72.3}$$

⋆ **1.1.73 Exercise** (Final and initial functors)**.** Show that for a functor $F : \mathsf{C} \to \mathsf{D}$, the following are equivalent:

(1.1.73.1) For every diagram $p : \mathsf{D} \to \mathsf{A}$ which has a limit, the pullback diagram $F^*p : \mathsf{C} \to \mathsf{A}$ also has a limit and the map $\lim_{\mathsf{D}} p \to \lim_{\mathsf{C}} F^*p$ is an isomorphism.

(1.1.73.2) For every $d \in \mathsf{D}$, the colimit $\operatorname{colim}_{\mathsf{C}_{/d}} *$ in Set is $*$.

A functor satisfying these properties is called *initial*. Show that a functor $d : * \to \mathsf{D}$ is initial iff d is an initial object of D. Although initial functors generalize initial objects, their use is somewhat different. The dual notion of initial is called *final* (F is final iff F^{op} is initial). Formulate precisely the duals of both properties above.

1.1.74 Exercise. Show that the inclusion of a final object is a final functor.

1.1.75 Lemma. *Every left adjoint functor is initial.*

Proof. A functor $F : \mathsf{C} \to \mathsf{D}$ has a right adjoint iff the category $\mathsf{C}_{/d}$ has a final object for every $d \in \mathsf{D}$. The colimit of the constant diagram $*$ over any category with a final object is $*$. □

⋆ **1.1.76 Definition** (Preservation of colimits)**.** A functor $F : \mathsf{C} \to \mathsf{D}$ is said to *preserve* a colimit diagram in C when it is sent to a colimit diagram in D by F. For example, we can ask that a functor preserve pushouts, initial objects, finite coproducts, all coproducts, finite colimits, filtered colimits, sifted colimits, simplicial realizations, all colimits, etc. A functor which preserves all colimits is called *cocontinuous*.

1.1.77 Exercise. Show that the forgetful functor $\mathsf{Ab} \to \mathsf{Set}$ preserves limits but not colimits.

Representability

⋆ **1.1.78 Definition** (Presheaf)**.** A *presheaf* on a category C is a functor $\mathsf{C}^{\mathsf{op}} \to \mathsf{Set}$. The category of presheaves on C is denoted $\mathsf{P}(\mathsf{C}) = \mathsf{Fun}(\mathsf{C}^{\mathsf{op}}, \mathsf{Set})$ (note that this is a large category (1.1.33) unless $\mathsf{C} = \varnothing$). More generally, a presheaf on C *valued in* E is a functor $\mathsf{C}^{\mathsf{op}} \to \mathsf{E}$, an object of the category $\mathsf{P}(\mathsf{C}; \mathsf{E}) = \mathsf{Fun}(\mathsf{C}^{\mathsf{op}}, \mathsf{E})$.

1.1.79 Example ($\mathsf{P}(\mathsf{Set})$ is not locally small)**.** Recall that a (possibly large) category C is called *locally small* when $\mathrm{Hom}_{\mathsf{C}}(x, y)$ is a set for every pair of objects $x, y \in \mathsf{C}$ (1.1.33). Most large categories of interest (such as Set, Grp, Top) are locally small. A notable exception is that the category of functors out of a large category is often not locally small. In particular, a certain amount of care is necessary when working with presheaf categories since they are often not locally small.

Here is an example to show that the large category $\mathsf{P}(\mathsf{Set}) = \mathsf{Fun}(\mathsf{Set}^{\mathsf{op}}, \mathsf{Set})$ is not locally small. Given a set S, consider the set of functions $\alpha : 2^S \to \mathrm{Card}$ assigning to each subset $A \subseteq S$ a cardinal number $\alpha(A)$ which is at most $|A|$ (note that the collection $Q(S)$ of all such functions α is indeed a set). Given a map of sets $f : T \to S$, we may define the pullback $f^*\alpha : 2^T \to \mathrm{Card}$ by $(f^*\alpha)(A) = \alpha(f(A))$ (note that $|f(A)| \leq |A|$). It is evident that $(gf)^* = f^* g^*$, so this defines a functor $Q : \mathsf{Set}^{\mathsf{op}} \to \mathsf{Set}$. Any endomorphism $\gamma : \mathrm{Card} \to \mathrm{Card}$ with the property that $\gamma(\kappa) \leq \kappa$ for all cardinals κ gives an endomorphism of the functor Q. These endomorphisms are all distinct, so $\mathrm{Hom}_{\mathsf{P}(\mathsf{Set})}(Q, Q)$ is large (consider, for instance, the endomorphisms $\max(\kappa_0, -) : \mathrm{Card} \to \mathrm{Card}$ for various cardinals κ_0).

⋆ **1.1.80 Definition** (Representable)**.** A presheaf $F \in \mathsf{P}(\mathsf{C})$ is called *representable* when it is isomorphic to a presheaf of the form $\mathrm{Hom}(-, X)$ for some $X \in \mathsf{C}$.

1.1.81 Exercise (Idempotent completion)**.** Let C be a category. An endomorphism π of an object $X \in \mathsf{C}$ is called *idempotent* when $\pi^2 = \pi$. Given a retraction $Y \to X \to Y$, the composition $X \to Y \to X$ is idempotent. Show that this idempotent, call it π, determines the retraction uniquely up to unique isomorphism by showing that $\mathrm{Hom}(Z, Y) = \mathrm{Hom}(Z, X)\pi \subseteq \mathrm{Hom}(Z, X)$ is the set of maps $Z \to X$ which factor as πf for some $f : Z \to X$ (thus the pair (X, π) determines the Yoneda presheaf of Y). We say that an idempotent π *splits* when it comes from a retraction (that is, when the functor $\mathrm{Hom}(-, X)\pi$ is representable). Split idempotents are preserved by any functor. A category is called *idempotent-complete* when every idempotent splits.

Given a category C, its *idempotent completion* $\Pi\mathsf{C}$ is defined as follows. An object of $\Pi\mathsf{C}$ is a pair (X, π) where $X \in \mathsf{C}$ is an object and $\pi \in \mathrm{Hom}_{\mathsf{C}}(X, X)$ is idempotent. Morphisms in $\Pi\mathsf{C}$ are given by

$$\mathrm{Hom}_{\Pi\mathsf{C}}((X, \pi), (X', \pi')) = \pi \mathrm{Hom}_{\mathsf{C}}(X, X')\pi' \subseteq \mathrm{Hom}_{\mathsf{C}}(X, X'), \tag{1.1.81.1}$$

namely the subset of $\mathrm{Hom}(X, X')$ consisting of morphisms which admit a factorization $\pi' f \pi$ (equivalently those morphisms g satisfying $g = \pi' g \pi$). There is an evident fully faithful embedding $\mathsf{C} \hookrightarrow \Pi\mathsf{C}$ given by $X \mapsto (X, \mathbf{1}_X)$. The maps $\pi : X \to (X, \pi)$ and $\pi : (X, \pi) \to X$ express (X, π) as a retract of X in the category $\Pi\mathsf{C}$. Show that $\Pi\mathsf{C}$ is idempotent-complete and that for any idempotent-complete category D, the restriction functor $\mathsf{Fun}(\Pi\mathsf{C}, \mathsf{D}) \to \mathsf{Fun}(\mathsf{C}, \mathsf{D})$ is an equivalence of categories.

1.1.82 Example. The idempotent completion of the category of free R-modules is the category of projective R-modules.

Adjoint functors

$\star$ **1.1.83 Lemma** (Adjoints and full faithfulness)**.** *Let (F, G) be a pair of adjoint functors $F : \mathsf{C} \rightleftarrows \mathsf{D} : G$. The left adjoint F is fully faithful iff the unit map $\eta : 1 \to GF$ is an isomorphism.*

Proof. Since η is a natural transformation, the following diagram commutes for every $x, y \in \mathsf{C}$.

$$\begin{tikzcd} \operatorname{Hom}_{\mathsf{C}}(x, y) \arrow[r, "F"] \arrow[d, "\eta(y)\circ"'] & \operatorname{Hom}_{\mathsf{D}}(Fx.Fy) \arrow[d, "G"] \\ \operatorname{Hom}_{\mathsf{C}}(x, GFy) & \operatorname{Hom}_{\mathsf{C}}(GFx, GFy) \arrow[l, "\circ\eta(x)"'] \end{tikzcd} \tag{1.1.83.1}$$

The anti-diagonal map $\operatorname{Hom}_{\mathsf{D}}(Fx, Fy) \to \operatorname{Hom}_{\mathsf{C}}(x, GFy)$ is an isomorphism by definition of η being the unit of an adjunction (F, G). We conclude that if $\eta(y) : y \to G(F(y))$ is an isomorphism, then the map $F(x, y) : \operatorname{Hom}_{\mathsf{C}}(x, y) \to \operatorname{Hom}_{\mathsf{D}}(Fx, Fy)$ is an isomorphism. Conversely, if $F(x, y) : \operatorname{Hom}_{\mathsf{C}}(x, y) \to \operatorname{Hom}_{\mathsf{D}}(Fx, Fy)$ is an isomorphism for all x, then $(- \circ \eta(y)) : \operatorname{Hom}_{\mathsf{C}}(x, y) \to \operatorname{Hom}_{\mathsf{C}}(x, GFy)$ is an isomorphism for all x, which implies $\eta(y)$ is an isomorphism (1.1.51). $\square$

1.1.84 Exercise (Internal Hom)**.** Let C be a category with binary products. Given a pair of objects $X, Y \in \mathsf{C}$, we may consider the presheaf $\underline{\operatorname{Hom}}(X, Y) \in \mathsf{P}(\mathsf{C})$ which sends $Z \in \mathsf{C}$ to the set of maps $X \times Z \to Y$. If this presheaf is representable, the resulting object $\underline{\operatorname{Hom}}(X, Y) \in \mathsf{C}$ (with its 'universal' map $X \times \underline{\operatorname{Hom}}(X, Y) \to Y$) is called the *internal Hom* between X and Y. Show there are canonical associative maps $\underline{\operatorname{Hom}}(X, Y) \times \underline{\operatorname{Hom}}(Y, Z) \to \underline{\operatorname{Hom}}(X, Z)$ (of presheaves, hence of internal Homs provided the latter exist).

1.1.85 Exercise. Consider an adjunction (f^*, f_*) of functors $f^* : X \rightleftarrows Y : f_*$. By composing with the unit and counit of the adjunction, define natural bijections $\operatorname{Hom}(f^*x, y) = \operatorname{Hom}(x, f_*y)$ and $\operatorname{Hom}(xf_*, y) = \operatorname{Hom}(x, yf^*)$, as illustrated in the following diagrams.

$$\begin{tikzcd} & X \arrow[dd, "f^*"] \\ A \arrow[ur, bend left, "x"] \arrow[dr, bend right, "y"'] \arrow[r, phantom, "\Downarrow"] & \\ & Y \end{tikzcd} \quad f^*x \to y \quad \leftrightsquigarrow \quad x \to f_*y \quad \begin{tikzcd} & X \\ A \arrow[ur, bend left, "x"] \arrow[dr, bend right, "y"'] \arrow[r, phantom, "\Downarrow"] & \\ & Y \arrow[uu, "f_*"'] \end{tikzcd} \tag{1.1.85.1}$$

$$\begin{tikzcd} X \arrow[dr, bend left, "x"] \arrow[dd, "f^*"'] & \\ \arrow[r, phantom, "\Downarrow"] & A \\ Y \arrow[ur, bend right, "y"'] & \end{tikzcd} \quad x \to yf^* \quad \leftrightsquigarrow \quad xf_* \to y \quad \begin{tikzcd} X \arrow[dr, bend left, "x"] & \\ \arrow[r, phantom, "\Downarrow"] & A \\ Y \arrow[ur, bend right, "y"'] \arrow[uu, "f_*"] & \end{tikzcd} \tag{1.1.85.2}$$

1.1.86 Exercise (Beck–Chevalley condition)**.** Consider adjoint pairs (f^*, f_*), (g^*, g_*), (x^*, x_*), (y^*, y_*), and a diagram of the following shape

$$\begin{tikzcd} A \arrow[r, shift left, "x_*"] \arrow[d, shift right, "g_*"'] & B \arrow[l, shift left, "x^*"] \arrow[d, shift left, "f_*"] \\ C \arrow[u, shift right, "g^*"'] \arrow[r, shift right, "y_*"'] & D \arrow[l, shift right, "y^*"'] \arrow[u, shift left, "f^*"] \end{tikzcd} \tag{1.1.86.1}$$

which does not necessarily commute. Apply (1.1.85) to show that a natural transformation of any of the forms $f_*x_* \to y_*g_*$, $y^*f_* \to g_*x^*$, $g^*y^* \to x^*f^*$ determines all the others in a canonical way.

$$\begin{tikzcd} A \arrow[r, "x_*"] \arrow[d, "g_*"'] \arrow[dr, phantom, "\swarrow"] & B \arrow[d, "f_*"] \\ C \arrow[r, "y_*"'] & D \end{tikzcd} \quad \leftrightsquigarrow \quad \begin{tikzcd} A \arrow[d, "g_*"'] \arrow[dr, phantom, "\nwarrow"] & B \arrow[l, "x^*"] \arrow[d, "f_*"] \\ C & D \arrow[l, "y^*"] \end{tikzcd} \quad \leftrightsquigarrow \quad \begin{tikzcd} A \arrow[dr, phantom, "\nearrow"] & B \arrow[l, "x^*"] \\ C \arrow[u, "g^*"] & D \arrow[l, "y^*"] \arrow[u, "f^*"'] \end{tikzcd} \tag{1.1.86.2}$$

A natural isomorphism $f_*x_* \to y_*g_*$ is said to satisfy the *Beck–Chevalley condition* when the resulting natural transformation $y^*f_* \to g_*x^*$ is also an isomorphism.

Reflective subcategories

⋆ **1.1.87 Definition** (Reflective subcategory)**.** A *reflective subcategory* is a full subcategory $i : \mathsf{A}_0 \subseteq \mathsf{A}$ whose inclusion functor i has a left adjoint r, called the *reflector*.

1.1.88 Example. The category of abelian groups Ab is a reflective subcategory of the category of all groups Grp. The reflector $\mathsf{Grp} \to \mathsf{Ab}$ is the abelianization functor $G \mapsto G/[G, G]$.

1.1.89 Example. Let hSpc denote the category of CW-complexes and homotopy classes of maps. Discrete spaces $\mathsf{Set} \subseteq \mathsf{hSpc}$ form a reflective subcategory, with reflector the π_0 functor $\mathsf{hSpc} \to \mathsf{Set}$.

1.1.90 Exercise. Show that if $\mathsf{A} \subseteq \mathsf{B}$ and $\mathsf{B} \subseteq \mathsf{C}$ are reflective, then $\mathsf{A} \subseteq \mathsf{C}$ is reflective and $r_{\mathsf{AC}} = r_{\mathsf{AB}}r_{\mathsf{BC}}$.

⋆ **1.1.91 Exercise** (Limits and colimits in a reflective subcategory)**.** Let $i : \mathsf{A}_0 \to \mathsf{A}$ be the inclusion of a reflective subcategory with left adjoint r. Show that $\mathsf{A}_0 \subseteq \mathsf{A}$ is closed under all limits (i.e. a limit of objects of A_0 which exists in A must in fact lie in A_0). Show that if a diagram in A_0 has a colimit in A, then it has a colimit in A_0, namely the image of the colimit in A under the reflector r.

1.1.92 Exercise (Cocontinuity on a reflective subcategory)**.** Let $\mathsf{A}_0 \subseteq \mathsf{A}$ be a reflective subcategory, and let E be a cocomplete category. Show that a functor $\mathsf{A} \to \mathsf{E}$ sending reflections to isomorphisms is cocontinuous iff its restriction to A_0 is cocontinuous.

1.1.93 Exercise. Let $\mathsf{A}_0 \subseteq \mathsf{A}$ be a reflective subcategory (1.1.87). Show that if the reflector $\mathsf{A} \to \mathsf{A}_0$ preserves κ-small products and κ-small products distribute over colimits in A, then κ-small products distribute over colimits in A_0.

1.1.94 Exercise (Characterizing a reflective subcategory)**.** Let $\mathsf{C}_0 \subseteq \mathsf{C}$ be a reflective subcategory. Let $\mathcal{P}$ be a property of objects in C which is satisfied by all objects of C_0. Show that if the reflector $\mathsf{C} \to \mathsf{C}_0$ reflects isomorphisms when restricted to the full subcategory $\mathsf{C}_\mathcal{P} \subseteq \mathsf{C}$ of objects satisfying $\mathcal{P}$, then conversely all objects satisfying $\mathcal{P}$ lie in C_0.

1.1.95 Exercise. Suppose $f_! : \mathsf{A} \rightleftarrows \mathsf{B} : f^*$ are adjoint $(f_!, f^*)$ and restrict to an equivalence between full subcategories $\mathsf{A}_0 \subseteq \mathsf{A}$ and $\mathsf{B}_0 \subseteq \mathsf{B}$. Let $\mathcal{P}$ be a property of objects of B which holds for all objects of B_0 and which implies being sent to A_0 by f^*. Show that if f^* restricted to $\mathsf{B}_\mathcal{P}$ reflects isomorphisms, then $\mathsf{B}_\mathcal{P} = \mathsf{B}_0$ (reduce to (1.1.94) by considering the reflective subcategory $\mathsf{A}_0 \rightleftarrows (f^*)^{-1}(\mathsf{A}_0)$).

1.1.96 Definition (Local object)**.** Let C be a category, and let Λ be a set of morphisms in C. An object $X \in \mathsf{C}$ is called *(right) Λ-local* when the functor $\operatorname{Hom}(-, X)$ sends morphisms in Λ to isomorphisms.

1.1.97 Lemma. *Let $\mathsf{C}_0 \subseteq \mathsf{C}$ be a reflective subcategory. An object $X \in \mathsf{C}$ lies in C_0 iff the functor* $\operatorname{Hom}(-, X)$ *sends all reflections $Y \to rY$ to isomorphisms.*

Proof. Suppose $\operatorname{Hom}(-, X)$ sends reflections to isomorphisms, and let us show that $X \in \mathsf{C}_0$ (the other direction is trivial). Apply the hypothesis on X to the reflection $\ell_X : X \to rX$ to see that $\operatorname{Hom}(rX, X) \xrightarrow{\circ \ell_X} \operatorname{Hom}(X, X)$ is an isomorphism. Lifting the identity map $\mathbf{1}_X$ produces a map $s : rX \to X$ for which the composition $X \xrightarrow{\ell_X} rX \xrightarrow{s} X$ is the identity. To show that the other composition $rX \xrightarrow{s} X \xrightarrow{\ell_X} rX$ is the identity, it suffices to show it is an isomorphism. Consider the commuting square obtained by applying the functor r to the morphism s.

$$\begin{tikzcd} rX \arrow[r, "s"] \arrow[d, "\ell_{rX}"'] & X \arrow[d, "\ell_X"] \\ rrX \arrow[r, "rs"] & rX \end{tikzcd} \tag{1.1.97.1}$$

The morphism ℓ_{rX} is an isomorphism, so it suffices to show that rs is an isomorphism. Now r sends ℓ_X to an isomorphism, so it must also send its retraction s to an isomorphism. $\square$

1.1.98 Definition (Passing a functor to reflective subcategories)**.** Let $\mathsf{A}_0 \subseteq \mathsf{A}$ and $\mathsf{B}_0 \subseteq \mathsf{B}$ be reflective subcategories. A functor $f : \mathsf{A} \to \mathsf{B}$ induces a functor $f_0 = rfi : \mathsf{A}_0 \to \mathsf{B}_0$. For functors $f : \mathsf{A} \to \mathsf{B}$ and $g : \mathsf{B} \to \mathsf{C}$, there is a canonical natural transformation $(gf)_0 = rgfi \xrightarrow{rg\eta fi} rgirfi = g_0 f_0$. For a third functor $h : \mathsf{C} \to \mathsf{D}$, the diagram of canonical natural transformations

$$\begin{tikzcd} (hgf)_0 \arrow[r] \arrow[d] & h_0(gf)_0 \arrow[d] \\ (hg)_0 f_0 \arrow[r] & h_0 g_0 f_0 \end{tikzcd} \tag{1.1.98.1}$$

commutes.

1.1.99 Exercise. Let $f : \mathsf{A} \to \mathsf{B}$ be a functor, and let $\mathsf{A}_0 \subseteq \mathsf{A}$ and $\mathsf{B}_0 \subseteq \mathsf{B}$ be reflective subcategories with reflectors r_A and r_B. Show that f preserves reflections (i.e. sends reflections to reflections) iff $f(\mathsf{A}_0) \subseteq \mathsf{B}_0$ and $r_\mathsf{B} f$ sends reflections to isomorphisms.

1.1.100 Exercise. Let $f_! : \mathsf{A} \rightleftarrows \mathsf{B} : f^*$ be adjoint $(f_!, f^*)$, and let $\mathsf{A}_0 \subseteq \mathsf{A}$ and $\mathsf{B}_0 \subseteq \mathsf{B}$ be reflective subcategories with reflectors r_A and r_B. Use (1.1.97) to show that $r_\mathsf{B} f_!$ sends reflections to isomorphisms iff $f^*(\mathsf{B}_0) \subseteq \mathsf{A}_0$.

1.1.101 Exercise (Passing an adjunction to reflective subcategories)**.** Let $f_! : \mathsf{A} \rightleftarrows \mathsf{B} : f^*$ be adjoint $(f_!, f^*)$. Let $i_\mathsf{A} : \mathsf{A}_0 \subseteq \mathsf{A}$ and $i_\mathsf{B} : \mathsf{B}_0 \subseteq \mathsf{B}$ be reflective subcategories with reflectors r_A and r_B. Show that if $f^*(\mathsf{B}_0) \subseteq \mathsf{A}_0$ then there is an adjunction $(r_\mathsf{B} f_!, f^*)$ of functors $r_\mathsf{B} f_! : \mathsf{A}_0 \rightleftarrows \mathsf{B}_0 : f^*$. More precisely, show that such an adjunction is given by the identifications

$$\operatorname{Hom}(r_\mathsf{B} f_! X, Y) \overset{(r_\mathsf{B}, i_\mathsf{B})}{=\!=\!=} \operatorname{Hom}(f_! X, Y) \overset{(f_!, f^*)}{=\!=\!=} \operatorname{Hom}(X, f^* Y) \tag{1.1.101.1}$$

for $X \in \mathsf{A}_0$ and $Y \in \mathsf{B}_0$, corresponding to the unit and counit maps

$$\mathbf{1} \xrightarrow{\eta} f^* f_! \xrightarrow{f^* \eta_B f_!} f^* r_\mathsf{B} f_! \quad : \mathsf{A}_0 \to \mathsf{A}_0 \tag{1.1.101.2}$$

$$r_\mathsf{B} f_! f^* \xrightarrow{r_\mathsf{B} \varepsilon} r_\mathsf{B} \underset{\sim}{\xleftarrow{\eta_\mathsf{B}}} \mathbf{1} \quad : \mathsf{B}_0 \to \mathsf{B}_0 \tag{1.1.101.3}$$

where $\eta : \mathbf{1} \to f^* f_!$ and $\varepsilon : f_! f^* \to \mathbf{1}$ are the unit and counit maps of the adjunction $(f_!, f^*)$ and $\eta_\mathsf{B} : \mathbf{1} \to r_\mathsf{B}$ is the unit of the reflection r_B.

1.1.102 Lemma. *In the setup of* (1.1.101)*, if $f_!$ is fully faithful and $r_\mathsf{A} f^*$ sends reflections to isomorphisms, then $r_\mathsf{B} f_!$ is fully faithful.*

Proof. It is equivalent (1.1.83) to show that the unit map $\mathbf{1} \to f^* r_\mathsf{B} f_!$ (1.1.101.2) is an isomorphism. We are given that the unit map $\eta : \mathbf{1} \to f^* f_!$ is an isomorphism (since $f_!$ is fully faithful), so it suffices to show that the map

$$f^* f_! \xrightarrow{f^* \eta_B f_!} f^* r_\mathsf{B} f_! \quad : \mathsf{A}_0 \to \mathsf{A}_0 \tag{1.1.102.1}$$

is an isomorphism. By hypothesis, the map

$$r_\mathsf{A} f^* \xrightarrow{r_\mathsf{A} f^* \eta_B} r_\mathsf{A} f^* r_\mathsf{B} \quad : \mathsf{B} \to \mathsf{A}_0 \tag{1.1.102.2}$$

is an isomorphism. Now simply precompose with $f_!$ to obtain the desired result (the additional r_A is harmless since the functors already land in A_0). □

Kan extension

⋆ **1.1.103 Lemma** (Kan extension and full faithfulness)**.** *If $f : \mathsf{C} \to \mathsf{D}$ is fully faithful, then $f_! : \mathsf{Fun}(\mathsf{C}, \mathsf{E}) \to \mathsf{Fun}(\mathsf{D}, \mathsf{E})$ is fully faithful on its domain of definition.*

Proof. It suffices to show that the natural map $F \to f^*f_!F$ is an isomorphism for every $F : \mathsf{C} \to \mathsf{E}$ for which $f_!F : \mathsf{D} \to \mathsf{E}$ exists (1.1.83). That is, we should show that the natural map $F(c) \to (f^*f_!F)(c) = \operatorname{colim}_{\mathsf{C}_{f(\cdot)/f(c)}} F$ is an isomorphism for every $c \in \mathsf{C}$. The indexing category $\mathsf{C}_{f(\cdot)/f(c)}$ is just $\mathsf{C}_{/c}$ since f is fully faithful, so the map in question is an isomorphism since $\mathbf{1}_c \in \mathsf{C}_{/c}$ is a final object (1.1.73)(1.1.74). □

1.1.104 Lemma. *Given an adjunction (g, f) of functors $f : \mathsf{C} \rightleftarrows \mathsf{D} : g$, we have $f^* = g_! : \mathsf{Fun}(\mathsf{D}, \mathsf{E}) \to \mathsf{Fun}(\mathsf{C}, \mathsf{E})$ (hence adjunctions $(f_!, f^* = g_!, f_* = g^*, g_*)$).*

Proof. For a functor $F : \mathsf{D} \to \mathsf{E}$, its left Kan extension $g_!F$ evaluated at $c \in \mathsf{C}$ is given by the colimit of F over $\mathsf{D}_{g(\cdot)/c}$. The adjunction (g, f) says that the functor $\operatorname{Hom}_{\mathsf{C}}(g(-), c)$ is represented by $f(c)$, that is $(f(c), g(f(c)) \to c) \in \mathsf{D}_{g(\cdot)/c}$ is a final object. The colimit $(g_!F)(c) = \operatorname{colim}_{\mathsf{D}_{g(\cdot)/c}} F$ is thus given by evaluation at this final object $F(f(c)) = (f^*F)(c)$. Now let us lift this objectwise isomorphism $(g_!F)(c) = (f^*F)(c)$ to an isomorphism of functors $g_! = f^*$.

The counit transformation $gf \to 1$ induces a natural transformation $f^*g^* \to 1$, which we may compose with $g_!$ to obtain a natural transformation $f^*g^*g_! \to g_!$. We may also compose the unit $1 \to g^*g_!$ with f^* to obtain a natural transformation $f^* \to f^*g^*g_!$. The composition of these two natural transformations is a natural transformation $f^* \to g_!$. Unwinding the definitions, we see that this natural transformation specializes on objects to the canonical isomorphism $(f^*F)(c) \xrightarrow{\sim} (g_!F)(c)$ defined above.

Alternatively, this result can be proven using the triangle identities (1.1.130.6) as follows. The unit $\eta : 1_{\mathsf{D}} \to fg$ and counit $\varepsilon : gf \to 1_{\mathsf{C}}$ induce natural transformations $\eta^* : 1_{\mathsf{Fun}(\mathsf{D},\mathsf{E})} \to (fg)^* = g^*f^*$ and $\varepsilon^* : f^*g^* = (gf)^* \to 1_{\mathsf{Fun}(\mathsf{C},\mathsf{E})}$. The triangle identities for η and ε imply the same for η^* and ε^*, showing that they induce an adjunction (f^*, g^*) (**??**). □

⋆ **1.1.105 Exercise** (Universal property of a reflective subcategory)**.** Deduce from (1.1.104) that for a reflective subcategory $i : \mathsf{A}_0 \subseteq \mathsf{A}$ with reflection $r : \mathsf{A} \to \mathsf{A}_0$, the functor $i_! = r^* : \mathsf{Fun}(\mathsf{A}_0, \mathsf{E}) \to \mathsf{Fun}(\mathsf{A}, \mathsf{E})$ is fully faithful with right adjoint i^*, for any category E. In particular, conclude that restriction of functors defines an equivalence between functors $\mathsf{A} \to \mathsf{E}$ sending reflections to isomorphisms and functors $\mathsf{A}_0 \to \mathsf{E}$.

Yoneda Lemma

It is difficult to overstate the importance of the Yoneda Lemma (1.1.106) and the series of results that follow it, although their proofs may appear to be trivial. While we already saw how universal properties may be used to specify individual objects of categories (**??**), the Yoneda Lemma goes further and allows one to perform manipulations with such objects using their universal properties. The Yoneda Lemma is the foundation for the study of the category of presheaves $\mathsf{P}(\mathsf{C}) = \mathsf{Fun}(\mathsf{C}^{\mathsf{op}}, \mathsf{Set})$ (1.1.78) on a category C and, in particular, the observation that the canonical functor $\mathsf{C} \hookrightarrow \mathsf{P}(\mathsf{C})$ is fully faithful (1.1.108) and is the 'free cocompletion' of C (1.1.118).

⋆ **1.1.106 Yoneda Lemma** ([60, 78])**.** *Let $F : \mathsf{C}^{\mathsf{op}} \to \mathsf{Set}$ be a presheaf, and let $c \in \mathsf{C}$ be an object. The maps*

$$\operatorname{Hom}_{\mathsf{P}(\mathsf{C})}(\mathsf{C}(-,c),F(-)) \to F(c) \qquad F(c) \to \operatorname{Hom}_{\mathsf{P}(\mathsf{C})}(\mathsf{C}(-,c),F(-)) \tag{1.1.106.1}$$
$$\gamma \mapsto \gamma(1_c) \qquad \xi \mapsto (f \mapsto f^*\xi) \tag{1.1.106.2}$$

are inverses of each other.

Proof. Inspection. □

⋆ **1.1.107 Definition** (Yoneda functor)**.** The hom functor $\operatorname{Hom}_{\mathsf{C}} : \mathsf{C}^{\mathsf{op}} \times \mathsf{C} \to \mathsf{Set}$ (1.1.19) corresponds to a functor

$$\mathsf{C} \to \mathsf{P}(\mathsf{C}) = \mathsf{Fun}(\mathsf{C}^{\mathsf{op}},\mathsf{Set}) \tag{1.1.107.1}$$
$$c \mapsto \operatorname{Hom}_{\mathsf{C}}(-,c) \tag{1.1.107.2}$$

which is called the *Yoneda functor*.

⋆ **1.1.108 Corollary.** *The Yoneda functor is fully faithful.*

Proof. A simple inspection shows that the action of the Yoneda functor (1.1.107) on morphisms

$$\operatorname{Hom}_{\mathsf{C}}(c,c') \to \operatorname{Hom}_{\mathsf{P}(\mathsf{C})}(\operatorname{Hom}_{\mathsf{C}}(-,c),\operatorname{Hom}_{\mathsf{C}}(-,c')) \tag{1.1.108.1}$$

coincides with the Yoneda transformation (1.1.106.1) in the case $F = \operatorname{Hom}_{\mathsf{C}}(-,c')$, hence is an isomorphism by the Yoneda Lemma (1.1.106). □

⋆ **1.1.109 Remark.** Every category admits a fully faithful continuous functor into a complete category (namely, the Yoneda embedding (1.1.108)(**??**)(**??**)). That is to say, every category C is a full subcategory of a category $\overline{\mathsf{C}}$ which has all limits, in such a way that all limits in C are also limits in $\overline{\mathsf{C}}$. This means that when reasoning about limits in a given category C, it is often the case that *we lose no generality by assuming that* C *has all limits* (by replacing C with $\overline{\mathsf{C}}$). By duality, the same holds for colimits. The same *does not* hold for reasoning which involves both limits and colimits.

1.1.110 Lemma. *The diagram*

$$\begin{tikzcd} \mathsf{C} \arrow[r, "f"] \arrow[d, "\text{よ}_{\mathsf{C}}"] & \mathsf{D} \arrow[d, "\text{よ}_{\mathsf{D}}"] \\ \mathsf{P}(\mathsf{C}) \arrow[r, "f_!"] & \mathcal{P}(\mathsf{D}) \end{tikzcd} \tag{1.1.110.1}$$

commutes up to canonical natural isomorphism.

Proof. The action of f on morphisms $\mathrm{Hom}_{\mathsf{C}}(-,-) \to \mathrm{Hom}_{\mathsf{D}}(f(-), f(-))$ is a morphism of functors $\mathsf{C}^{\mathsf{op}} \times \mathsf{C} \to \mathsf{Set}$. When regarded as a morphism of functors $\mathsf{C} \to \mathsf{P}(\mathsf{C})$, it is a morphism $よ_{\mathsf{C}} \to f^*よ_{\mathsf{D}}f$. The adjunction $(f_!, f^*)$ thus determines a morphism $f_!よ_{\mathsf{C}} \to よ_{\mathsf{D}}f$, and it is this morphism which we will show is an isomorphism.

Fix $c \in \mathsf{C}$, and let us check that $f_!\mathrm{Hom}_{\mathsf{C}}(-,c) \to \mathrm{Hom}_{\mathsf{D}}(-, f(c))$ is an isomorphism. By the definition of adjoint functors, this map being an isomorphism means that the presheaf $\mathrm{Hom}_{\mathsf{D}}(-, f(c))$ equipped with the map $\mathrm{Hom}_{\mathsf{C}}(-,c) \to \mathrm{Hom}_{\mathsf{D}}(f(-), f(c))$ is an initial object in the category of presheaves F on D equipped with a map $\mathrm{Hom}_{\mathsf{C}}(-,c) \to F(f(-))$ in $\mathsf{P}(\mathsf{C})$. By Yoneda (1.1.106) for C, the functor sending F to the set of maps $\mathrm{Hom}_{\mathsf{C}}(-,c) \to F(f(-))$ coincides with the functor sending F to $F(f(c))$. Thus we are to show that $\mathrm{Hom}_{\mathsf{D}}(-, f(c))$ is the initial object in the category of presheaves F on D equipped with an element of $F(f(c))$. This is Yoneda for D. □

1.1.111 Remark (Small presheaves)**.** The category of presheaves $\mathsf{P}(\mathsf{C})$ on a large category C is a somewhat wild object; for example, it need not be locally small (1.1.79). A presheaf is called *small* when it is a colimit of representable presheaves, and we denote by $\mathsf{P}(\mathsf{C})_{\mathsf{small}} \subseteq \mathsf{P}(\mathsf{C})$ the full subcategory spanned by small presheaves. When C is essentially small, every presheaf on C is small (**??**). When C is large, the inclusion $\mathsf{P}(\mathsf{C})_{\mathsf{small}} \subseteq \mathsf{P}(\mathsf{C})$ can be proper (indeed, we will see below that $\mathsf{P}(\mathsf{C})_{\mathsf{small}}$ is always locally small). Apparently, $\mathsf{P}(\mathsf{C})_{\mathsf{small}}$ is better behaved than $\mathsf{P}(\mathsf{C})$, though it is useful to have both notions available (even if, in the end, one is only ever interested in objects of $\mathsf{P}(\mathsf{C})_{\mathsf{small}}$, it may be necessary to reason inside $\mathsf{P}(\mathsf{C})$ somewhere along the way to proving they indeed lie in $\mathsf{P}(\mathsf{C})_{\mathsf{small}}$).

The notion of smallness comes up naturally when we try to extend to the setting of large categories the adjunction $(f_!, f^*)$ of functors $f_! : \mathsf{P}(\mathsf{C}) \rightleftarrows \mathsf{P}(\mathsf{D}) : f^*$ associated to a functor of essentially small categories $f : \mathsf{C} \to \mathsf{D}$. When f is a functor of large categories, the presheaf pullback functor $f^* : \mathsf{P}(\mathsf{D}) \to \mathsf{P}(\mathsf{C})$ evidently makes sense, while its left adjoint *a priori* only makes sense on small presheaves $f_! : \mathsf{P}(\mathsf{C})_{\mathsf{small}} \to \mathsf{P}(\mathsf{D})_{\mathsf{small}}$ (recall that $f_!$ is compatible with Yoneda (1.1.110) and that as a left adjoint it is cocontinuous).

The relation between small/all presheaves and presheaf pushforward/pullback extends further. Given a large category C, we may express the category of all presheaves $\mathsf{P}(\mathsf{C})$ as the inverse limit over all essentially small full subcategories $\mathsf{C}_\kappa \subseteq \mathsf{C}$ of the categories of presheaves on C_κ.

$$\mathsf{P}(\mathsf{C}) \xrightarrow{\sim} \varprojlim_{\substack{\mathsf{C}_\kappa \subseteq \mathsf{C} \\ \text{ess. small}}} \mathsf{P}(\mathsf{C}) \tag{1.1.111.1}$$

The structure maps defining the above diagram are restriction of presheaves. Passing to their left adjoints (presheaf pushforward) (**??**), we obtain a dual functor

$$\varinjlim_{\substack{\mathsf{C}_\kappa \subseteq \mathsf{C} \\ \text{ess. small}}} \mathsf{P}(\mathsf{C}_\kappa) \to \mathsf{P}(\mathsf{C}) \tag{1.1.111.2}$$

which by (1.1.103) is fully faithful with image precisely $\mathsf{P}(\mathsf{C})_{\mathsf{small}}$. In particular, it follows that $\mathsf{P}(\mathsf{C})_{\mathsf{small}}$ is always locally small.

1.1.112 Lemma. *If* C *has finite products and* $f : \mathsf{C} \to \mathsf{D}$ *preserves finite products, then* $f_! : \mathsf{P}(\mathsf{C}) \to \mathsf{P}(\mathsf{D})$ *preserves finite products.*

Proof. By the compatibility of $f_!$ with Yoneda (1.1.110), we know by hypothesis that $f_!$ preserves finite products of objects of $\mathsf{C} \subseteq \mathsf{P}(\mathsf{C})$. To deduce from this that $f_!$ preserves the product of an arbitrary pair of presheaves $F, G \in \mathsf{P}(\mathsf{C})$, write $F = \operatorname{colim}_K D$ and $G = \operatorname{colim}_L E$ as colimits in $\mathsf{P}(\mathsf{C})$ of diagrams in C (**??**) and consider the following diagram.

$$\begin{tikzcd}
f(\operatorname{colim}_K D \times \operatorname{colim}_L E) \arrow[d] & f(\operatorname{colim}_{K\times L} D \times E) \arrow[l, "\sim"'] \\
f(\operatorname{colim}_K D) \times f(\operatorname{colim}_L E) & \operatorname{colim}_{K\times L} f(D \times E) \arrow[l] \arrow[u, "\sim"'] \arrow[d, "\sim"] \\
\operatorname{colim}_K f(D) \times \operatorname{colim}_L f(E) \arrow[u, "\sim"] & \operatorname{colim}_{K\times L} f(D) \times f(E) \arrow[l, "\sim"']
\end{tikzcd} \tag{1.1.112.1}$$

The top and bottom arrows are isomorphisms since products distribute over colimits in $\mathsf{P}(\mathsf{C})$ and $\mathsf{P}(\mathsf{D})$ (since limits and colimits in presheaf categories are computed pointwise (**??**) and products distribute over colimits in the category of sets (**??**)). The upper right and lower left vertical arrows are isomorphisms since $f_!$ is cocontinuous (as it is a left adjoint (**??**)). The lower right vertical arrow is an isomorphism since f preserves finite products of objects of $\mathsf{C} \subseteq \mathsf{P}(\mathsf{C})$. We conclude that the upper left vertical arrow is an isomorphism. □

1.1.113 Lemma. *A functor* $f : \mathsf{C} \to \mathsf{D}$ *is fully faithful iff* $f_! : \mathsf{P}(\mathsf{C}) \to \mathsf{P}(\mathsf{D})$ *is fully faithful.*

Proof. If f is fully faithful, then $f_!$ is fully faithful by (1.1.103). Conversely, if $f_!$ is fully faithful, then full faithfulness of Yoneda (**??**) and compatibility of Yoneda with $f_!$ (1.1.110) implies that f is fully faithful. □

1.1.114 Definition (Morita equivalence)**.** A functor $f : \mathsf{C} \to \mathsf{D}$ is called a *Morita equivalence* when the induced functor $f_! : \mathsf{P}(\mathsf{C}) \to \mathsf{P}(\mathsf{D})$ (equivalently, its right adjoint $f^* : \mathsf{P}(\mathsf{D}) \to \mathsf{P}(\mathsf{C})$) is an equivalence of categories.

1.1.115 Definition (Dominant)**.** A functor $f : \mathsf{C} \to \mathsf{D}$ is called *dominant* when every object of D is a retract of an object in the image of f.

1.1.116 Lemma. *A functor* $f : \mathsf{C} \to \mathsf{D}$ *is a Morita equivalence iff it is fully faithful and dominant.*

Proof. We saw just above that f is fully faithful iff $f_!$ is fully faithful (1.1.113). Now let us show, in the case that f and $f_!$ are fully faithful, that $f_!$ is essentially surjective iff every object of D is a retract of an object of C. The functor $f_!$ is cocontinuous (since it is a left adjoint (**??**)) and its domain $\mathsf{P}(\mathsf{C})$ is cocomplete (**??**), hence so is its essential image. Thus $f_!$ is essentially surjective iff every object of $\mathsf{D} \subseteq \mathsf{P}(\mathsf{D})$ lies in its image. Every object of $\mathsf{P}(\mathsf{C})$ is a colimit of objects of C (**??**), so $f_!$ is essentially surjective when every $d \in \mathsf{D}$ is isomorphic in

$\mathsf{P}(\mathsf{D})$ to $\operatorname{colim}_\alpha c_\alpha$ for some diagram in C. Since colimits in presheaf categories are computed pointwise, every map $d \to \operatorname{colim}_\alpha c_\alpha$ factors through the tautological map $c_{\alpha_0} \to \operatorname{colim}_\alpha c_\alpha$ for some α_0. In the event the given map $d \to \operatorname{colim}_\alpha c_\alpha$ is an isomorphism, we conclude that d is a retract of c_{α_0}. Conversely, if d is a retract of some $c \in \mathsf{C}$, it remains so in $\mathsf{P}(\mathsf{D})$, and a retract of an object is the colimit of a diagram involving just that object (**??**). $\square$

1.1.117 Lemma (Left Kan extensions along Yoneda are cocontinuous)**.** *Let $f : \mathsf{C} \to \mathsf{E}$ be a functor to a cocomplete category E.*

$$\begin{tikzcd} \mathsf{C} \arrow[r, "\text{よ}"] \arrow[d, "f"'] \arrow[dr, phantom, "\Rightarrow"] & \mathsf{P}(\mathsf{C}) \arrow[dl, dashed, "\text{よ}_! f"] \\ \mathsf{E} & \end{tikzcd} \tag{1.1.117.1}$$

The left Kan extension $\text{よ}_! f : \mathsf{P}(\mathsf{C}) \to \mathsf{E}$ of f along the Yoneda embedding よ of C is left adjoint to the composition $\mathsf{E} \xrightarrow{\text{よ}_\mathsf{E}} \mathsf{P}(\mathsf{E}) \xrightarrow{f^*} \mathsf{P}(\mathsf{C})$. In particular, $\text{よ}_! f$ is cocontinuous.

Proof. Recall that $\text{よ}_\mathsf{E} \circ f = f_! \circ \text{よ}$ (1.1.110). Applying the left adjoint $\operatorname{colim}_\mathsf{E}$ of よ_E to both sides and noting that the counit $\operatorname{colim}_\mathsf{E} \circ \text{よ}_\mathsf{E} \to \mathbf{1}_\mathsf{E}$ is an isomorphism since よ_E is fully faithful, we conclude that $f = \operatorname{colim}_\mathsf{E} \circ f_! \circ \text{よ}$. Since $\operatorname{colim}_\mathsf{E}$ and $f_!$ are cocontinuous (as they are left adjoints), we have

$$\text{よ}_! f = \text{よ}_!(\operatorname{colim}_\mathsf{E} \circ f_! \circ \text{よ}) = \operatorname{colim}_\mathsf{E} \circ f_! \circ \text{よ}_! \text{よ}. \tag{1.1.117.2}$$

Now $\text{よ}_! \text{よ} = \mathbf{1}_{\mathsf{P}(\mathsf{C})}$ (**??**), which gives the desired result. $\square$

$\star$ **1.1.118 Proposition** (Universal property of presheaves)**.** *For any category C and any cocomplete category E, the pair of adjoint functors*

$$\text{よ}_! : \mathsf{Fun}(\mathsf{C}, \mathsf{E}) \rightleftarrows \mathsf{Fun}(\mathsf{P}(\mathsf{C}), \mathsf{E}) : \text{よ}^* \tag{1.1.118.1}$$

restrict to an equivalence between the following full subcategories of $\mathsf{Fun}(\mathsf{C}, \mathsf{E})$ and $\mathsf{Fun}(\mathsf{P}(\mathsf{C}), \mathsf{E})$:

(1.1.118.2) *Functors $\mathsf{C} \to \mathsf{E}$.*

(1.1.118.3) *Functors $\mathsf{P}(\mathsf{C}) \to \mathsf{E}$ which send the diagrams $(\mathsf{C} \downarrow F)^\triangleright \to \mathsf{P}(\mathsf{C})$ to colimit diagrams for all $F \in \mathsf{P}(\mathsf{C})$.*

(1.1.118.4) *Functors $\mathsf{P}(\mathsf{C}) \to \mathsf{E}$ which are cocontinuous.*

More generally, the same holds for E not assumed cocomplete, once we restrict to those functors $\mathsf{C} \to \mathsf{E}$ which send every diagram in C to a diagram in E whose colimit exists.

Proof. Since E is cocomplete, the left Kan extension functor $\text{よ}_!$ exists (**??**). Since よ is fully faithful, left Kan extension $\text{よ}_!$ is as well (1.1.103). The essential image of left Kan extension $\text{よ}_!$ is, by definition, the functors $\mathsf{P}(\mathsf{C}) \to \mathsf{E}$ which send the diagrams $(\mathsf{C} \downarrow F)^\triangleright \to \mathsf{P}(\mathsf{C})$ to colimit diagrams for all $F \in \mathsf{P}(\mathsf{C})$. Everything in the essential image of left Kan extension $\text{よ}_!$ is cocontinuous (1.1.117). To see the converse (that everything cocontinuous is in the essential image of left Kan extension), it suffices (1.1.94) to check that よ^* reflects isomorphisms of cocontinuous functors $\mathsf{P}(\mathsf{C}) \to \mathsf{E}$, and this holds since each diagram $(\mathsf{C} \downarrow F)^\triangleright \to \mathsf{P}(\mathsf{C})$ is a colimit diagram (**??**).

To deduce the result for E not assumed cocomplete, it suffices to choose a cocontinuous embedding into a cocomplete category $\mathsf{E} \hookrightarrow \overline{\mathsf{E}}$ (1.1.109) and apply the result to the cocomplete category $\overline{\mathsf{E}}$. Alternatively, it is also straightforward to adapt the above argument to treat the case of general E. □

1.1.119 Exercise (Universal property of a full subcategory of presheaves)**.** Let $\mathsf{P}(\mathsf{C})_0 \subseteq \mathsf{P}(\mathsf{C})$ be a full subcategory containing $\mathsf{C} \subseteq \mathsf{P}(\mathsf{C})$. Conclude from the universal property of presheaves (1.1.118) and full faithfulness of Kan extension along fully faithful functors (1.1.103) that the restriction and left Kan extension functors along $\mathsf{C} \hookrightarrow \mathsf{P}(\mathsf{C})_0 \hookrightarrow \mathsf{P}(\mathsf{C})$ induce equivalences between:

(1.1.119.1) Functors $\mathsf{C} \to \mathsf{E}$.

(1.1.119.2) Functors $\mathsf{P}(\mathsf{C})_0 \to \mathsf{E}$ which have a cocontinuous extension to $\mathsf{P}(\mathsf{C})$.

(1.1.119.3) Functors $\mathsf{P}(\mathsf{C}) \to \mathsf{E}$ which are cocontinuous.

Now use (1.1.94) to argue that the condition on functors $\mathsf{P}(\mathsf{C})_0 \to \mathsf{E}$ (1.1.119.2) is equivalent to *any* property $\mathcal{P}$ of functors $\mathsf{P}(\mathsf{C})_0 \to \mathsf{E}$ which holds for the restriction of cocontinuous functors on $\mathsf{P}(\mathsf{C})$ and for which a morphism of $\mathcal{P}$-functors $F \to G : \mathsf{P}(\mathsf{C})_0 \to \mathsf{E}$ is an isomorphism if it is so restricted to C. For example, we could take $\mathcal{P}$ to be 'preserves $\mathcal{K}$-colimits' for any collection $\mathcal{K}$ of colimit diagrams in $\mathsf{P}(\mathsf{C})$ contained in $\mathsf{P}(\mathsf{C})_0$ for which $\mathsf{P}(\mathsf{C})_0$ is generated under $\mathcal{K}$-colimits by $\mathsf{C} \subseteq \mathsf{P}(\mathsf{C})_0$.

A reflective subcategory of presheaves satisfies *another* universal property, distinct from (1.1.119). Note that reflective subcategories $\mathsf{P}'(\mathsf{C}) \subseteq \mathsf{P}(\mathsf{C})$ do not necessarily contain $\mathsf{C} \subseteq \mathsf{P}(\mathsf{C})$, hence are more general than the full subcategories considered in (1.1.119).

⋆ **1.1.120 Proposition** (Universal property of a reflective subcategory of presheaves)**.** *Let* $\mathsf{P}'(\mathsf{C}) \subseteq \mathsf{P}(\mathsf{C})$ *be a reflective subcategory with reflector* r*. For any cocomplete category* E*, the adjoint functors*

$$\mathsf{Fun}(\mathsf{C},\mathsf{E}) \underset{\mathcal{Y}^*}{\overset{\mathcal{Y}_!}{\rightleftarrows}} \mathsf{Fun}(\mathsf{P}(\mathsf{C}),\mathsf{E}) \underset{r^*}{\overset{r_!}{\rightleftarrows}} \mathsf{Fun}(\mathsf{P}'(\mathsf{C}),\mathsf{E}) \tag{1.1.120.1}$$

restrict to equivalences between the following categories of functors:

(1.1.120.2) *Functors* $\mathsf{P}'(\mathsf{C}) \to \mathsf{E}$ *which are cocontinuous.*

(1.1.120.3) *Functors* $\mathsf{P}(\mathsf{C}) \to \mathsf{E}$ *which are cocontinuous and send reflections to isomorphisms.*

(1.1.120.4) *Functors* $\mathsf{C} \to \mathsf{E}$ *whose unique cocontinuous extension to* $\mathsf{P}(\mathsf{C})$ *sends reflections to isomorphisms.*

Proof. By the universal property of a reflective subcategory (1.1.105), functors $\mathsf{P}'(\mathsf{C}) \to \mathsf{E}$ are equivalent via pullback along r to functors $\mathsf{P}(\mathsf{C}) \to \mathsf{E}$ sending reflections to isomorphisms. This equivalence respects cocontinuity by (1.1.92). Now use the universal property of presheaves to identify cocontinuous functors $\mathsf{P}(\mathsf{C}) \to \mathsf{E}$ with functors $\mathsf{C} \to \mathsf{E}$ via pullback along $\mathcal{Y}_\mathsf{C}$ (1.1.118). □

1.1.121 Exercise (Universal property of a full subcategory of a reflective subcategory of presheaves)**.** Let $\mathsf{P}'(\mathsf{C}) \subseteq \mathsf{P}(\mathsf{C})$ be a reflective subcategory, and let $\mathsf{P}'(\mathsf{C})_0 \subseteq \mathsf{P}'(\mathsf{C})$ be a full subcategory containing the image of $\mathsf{C} \hookrightarrow \mathsf{P}(\mathsf{C}) \to \mathsf{P}'(\mathsf{C})$. Conclude from the universal property of a reflective subcategory of presheaves (1.1.120) and full faithfulness of Kan extension along fully faithful functors (1.1.103) that the pullback and left Kan extension functors along $\mathsf{C} \to \mathsf{P}'(\mathsf{C})_0 \hookrightarrow \mathsf{P}'(\mathsf{C})$ induce equivalences between:

(1.1.121.1) Functors $\mathsf{C} \to \mathsf{E}$ whose unique cocontinuous extension to $\mathsf{P}(\mathsf{C})$ sends reflections to isomorphisms.

(1.1.121.2) Functors $\mathsf{P}'(\mathsf{C})_0 \to \mathsf{E}$ which have a cocontinuous extension to $\mathsf{P}'(\mathsf{C})$.

(1.1.121.3) Functors $\mathsf{P}'(\mathsf{C}) \to \mathsf{E}$ which are cocontinuous.

Now use (1.1.95) to argue that the condition on functors $\mathsf{P}'(\mathsf{C})_0 \to \mathsf{E}$ (1.1.121.2) is equivalent to *any* property $\mathcal{P}$ of functors $\mathsf{P}'(\mathsf{C})_0 \to \mathsf{E}$ which holds for the restriction of cocontinuous functors on $\mathsf{P}'(\mathsf{C})$, which implies the pullback to C satisfies (1.1.121.1), and with the property that a morphism of $\mathcal{P}$-functors $F \to G : \mathsf{P}'(\mathsf{C})_0 \to \mathsf{E}$ is an isomorphism if it is so when pulled back to C.

Representable morphisms

⋆ **1.1.122 Definition** (Representable morphism)**.** A morphism $X \to Y$ in $\mathsf{P}(\mathsf{C})$ is called *representable* when the fiber product $X \times_Y c$ is representable for every map $c \to Y$ from an object $c \in \mathsf{C} \subseteq \mathsf{P}(\mathsf{C})$.

1.1.123 Exercise. Show that representability is preserved under pullback and closed under composition.

⋆ **1.1.124 Definition** (Induced property)**.** Let $\mathcal{P}$ be a property of morphisms in C which is preserved under pullback. A representable morphism $X \to Y$ in $\mathsf{P}(\mathsf{C})$ is said to have the 'induced' property $\overline{\mathcal{P}}$ (usually just said $\mathcal{P}$) when for every map $c \to Y$ from an object $c \in \mathsf{C}$, the pullback $X \times_Y c \to c$ has $\mathcal{P}$.

1.1.125 Warning. When discussing induced properties, we often shorten 'representable and $\overline{\mathcal{P}}$' to just '$\mathcal{P}$'. This is potentially dangerous: sometimes there is a reasonable generalization of $\mathcal{P}$ to (not necessarily representable) morphisms in $\mathsf{P}(\mathsf{C})$ which agrees with the induction (1.1.124) for representable morphisms (in which case 'representable and $\mathcal{P}$' is strictly stronger than just '$\mathcal{P}$').

1.1.126 Exercise. Let $\mathcal{P}$ be a property of morphisms in C which is preserved under pullback. Show that the induced property for morphisms in $\mathsf{P}(\mathsf{C})$ is preserved under pullback. Show that if $\mathcal{P}$ moreover closed under composition, then so is the induced property for morphisms in $\mathsf{P}(\mathsf{C})$.

1.1.127 Lemma. *Let $\mathcal{P}$ be a property of morphisms in C preserved under pullback, and let $X \to Y \to B$ be morphisms in $\mathsf{P}(\mathsf{C})$. The morphism $X \to Y$ has $\mathcal{P}$ iff every pullback $X \times_B c \to Y \times_B c$ has $\mathcal{P}$.*

Proof. Let $c \to Y$ be a morphism from $c \in \mathsf{C}$, and consider the following diagram.

$$\begin{tikzcd}
X \times_Y c \arrow[r] \arrow[d] & X \times_B c \arrow[d] \\
c = Y \times_Y c \arrow[r] \arrow[d] & Y \times_B c \arrow[d] \\
Y = Y \times_Y Y \arrow[r] & Y \times_B Y
\end{tikzcd} \tag{1.1.127.1}$$

The bottom square and the composite square are both pullbacks (**??**), so the top square is a pullback by cancellation (1.1.57). □

1.1.128 Definition (Group object)**.** Let $\mathsf{Grp}^{\mathsf{finfree}} \subseteq \mathsf{Grp}$ denote the full subcategory spanned by finitely generated free groups. A *group object* in a category C is a functor $(\mathsf{Grp}^{\mathsf{finfree}})^{\mathsf{op}} \to \mathsf{C}$ which sends finite coproducts in $\mathsf{Grp}^{\mathsf{finfree}}$ to products in C. Group objects form a full subcategory $\mathsf{Fun}_\times((\mathsf{Grp}^{\mathsf{finfree}})^{\mathsf{op}}, \mathsf{C}) \subseteq \mathsf{Fun}((\mathsf{Grp}^{\mathsf{finfree}})^{\mathsf{op}}, \mathsf{C})$.

1.1.129 Exercise. Show that for a group object G in any category C, the following are pullback squares.

$$\begin{tikzcd}
G \arrow[r, "x \mapsto (x,x)"] \arrow[d] & G \times G \arrow[r, "(x,y) \mapsto x"] \arrow[d, "(x,y) \mapsto x^{-1}y"] & G \arrow[d] \\
* \arrow[r, "1"] & G \arrow[r] & *
\end{tikzcd} \tag{1.1.129.1}$$

Monoidal categories and enriched categories

1.1.130 Definition (Monoidal category)**.** A *monoidal structure* $\otimes$ on a category C consists of the following data:

(1.1.130.1) A functor $\otimes : \mathsf{C} \times \mathsf{C} \to \mathsf{C}$.

(1.1.130.2) A natural isomorphism of functors $\otimes \circ (\mathbf{1}_{\mathsf{C}} \times \otimes) = \otimes \circ (\otimes \times \mathbf{1}_{\mathsf{C}})$, namely a chosen isomorphism $X \otimes (Y \otimes Z) = (X \otimes Y) \otimes Z$ functorial in $X, Y, Z \in \mathsf{C}$.

(1.1.130.3) The cyclic composition

$$\begin{tikzcd}
& (X \otimes Y) \otimes (Z \otimes W) \arrow[dl, leftrightarrow] \arrow[dr, leftrightarrow] & \\
((X \otimes Y) \otimes Z) \otimes W \arrow[d, leftrightarrow] & & X \otimes (Y \otimes (Z \otimes W)) \arrow[d, leftrightarrow] \\
(X \otimes (Y \otimes Z)) \otimes W \arrow[rr, leftrightarrow] & & X \otimes ((Y \otimes Z) \otimes W)
\end{tikzcd}$$

must be the identity map for all $X, Y, Z, W \in \mathsf{C}$.

(1.1.130.4) An object $\mathbf{1} \in \mathsf{C}$.

(1.1.130.5) Natural isomorphisms of functors $(\mathbf{1}\otimes -) = 1_{\mathsf{C}} = (-\otimes \mathbf{1})$, namely chosen isomorphisms $\mathbf{1}\otimes X = X = X\otimes \mathbf{1}$ functorial in X.

(1.1.130.6) The cyclic composition

$$\begin{tikzcd}
(X\otimes\mathbf{1})\otimes Y \arrow[rr, leftrightarrow] \arrow[dr, leftrightarrow] & & X\otimes(\mathbf{1}\otimes Y) \arrow[dl, leftrightarrow] \\
& X\otimes Y &
\end{tikzcd}$$

must be the identity map for all $X, Y \in \mathsf{C}$.

A *monoidal category* $(\mathsf{C},\otimes)$ is a category C equipped with a monoidal structure $\otimes$. Given a monoidal structure $\otimes$, there is an associated *opposite* monoidal structure $\otimes^{\mathsf{op}}$ defined by reversing the order everywhere (for example $X\otimes^{\mathsf{op}} Y = Y\otimes X$).

1.1.131 Exercise (Mac Lane [77])**.** Show that any two parenthesizations of $X_1\otimes\cdots\otimes X_n$ are related by a sequence of associator moves $X\otimes(Y\otimes Z)\rightleftarrows(X\otimes Y)\otimes Z$ (1.1.130.2). Show that any loop of associator moves (i.e. starting at a given parenthesization and returning to the same) may be trivialized by repeated applications of the pentagon relation (1.1.130.3). Conclude that $X_1\otimes\cdots\otimes X_n$ expresses a well-defined object of C for any sequence of objects $X_1,\ldots,X_n$ in a monoidal category $(\mathsf{C},\otimes)$.

Show that the unitor isomorphism $\mathbf{1}\otimes(X\otimes Y) = (X\otimes Y)$ (1.1.130.5) for $X\otimes Y$ agrees with the unitor isomorphism $(\mathbf{1}\otimes X = X)\otimes Y$ (show that this holds after applying $Z\otimes -$ by appealing to (1.1.130.6), then take $Z=\mathbf{1}$; this trick is useful below as well). Conclude that there is a canonical isomorphism $X_n\otimes\cdots\otimes X_1\otimes\mathbf{1}\otimes Y_1\otimes\cdots\otimes Y_m = X_n\otimes\cdots\otimes X_1\otimes Y_1\otimes\cdots\otimes Y_m$ (removing a $\mathbf{1}$ from a tensor string) for $n, m\geq 0$, where the tensor product of the empty string is defined to be $\mathbf{1}$. Show that removing two $\mathbf{1}$'s from a tensor string in either order gives the same isomorphism, and conclude that there is a canonical isomorphism associated to removing any number of $\mathbf{1}$'s from a tensor string. Conclude that removing either one of a pair of adjacent $\mathbf{1}$'s gives the same isomorphism (remove the second one and appeal to the previous assertion). Conclude that there is a canonical isomorphism associated to shortening/lengthening consecutive occurrences of $\mathbf{1}$ in any tensor string (in particular, conclude that the left and right unitor isomorphisms $\mathbf{1}\otimes\mathbf{1}=\mathbf{1}$ (1.1.130.5) coincide).

1.1.132 Example. If C has finite products, then finite products define a monoidal structure on C.

1.1.133 Example. Tensor product is a monoidal structure on the category of vector spaces over a given field k.

1.1.134 Definition (Internal Hom)**.** Given a monoidal category $(\mathsf{C},\otimes)$ and a pair of objects $X, Y\in\mathsf{C}$, we may consider the presheaf $\underline{\operatorname{Hom}}(X,Y)\in\mathsf{P}(\mathsf{C})$ which sends $Z\in\mathsf{C}$ to the set of maps $X\otimes Z\to Y$ (this generalizes (1.1.84)). If this presheaf is representable, the resulting object $\underline{\operatorname{Hom}}(X,Y)\in\mathsf{C}$ (with its 'universal' map $X\otimes\underline{\operatorname{Hom}}(X,Y)\to Y$) is called the *internal Hom* between X and Y. Note that the internal Hom with respect to $\otimes$ is not *a priori* related to the internal Hom with respect to $\otimes^{\mathsf{op}}$.

1.1.135 Definition (Monoidal functor)**.** Let $(\mathsf{C}, \otimes)$ and $(\mathsf{D}, \otimes)$ be monoidal categories. A *lax monoidal functor* $F : (\mathsf{C}, \otimes) \to (\mathsf{D}, \otimes)$ is a functor $F : \mathsf{C} \to \mathsf{D}$ together with a map $\mathbf{1} \to F(\mathbf{1})$ and natural maps

$$F(X) \otimes F(Y) \to F(X \otimes Y), \tag{1.1.135.1}$$

namely a natural transformation $\otimes_{\mathsf{D}} \circ (F \times F) \to F \circ \otimes_{\mathsf{C}}$ of functors $\mathsf{C} \times \mathsf{C} \to \mathsf{D}$, such that the following diagram commutes

$$\begin{tikzcd} F(X) \otimes F(Y) \otimes F(Z) \arrow[r] \arrow[d] & F(X \otimes Y) \otimes F(Z) \arrow[d] \\ F(X) \otimes F(Y \otimes Z) \arrow[r] & F(X \otimes Y \otimes Z) \end{tikzcd} \tag{1.1.135.2}$$

for all $X, Y, Z \in \mathsf{C}$, and the maps

$$F(X) = F(X) \otimes \mathbf{1} \to F(X) \otimes F(\mathbf{1}) \to F(X \otimes \mathbf{1}) = F(X) \tag{1.1.135.3}$$
$$F(X) = \mathbf{1} \otimes F(X) \to F(\mathbf{1}) \otimes F(X) \to F(\mathbf{1} \otimes X) = F(X) \tag{1.1.135.4}$$

are the identity for all $X \in \mathsf{C}$. An *oplax monoidal functor* $(\mathsf{C}, \otimes) \to (\mathsf{D}, \otimes)$ is a lax monoidal functor $(\mathsf{C}, \otimes)^{\mathsf{op}} \to (\mathsf{D}, \otimes)^{\mathsf{op}}$ (concretely, this means the direction of the above maps are all reversed). A *monoidal functor* is a lax (equivalently, oplax) monoidal functor whose constituent maps $\mathbf{1} \to F(\mathbf{1})$ and $F(X) \otimes F(Y) \to F(X \otimes Y)$ are all isomorphisms.

1.1.136 Exercise. Show that for a lax monoidal functor $F : (\mathsf{C}, \otimes) \to (\mathsf{D}, \otimes)$, there are canonical isomorphisms $F(X_1) \otimes \cdots \otimes F(X_n) \to F(X_1 \otimes \cdots \otimes X_n)$ for all $n \geq 0$, and show that they respect the canonical isomorphisms associated to removing copies of $\mathbf{1}$ from tensor strings (after applying the map $\mathbf{1} \to F(\mathbf{1})$).

1.1.137 Example. The forgetful functor $(\mathsf{Vect}, \otimes) \to (\mathsf{Set}, \times)$ is lax monoidal via the canonical maps $V \times W \to V \otimes W$ for vector spaces V and W.

1.1.138 Definition (Enriched category)**.** Let $(\mathsf{C}, \otimes)$ be a monoidal category. The notion of a $(\mathsf{C}, \otimes)$-*enriched category* is a generalization of the notion of a category (1.1.1). A $(\mathsf{C}, \otimes)$-enriched category D has a set of objects, but morphisms in D consist of objects $\operatorname{Hom}(X, Y) \in \mathsf{C}$ for pairs $X, Y \in \mathsf{D}$. Composition in D consists of maps $\operatorname{Hom}(X, Y) \otimes \operatorname{Hom}(Y, Z) \to \operatorname{Hom}(X, Z)$, and associativity of composition involves the associator isomorphisms (1.1.130.2) of $(\mathsf{C}, \otimes)$ and implies that iterated composition

$$\operatorname{Hom}(X_0, X_1) \otimes \cdots \otimes \operatorname{Hom}(X_{n-1}, X_n) \to \operatorname{Hom}(X_0, X_n) \tag{1.1.138.1}$$

is well defined. Identity morphisms in D are maps $\mathbf{1}_X : \mathbf{1}_{\mathsf{C}} \to \operatorname{Hom}(X, X)$, and composition with $\mathbf{1}_X$ or $\mathbf{1}_Y$ on $\operatorname{Hom}(X, Y)$ must yield $\mathbf{1}_{\operatorname{Hom}(X,Y)}$ when combined with the unitor isomorphisms (1.1.130.5) of C.

A $(\mathsf{Set}, \times)$-enriched category is simply a category in the usual sense. A lax monoidal functor $(\mathsf{A}, \otimes) \to (\mathsf{B}, \otimes)$ turns $(\mathsf{A}, \otimes)$-enriched categories into $(\mathsf{B}, \otimes)$-enriched categories. In particular, we may regard a $(\mathsf{C}, \otimes)$-enriched category as having an 'underlying category' in the presence of a chosen lax monoidal functor $(\mathsf{C}, \otimes) \to (\mathsf{Set}, \times)$.

1.1.139 Exercise. Show that a pointed category C (1.1.52) is *uniquely* enriched over $(\mathsf{Set}_*, \times)$ (pointed sets with the product symmetric monoidal structure, with the obvious forgetful functor to $(\mathsf{Set}, \times)$). Conversely, show that if C is enriched in pointed sets and C has an initial object and a final object, then C is pointed.

1.1.140 Definition (Differential graded category)**.** Let k be a field. A k-linear *differential graded category* (or *dg-category*) C is a category enriched over complexes of vector spaces over k. In other words, it consists of a set C of objects, a *morphism complex* $\operatorname{Hom}(X, Y) \in \mathsf{Kom}(\mathsf{Vect}_k)$ for each $X, Y \in \mathsf{C}$, and composition maps $\operatorname{Hom}(X, Y) \otimes \operatorname{Hom}(Y, Z) \to \operatorname{Hom}(X, Z)$ which are associative and unital.

1.1.141 Definition (Complex)**.** Let C be a pointed category (1.1.52)(1.1.139) together with an auto-equivalence $\Sigma : \mathsf{C} \to \mathsf{C}$ called 'suspension'. A *complex* in (C, Σ) is a pair (X, d) consisting of an object $X \in \mathsf{C}$ and a morphism $d : X \to \Sigma X$ whose square

$$X \xrightarrow{d} \Sigma X \xrightarrow{\Sigma d} \Sigma^2 X \tag{1.1.141.1}$$

vanishes (is the basepoint in $\operatorname{Hom}(X, \Sigma^2 X)$. Complexes in (C, Σ) form a category $\mathsf{Kom}(\mathsf{C}, \Sigma)$ in which a morphism $(X, d) \to (Y, d)$ is a map $X \to Y$ which commutes with d.

A functor $F : (\mathsf{C}, \Sigma) \to (\mathsf{D}, \Sigma)$ (meaning equipped with an isomorphism $F \circ \Sigma = \Sigma \circ F$) induces a functor $\mathsf{Kom}(\mathsf{C}, \Sigma) \to \mathsf{Kom}(\mathsf{D}, \Sigma)$. In particular, the shift functor Σ on C induces an autoequivalence of $\mathsf{Kom}(\mathsf{C}, \Sigma)$, also denoted Σ.

1.1.142 Definition (Sign functor)**.** The sign function $\operatorname{sgn} : \mathbb{R}^\times \to \mathbb{Z}^\times$ is given by $\operatorname{sgn}(\lambda) = \lambda/|\lambda|$. The sign functor sgn is a functor from the category of one-dimensional real vector spaces and isomorphisms to the category of free abelian groups of rank one and isomorphisms. It is defined by declaring that $\operatorname{sgn}(\mathbb{R}) = \mathbb{Z}$ and $\operatorname{sgn}(\mathbb{R} \xrightarrow{\lambda} \mathbb{R}) = (\mathbb{Z} \xrightarrow{\operatorname{sgn}(\lambda)} \mathbb{Z})$.

⋆ **1.1.143 Definition** (Orientation line)**.** Let V be a finite-dimensional real vector space. Its top wedge power $\wedge^{\dim V} V$ is a one-dimensional real vector space. The orientation line of V is

$$\mathfrak{o}(V) = \operatorname{sgn}(\wedge^{\dim V} V)[\dim V] \tag{1.1.143.1}$$

where sgn is the sign functor (1.1.142) and $[\dim V]$ indicates placement in homological degree $\dim V$. There are canonical associative isomorphisms $\mathfrak{o}(V \oplus W) = \mathfrak{o}(V) \otimes \mathfrak{o}(W)$ which are symmetric with respect to the super tensor product on graded $\mathbb{Z}$-modules. Thus the orientation line is a symmetric monoidal functor

$$((\mathsf{Vect}_{\mathbb{R}})_{\simeq}, \oplus) \to ((\mathsf{Ab}^{\mathbb{Z}})_{\simeq}, \otimes) \tag{1.1.143.2}$$

$$V \mapsto \mathfrak{o}(V) \tag{1.1.143.3}$$

There is a canonical isomorphism $\mathfrak{o}(V) = \mathfrak{o}(V^*)$.

Complex vector spaces are canonically oriented by taking, for any ordered $\mathbb{C}$-basis $v_1, \ldots, v_n \in V$, the generator $v_1 \wedge iv_1 \wedge \cdots \wedge v_n \wedge iv_n$ of $\wedge_{\mathbb{R}}^{2\dim_{\mathbb{C}} V} V$, which is independent up to positive scaling of the choice of basis. This establishes an isomorphism of symmetric monoidal functors between the pre-composition of the orientation line functor with the forgetful functor $\mathsf{Vect}_{\mathbb{C}} \to \mathsf{Vect}_{\mathbb{R}}$ and the functor $V \mapsto \mathbb{Z}[2\dim_{\mathbb{C}} V]$. This isomorphism is not unique: we have followed the usual convention by orienting $\mathbb{C}$ using $1 \wedge i$, but we could just as well have taken its opposite. This freedom is precisely the automorphism group of the symmetric monoidal functor $V \mapsto \mathbb{Z}[2\dim_{\mathbb{C}} V]$, namely $\mathbb{Z}/2$ generated by $(-1)^{\dim_{\mathbb{C}} V}$.

The definition of the orientation line of a vector space (1.1.143) carries over without change to the setting of vector bundles.

1.1.144 Definition (Mittag-Leffler inverse system)**.** An inverse system of sets $\cdots \to S_2 \to S_1 \to S_0$ is said to satisfy the *Mittag-Leffler condition* when the infinite decreasing intersection $S_i' = \bigcap_{j \geq i} \operatorname{im}(S_j \to S_i)$ is achieved at some finite stage: $S_i' = \operatorname{im}(S_j \to S_i)$ for some $j = j(i)$.

1.1.145 Lemma (Mittag-Leffler)**.** *Let $\{A_i\}_i \to \{B_i\}_i \to \{C_i\}_i$ be a sequence of maps inverse systems of abelian groups which is exact in the middle. If $\{A_i\}_i$ is Mittag-Leffler, then the sequence of inverse limits $\varprojlim_i A_i \to \varprojlim_i B_i \to \varprojlim_i C_i$ is also exact in the middle.*

2-categories

A 2-category is like a category, except that $\operatorname{Hom}(X, Y)$ is a groupoid (1.1.9) instead of a set. Because of this, the associativity axiom needs modification: a natural isomorphism between the two ways of composing a triple of morphisms $\operatorname{Hom}(X, Y) \times \operatorname{Hom}(Y, Z) \times \operatorname{Hom}(Z, W) \to \operatorname{Hom}(X, W)$ is specified, and these 'associators' are required to satisfy a certain 'pentagon identity' for quadruples of morphisms (1.1.146.4) which ensures that composition of any tuple of morphisms is well defined up to well defined isomorphism.

The theory of 2-categories contains the theory of categories as a special case, namely when all morphism groupoids are discrete (1.1.30). Most concepts and results in category theory carry over directly to 2-category theory, albeit with the caveat that there are often many more diagrams to chase. A detailed treatment of the theory of 2-categories can be found in Johnson–Yau [46] (though the reader should beware of various terminological differences with our presentation here).

1.1.146 Definition (2-category)**.** A *2-category* C consists of the following data:

(1.1.146.1) A set C, whose elements are called the *objects* of $\mathcal{C}$.

(1.1.146.2) For every pair of objects $X, Y \in \mathsf{C}$, a groupoid $\operatorname{Hom}(X, Y)$, whose objects are called the *morphisms* $X \to Y$ in C and whose morphisms are called the *2-morphisms* in $\mathcal{C}$.

(1.1.146.3) For every triple of objects $X, Y, Z \in \mathsf{C}$, a functor

$$\operatorname{Hom}(X, Y) \times \operatorname{Hom}(Y, Z) \to \operatorname{Hom}(X, Z)$$

called *composition.*

(1.1.146.4) For every quadruple of objects $X, Y, Z, W \in \mathsf{C}$, a natural isomorphism between the two ways of composing twice to obtain a functor

$$\mathrm{Hom}(X,Y) \times \mathrm{Hom}(Y,Z) \times \mathrm{Hom}(Z,W) \to \mathrm{Hom}(X,W)$$

such that for every quadruple of morphisms a, b, c, d, the cyclic composition

$$\begin{tikzcd}[column sep=tiny] & (ab)(cd) \arrow[dl, leftrightarrow] \arrow[dr, leftrightarrow] & \\ ((ab)c)d \arrow[d, leftrightarrow] & & a(b(cd)) \arrow[d, leftrightarrow] \\ (a(bc))d \arrow[rr, leftrightarrow] & & a((bc)d) \end{tikzcd}$$

is the identity map.

(1.1.146.5) For every object $X \in \mathsf{C}$, an object $\mathbf{1}_X \in \mathrm{Hom}(X,X)$ together with natural isomorphisms between composition with $\mathbf{1}_X$ and the identity functors on $\mathrm{Hom}(X,Y)$ and $\mathrm{Hom}(Z,X)$, such that for every pair of morphisms a, b, the cyclic composition

$$\begin{tikzcd} & ab \arrow[dl, leftrightarrow] \arrow[dr, leftrightarrow] & \\ (a\mathbf{1})b \arrow[rr, leftrightarrow] & & a(\mathbf{1}b) \end{tikzcd}$$

is the identity map.

1.1.147 Example (Categories are 2-categories)**.** Every category may be regarded as a 2-category by regarding each set $\mathrm{Hom}(X,Y)$ as a groupoid as in (1.1.30).

1.1.148 Example (Homotopy category of a 2-category)**.** A 2-category C gives rise to a category $\pi_0\mathsf{C}$ by replacing each morphism groupoid $\mathrm{Hom}(X,Y)$ with its set of isomorphism classes.

1.1.149 Example (2-category of categories)**.** Categories form a 2-category Cat in which $\mathrm{Hom}(\mathsf{C},\mathsf{D}) = \mathsf{Fun}(\mathsf{C},\mathsf{D})_{\simeq}$. That is, a morphism $\mathsf{C} \to \mathsf{D}$ is a functor, and a 2-morphism is a natural isomorphism of functors. The homotopy category of the 2-category Cat is the category denoted hCat discussed in (1.1.35).

1.2 Simplicial objects

⋆ **1.2.1 Definition** (Simplex category $\mathbf{\Delta}$)**.** Let $\mathbf{\Delta}$ denote the category whose objects are non-empty finite ordered sets and whose morphisms are weakly order preserving maps. In other words, every object of $\mathbf{\Delta}$ is isomorphic to $[n] = \{0, \ldots, n\}$ for some integer $n \geq 0$, and a morphism $f : [n] \to [m]$ is a map of sets satisfying $f(i) \leq f(j)$ for $i \leq j$.

1.2.2 Example (Simplices as categories)**.** We may regard $[n]$ as the category with set of objects $\{0, \ldots, n\}$ and a single morphism $i \to j$ for $i \leq j$. Now a map $[n] \to [m]$ in $\mathbf{\Delta}$ is the same as a functor $[n] \to [m]$. Thus $\mathbf{\Delta} \subseteq \mathsf{Cat}$ is a full subcategory; compare (1.1.3)(1.1.31).

1.2.3 Example (Complete simplex)**.** Given a finite set S, the *complete simplex* on S is the subspace of $\mathbb{R}^S$ defined by the conditions $\sum_s x_s = 1$ and $x_s \geq 0$. A map of finite sets $f : S \to T$ induces a map $\mathbb{R}^S \to \mathbb{R}^T$ by pushforward $y_t = \sum_{f(s)=t} x_s$, hence a map $\Delta^S \to \Delta^T$ by restriction. This defines a functor $\mathsf{Set}^{\mathsf{fin}} \to \mathsf{Top}$; by pre-composing with the forgetful functor $\mathbf{\Delta} \to \mathsf{Set}^{\mathsf{fin}}$, we obtain a functor $\mathbf{\Delta} \to \mathsf{Top}$.

⋆ **1.2.4 Definition** (Simplicial object)**.** For any category C, a *simplicial object* of C is a functor $\mathbf{\Delta}^{\mathsf{op}} \to \mathsf{C}$ (dually, a functor $\mathbf{\Delta} \to \mathsf{C}$ is termed a *cosimplicial object*). A simplicial object $X_\bullet : \mathbf{\Delta}^{\mathsf{op}} \to \mathsf{C}$ thus consists of a sequence of objects $X_0, X_1, \ldots$ of C and maps $f^* : X_m \to X_n$ associated to maps $f : [n] \to [m]$, satisfying $(fg)^* = g^* f^*$. Simplicial objects of C form a category denoted $\mathsf{sC} = \mathsf{Fun}(\mathbf{\Delta}^{\mathsf{op}}, \mathsf{C})$ (and $\mathsf{csC} = \mathsf{Fun}(\mathbf{\Delta}, \mathsf{C})$ for cosimplicial objects). Note that a simplicial object of C is, despite the terminology, evidently not an object of C.

⋆ **1.2.5 Definition** (Levelwise property)**.** Let $\mathcal{P}$ be a property of morphisms in a category C. A morphism of simplicial objects $X_\bullet \to Y_\bullet$ in C is called *(levelwise)* $\mathcal{P}$ when each of its constituent maps $X_k \to Y_k$ has $\mathcal{P}$.

Simplicial sets

⋆ **1.2.6 Definition** (Simplicial set)**.** The category of simplicial sets is $\mathsf{sSet} = \mathsf{Fun}(\mathbf{\Delta}^{\mathsf{op}}, \mathsf{Set})$.

The Yoneda functor of $\mathbf{\Delta}$ is an embedding $\mathbf{\Delta} \to \mathsf{sSet}$, and the image of $[n]$ under this embedding is also denoted Δ^n. For any simplicial set $X_\bullet$, the Yoneda Lemma (1.1.106) identifies elements of $X_n = X([n])$ with maps $\Delta^n \to X$; these are called the 'n-simplices of X'. One should view a simplicial set as a combinatorial/categorical specification of a way to 'glue' together these simplices along simplicial maps (more formally, the category sSet is the free cocompletion of $\mathbf{\Delta}$ (1.1.118)).

1.2.7 Exercise. Describe the k-simplices of Δ^1 (there are $k + 2$ of them).

1.2.8 Exercise (Simplicial mapping space)**.** Show that for every pair of simplicial sets $X, Y \in \mathsf{sSet}$, there is a simplicial set $\underline{\mathrm{Hom}}(X, Y)$ defined by the universal property that a map $Z \to \underline{\mathrm{Hom}}(X, Y)$ is the same as a map $Z \times X \to Y$. Show that there is a tautological composition map $\underline{\mathrm{Hom}}(X, Y) \times \underline{\mathrm{Hom}}(Y, Z) \to \underline{\mathrm{Hom}}(X, Z)$, which is associative for quadruples (X, Y, Z, W).

⋆ **1.2.9 Definition** (Non-degenerate simplex)**.** Let X be a simplicial set. A simplex $[n] \to X$ is called *non-degenerate* when it has no factorization as $[n] \to [m] \to X$ with $m < n$; otherwise, it is called *degenerate*.

⋆ **1.2.10 Definition** ([28, (8.3)])**.** Let X be a simplicial set. Every simplex $[n] \to X$ admits a *unique* factorization $[n] \twoheadrightarrow [r] \to X$ with $[r] \to X$ is non-degenerate.

Proof. The existence of a factorization of the desired form is trivial, so the content is to prove uniqueness.

A surjection out of $[n]$ is determined uniquely by the set of arrows in $0 \to \cdots \to n$ which are collapsed. Fix a pair of surjections $f : [n] \twoheadrightarrow [r]$ and $g : [n] \twoheadrightarrow [s]$, and let $[n] \twoheadrightarrow [a]$ be the surjection which collapses the union of the arrows collapsed by f and g. This determines a diagram of the following shape.

$$\begin{tikzcd} {[n]} \arrow[r, two heads, "f"] \arrow[d, two heads, "g"'] & {[r]} \arrow[d, two heads] \\ {[s]} \arrow[r, two heads] & {[a]} \end{tikzcd} \tag{1.2.10.1}$$

We will show that this diagram is pushout in the category of simplicial sets, from which the desired uniqueness assertion follows immediately. A simple inspection shows that (1.2.10.1) is a pushout in the simplex category $\mathbf{\Delta}$, but this does not imply that it is a pushout in the category of simplicial sets sSet (Yoneda typically does not preserve colimits).

To show that (1.2.10.1) is a pushout in sSet, it is equivalent (since colimits in diagram categories are computed pointwise (**??**)) to show that the induced diagram

$$\begin{tikzcd} \operatorname{Hom}([k],[n]) \arrow[r] \arrow[d] & \operatorname{Hom}([k],[r]) \arrow[d] \\ \operatorname{Hom}([k],[s]) \arrow[r] & \operatorname{Hom}([k],[a]) \end{tikzcd} \tag{1.2.10.2}$$

is a pushout for every $[k] \in \mathbf{\Delta}$. This can be checked by the following explicit argument.

Every surjection in $\mathbf{\Delta}$ has a section, and having a section is preserved by the functor $\operatorname{Hom}([k], -)$, so the maps in (1.2.10.2) are surjective. In particular, the induced map from the colimit C of the $(\bullet \leftarrow \bullet \to \bullet)$ part of the square to its lower right corner is surjective. To show injectivity of this map, we need to show that if two maps $[k] \to [n]$ agree upon post-composition with the surjection $[n] \twoheadrightarrow [a]$, then they coincide in the colimit C. Denote by A the endomorphism of $\operatorname{Hom}([k],[n])$ obtained by post-composing with $f : [n] \twoheadrightarrow [r]$ and then with the section $[r] \to [n]$ of f sending an element $i \in [r]$ to the smallest element of $f^{-1}(i) \subseteq [n]$. Similarly, define an endomorphism B of $\operatorname{Hom}([k],[n])$ using g in place of f. Now it is simple to check that if two elements of $\operatorname{Hom}([k],[n])$ coincide upon post-composition with $[n] \twoheadrightarrow [a]$, then they can be made to coincide in $\operatorname{Hom}([k],[n])$ by applying A and B sufficiently many times. This gives the desired injectivity assertion. □

1.2.11 Definition (Cardinality of a simplicial set)**.** The *cardinality* of a simplicial set is the cardinality of the set of its *non-degenerate* simplices. If a simplicial set has cardinality κ, then the set of (all of) its simplices has cardinality $\aleph_0 \cdot \kappa$, which equals $\max(\aleph_0, \kappa)$ when $\kappa > 0$.

1.2.12 Exercise. Show that there are exactly $\frac{(n+m)!}{n!m!}$ non-degenerate $(n+m)$-simplices in $\Delta^n \times \Delta^m$. Identify these simplices with paths from $(0,0)$ to (n,m) in the $n \times m$ unit grid.

Filtering by dimension

1.2.13 Definition (Truncated simplicial object)**.** Denote by $\mathbf{\Delta}_{\leq k} \subseteq \mathbf{\Delta}$ the full subcategory spanned by the objects $[a]$ for $a \leq k$. A k-truncated simplicial object of a category C is a functor $\mathbf{\Delta}_{\leq k}^{\mathsf{op}} \to \mathsf{C}$. There is a tautological restriction ('truncation') functor

$$\mathsf{Fun}(\mathbf{\Delta}^{\mathsf{op}}, \mathsf{C}) \to \mathsf{Fun}(\mathbf{\Delta}_{\leq k}^{\mathsf{op}}, \mathsf{C}) \tag{1.2.13.1}$$

from simplicial objects to k-truncated simplicial objects. When C is cocomplete, the truncation functor has a left adjoint given by left Kan extension.

$$\mathsf{Fun}(\mathbf{\Delta}_{\leq k}^{\mathsf{op}}, \mathsf{C}) \to \mathsf{Fun}(\mathbf{\Delta}^{\mathsf{op}}, \mathsf{C}) \tag{1.2.13.2}$$

$$X_\bullet \mapsto \Big([a] \mapsto \operatorname*{colim}_{([a]\downarrow \mathbf{\Delta}_{\leq k})^{\mathsf{op}}} X_\bullet\Big) \tag{1.2.13.3}$$

Since $\mathbf{\Delta}_{\leq k} \subseteq \mathbf{\Delta}$ is a full subcategory, this left adjoint is fully faithful (1.1.103). We implicitly identify k-truncated simplicial objects with the full subcategory of simplicial objects given by the essential image of this functor. Being k-truncated is thus a *property* of a simplicial object, and a simplicial object will be called *truncated* when it is k-truncated for some $k < \infty$. The truncation functor from simplicial objects to k-truncated simplicial objects is also called the *k-skeleton* functor.

⋆ **1.2.14 Definition** (Latching object)**.** Let $X_\bullet : \mathbf{\Delta}^{\mathsf{op}} \to \mathsf{C}$ be a simplicial object. The nth *latching object* of $X_\bullet$ is the colimit

$$L_n X_\bullet = \operatorname*{colim}_{([n]\downarrow \mathbf{\Delta}_{<n})^{\mathsf{op}}} X_\bullet. \tag{1.2.14.1}$$

There is a tautological map $L_n X_\bullet \to X_n$ called the nth *latching map* of $X_\bullet$. More generally, the nth latching map of a map of simplicial objects $X_\bullet \to Y_\bullet$ is the tautological map

$$X_n \bigsqcup_{L_n X_\bullet} L_n Y_\bullet \to Y_n \tag{1.2.14.2}$$

(when $X_\bullet = \varnothing$ is the initial object, this evidently reduces to the latching map of $Y_\bullet$).

The dual notion (i.e. for cosimplicial objects) is called *matching* and is denoted M^n.

1.2.15 Remark. We note that the full subcategory $([n] \twoheaddownarrow \mathbf{\Delta}_{<n}) \subseteq ([n] \downarrow \mathbf{\Delta}_{<n})$ spanned by surjections $[n] \twoheadrightarrow [a]$ is initial, since its inclusion has a right adjoint (sending a map $[n] \to [a]$ to the surjection $[n] \twoheadrightarrow \operatorname{im}([n] \to [a])$) (1.1.75). Thus the latching object is equivalently given by the colimit

$$L_n X_\bullet = \operatorname*{colim}_{([n]\twoheaddownarrow \mathbf{\Delta}_{<n})^{\mathsf{op}}} X_\bullet. \tag{1.2.15.1}$$

This category $([n] \downarrow \mathbf{\Delta}_{<n})$ has a quite simple structure. A surjection $f : [n] \twoheadrightarrow [a]$ is determined uniquely by the sequence of n bits $\varepsilon_i(f) = f(i) - f(i-1) \in \{0, 1\}$ for $i = 1, \dots, n$. The category $([n] \downarrow \mathbf{\Delta}_{\leq n})$ is thus the poset category $\{0 \leftarrow 1\}^n$, and its full subcategory $([n] \downarrow \mathbf{\Delta}_{<n})$ is the complement of the initial vertex $(1, \dots, 1)$ (corresponding to the identity surjection $[n] \twoheadrightarrow [n]$).

1.2.16 Lemma. *A simplicial object is k-truncated iff its latching maps in all degrees $> k$ are isomorphisms.*

Proof. The counit map $\mathrm{sk}_{r-1} X_\bullet \to X_\bullet$ in degree r is precisely the rth latching map $L_r X_\bullet \to X_r$. Thus if X is $(r-1)$-truncated, then the rth latching map is an isomorphism. Since being k-truncated implies being i-truncated for all $i \geq k$, we conclude that being k-truncated implies the latching maps in all degrees $> k$ are isomorphisms.

For the converse, we apply the criterion (1.1.94) for identifying a reflective subcategory. It thus suffices to show, for any pair of simplicial objects $X_\bullet$ and $Y_\bullet$ whose latching maps are isomorphisms in degrees $> k$, that a morphism $X_\bullet \to Y_\bullet$ is an isomorphism iff it is an isomorphism in degrees $\leq k$. This may be proven by induction. □

⋆ **1.2.17 Definition** (Reedy property [94])**.** Let $\mathcal{P}$ be a property of morphisms in a category C. A simplicial object $X_\bullet : \mathbf{\Delta}^{\mathsf{op}} \to \mathsf{C}$ is said to be *Reedy* $\mathcal{P}$ when its latching maps $L_i X_\bullet \to X_i$ have property $\mathcal{P}$. More generally, a morphism of simplicial objects $X_\bullet \to Y_\bullet$ is called Reedy $\mathcal{P}$ when its relative latching maps (1.2.14.2) have $\mathcal{P}$ (when $X_\bullet = \varnothing$ is the initial object, this is evidently the same as $Y_\bullet$ being Reedy $\mathcal{P}$).

1.2.18 Lemma. *Let $X^\bullet \to Y^\bullet$ be a map of cosimplicial objects. The map on nth matching objects $M^n X^\bullet \to M^n Y^\bullet$ is (functorially in $X^\bullet \to Y^\bullet$) a finite composition of pullbacks of ith matching maps $X^i \to M^i X^\bullet \times_{M^i Y^\bullet} Y^i$ for $i < n$.*

Proof. Write matching objects as limits over the categories $([n] \downarrow \mathbf{\Delta}_{<n})$ as in (1.2.15). Consider the category $([n] \downarrow \mathbf{\Delta}_{<n}) \times (x \to y)$ and the evident diagram from it associated to $X^\bullet \to Y^\bullet$. The limit of this diagram is $M^n X^\bullet$ (since $([n] \downarrow \mathbf{\Delta}_{<n}) \times x$ is initial) while the limit of its restriction to $([n] \downarrow \mathbf{\Delta}_{<n}) \times y$ is $M^n Y^\bullet$. Now let us build $([n] \downarrow \mathbf{\Delta}_{<n}) \times (x \to y)$ from its full subcategory $([n] \downarrow \mathbf{\Delta}_{<n}) \times y$ by iteratively adding maximal objects not already present. The effect on the limit of adding such a maximal object $([n] \twoheadrightarrow [i]) \times x$ is to form a pullback of the ith matching map of $X^\bullet \to Y^\bullet$ (use Mayer–Vietoris (**??**) twice). □

1.2.19 Exercise. Let $X_\bullet : \mathbf{\Delta}^{\mathsf{op}} \to \mathsf{C}$ be a simplicial object which is Reedy $\mathcal{P}$. Conclude from (1.2.18) that if $F : \mathsf{C} \to \mathsf{D}$ preserves pushouts of $\mathcal{P}$-morphisms, then it preserves the latching objects $L_i X_\bullet$ (in the sense that the natural map $L_i F(X_\bullet) \to F(L_i X_\bullet)$ is an isomorphism (1.1.76)).

Simplicial abelian groups

We now study simplicial abelian groups (and, more generally, simplicial objects in additive categories (**??**)).

The following classical result identifies simplicial abelian groups with complexes of abelian groups supported in non-negative homological degree. It is an 'abelian' analogue of the fundamental result on non-degenerate simplices for simplicial sets (1.2.10) and the resulting fact that every simplicial set is the ascending union of its skeleta, each of which is obtained from the previous by attaching some set of non-degenerate simplices.

⋆ **1.2.20 Dold–Kan Correspondence** ([21][58]). *The functor*

$$\begin{aligned} \mathrm{DK} : \mathsf{Kom}_{\geq 0}(\mathsf{Ab}) &\to \mathsf{sAb} && (1.2.20.1) \\ K_\bullet &\mapsto \Big([n] \mapsto \mathrm{Hom}(C_\bullet^{\mathrm{cell}}(\Delta^n), K_\bullet)\Big) && (1.2.20.2) \end{aligned}$$

is an equivalence of categories.

Proof. The key to the Dold–Kan correspondence is to express the group of homomorphisms $\mathrm{Hom}(C_\bullet^{\mathrm{cell}}(\Delta^n), K_\bullet)$ using a 'shelling' (filtration by pushouts of horns) of Δ^n. Fix such a filtration (there are many) $\varnothing = F_0 \subseteq F_1 \subseteq \cdots \subseteq F_{2^n} = \Delta^n$, which necessarily contains exactly $\binom{n}{k}$ pushouts of k-dimensional horns for all k. Applying $\mathrm{Hom}(C_\bullet^{\mathrm{cell}}(-), K_\bullet)$ yields a sequence of maps.

$$\begin{aligned} &\mathrm{Hom}(C_\bullet^{\mathrm{cell}}(\Delta^n), K_\bullet) = \mathrm{Hom}(C_\bullet^{\mathrm{cell}}(F_{2^n}), K_\bullet) \to \cdots \\ &\qquad \cdots \to \mathrm{Hom}(C_\bullet^{\mathrm{cell}}(F_1), K_\bullet) \to \mathrm{Hom}(C_\bullet^{\mathrm{cell}}(F_0), K_\bullet) = 0 \quad (1.2.20.3) \end{aligned}$$

Every restriction map

$$\mathrm{Hom}(C_\bullet^{\mathrm{cell}}(F_i), K_\bullet) \to \mathrm{Hom}(C_\bullet^{\mathrm{cell}}(F_{i-1}), K_\bullet) \tag{1.2.20.4}$$

has a canonical section: when (F_i, F_{i-1}) is a pushout of a k-dimensional horn, a chain map $C_\bullet^{\mathrm{cell}}(F_{i-1}) \to K_\bullet$ may be extended to $C_\bullet^{\mathrm{cell}}(F_i)$ by declaring it should vanish on the new k-simplex, and this uniquely determines its value on the new $(k-1)$-simplex. The kernel of the restriction map is $\mathrm{Hom}(C_\bullet^{\mathrm{cell}}(F_i, F_{i-1}), K_\bullet) = \mathrm{Hom}(C_\bullet^{\mathrm{cell}}(\Delta^k, \Lambda_j^k), K_\bullet) = K_k$. A choice of shelling of Δ^n thus defines an isomorphism

$$\mathrm{Hom}(C_\bullet^{\mathrm{cell}}(\Delta^n), K_\bullet) \cong \bigoplus_k K_k^{\oplus \binom{n}{k}}. \tag{1.2.20.5}$$

It follows immediately that the Dold–Kan functor is faithful. ☐

1.2.21 Example. Let X be a topological space, and let $\mathsf{Vect}(X)$ denote the additive category of finite-dimensional real vector bundles on X. The category $\mathsf{Vect}(X)$ is idempotent-complete (it suffices to treat the 'universal' case which consists of showing that $\ker \pi$ is a vector bundle over $\{\pi : \mathbb{R}^n \to \mathbb{R}^n \mid \pi^2 = \pi\}$). The Dold–Kan Correspondence (1.2.20) thus provides an equivalence between complexes of vector bundles supported in non-negative homological degrees $\mathsf{Kom}_{\geq 0}(\mathsf{Vect}(X))$ and simplicial vector bundles $\mathsf{sVect}(X)$.

1.2.22 Example. Consider an idempotent-complete additive category A. idempotent-completeness is invariant under passing to opposites, so A^{op} is also idempotent-complete. The Dold–Kan Correspondence (1.2.20) for A is an equivalence between $\mathsf{Kom}^{\geq 0}(\mathsf{A})$ and csA.

1.2.23 Exercise. Show that under the Dold–Kan Correspondence (1.2.20):
(1.2.23.1) $[\mathbb{Z}[k+1] \to \mathbb{Z}[k]] \in \mathsf{Kom}_{\geq 0}(\mathsf{Ab})$ corresponds to $C^k_{\mathrm{cell}}(\Delta^\bullet) \in \mathsf{sAb}$.
(1.2.23.2) $\mathbb{Z}[k] \in \mathsf{Kom}_{\geq 0}(\mathsf{Ab})$ corresponds to $Z^k_{\mathrm{cell}}(\Delta^\bullet) \in \mathsf{sAb}$.

1.2.24 Corollary. *Let C be an idempotent-complete additive category, and let $\mathcal{P}$ be any property of morphisms in C which is closed under direct sums and retracts. For any map $A_\bullet \to B_\bullet$ in sC and any $n \geq 0$, the following are equivalent:*
(1.2.24.1) *The map $A_n \to B_n$ has $\mathcal{P}$.*
(1.2.24.2) *The map $N_i A_\bullet \to N_i B_\bullet$ has $\mathcal{P}$ for all $i \leq n$.*
In particular $A_k \to B_k$ has $\mathcal{P}$ for all $k \geq 0$ iff $N_k A_\bullet \to N_k B_\bullet$ has $\mathcal{P}$ for all $k \geq 0$.

Proof. A shelling of Δ^n fixes a functorial isomorphism $A_n = \bigoplus_{i=0}^n (N_i A_\bullet)^{\binom{n}{i}}$ (1.2.20.5). □

The next result is a linear analogue of the theory of non-degene simplices in simplicial sets (1.2.10) (compare (**??**)). It appears that it does not follow formally from (1.2.10), since the forgetful functor $\mathsf{Vect} \to \mathsf{Set}$ does not preserve colimits.

1.2.25 Corollary. *For any simplicial object $A_\bullet$ in an idempotent-complete additive category, there is a functorial short exact sequence*

$$0 \to L_k A_\bullet \to A_k \to N_k A_\bullet \to 0 \tag{1.2.25.1}$$

for every $k \geq 0$. This short exact sequence has a functorial splitting associated to any choice of codimension one face of Δ^k.

Proof. □

1.2.26 Corollary. *Let $A_\bullet \to B_\bullet$ be a map of simplicial objects in an idempotent-complete additive category. The cone of the kth latching map $A_k \sqcup_{L_k A_\bullet} L_k B_\bullet \to B_k$ is (functorially) homotopy equivalent to the cone of the map $N_k A_\bullet \to N_k B_\bullet$ on normalized chains in degree k.*

Proof. The map on short exact sequences (1.2.25)

$$\begin{tikzcd} L_k A_\bullet \arrow[r] \arrow[d] & A_\bullet \arrow[r] \arrow[d] & N_k A_\bullet \arrow[d] \\ L_k B_\bullet \arrow[r] & B_\bullet \arrow[r] & N_k B_\bullet \end{tikzcd} \tag{1.2.26.1}$$

induces a map from the total complex of the square on the left to the cone of $N_k A_\bullet \to N_k B_\bullet$. It suffices to show that this map is a homotopy equivalence and so is the natural map from the total complex of the square on the left to the cone of $A_k \sqcup_{L_k A_\bullet} L_k B_\bullet \to B_k$. The above map on short exact sequences is functorially split (1.2.25), hence may be written as

$$\begin{array}{ccccc} L_k A_\bullet & \longrightarrow & L_k A_\bullet \oplus N_k A_\bullet & \longrightarrow & N_k A_\bullet \\ \downarrow & & \downarrow & & \downarrow \\ L_k B_\bullet & \longrightarrow & L_k B_\bullet \oplus N_k B_\bullet & \longrightarrow & N_k B_\bullet \end{array} \tag{1.2.26.2}$$

with the evident inclusion and projection maps, from which the two desired homotopy equivalence assertions are immediate. □

1.2.27 Corollary. *A simplicial object in an additive category is n-truncated* (1.2.13) *iff the corresponding chain complex is supported in degrees $\leq n$.*

Proof. Combine (1.2.16) and (1.2.26). □

1.3 Simplicial homotopy theory

Basic building blocks

1.3.1 Definition (Boundary and horns)**.** The boundary $\partial\Delta^n \subseteq \Delta^n$ consists of those simplices of Δ^n which omit at least one vertex of Δ^n. The ith horn $\Lambda^n_i \subseteq \Delta^n$ ($0 \leq i \leq n$ and $n \geq 1$) consists of those simplices of Δ^n which omit at least one vertex other than vertex i.

1.3.2 Exercise. Draw $\Lambda^n_i \subseteq \Delta^n$ and $\partial\Delta^n \subseteq \Delta^n$ for all $n \leq 3$.

1.3.3 Exercise. Show that the map $\Lambda^n_i \to \Delta^n$ does not have a retraction (except for $n = 1$), but does after applying geometric realization (it will be helpful to use (**??**)).

1.3.4 Definition (Transfinite composition of morphisms)**.** Let α be an ordinal, and consider a sequence of objects $X_0, X_1, \ldots$ of a category C indexed by ordinals $\nu < \alpha$ along with morphisms

$$\operatorname*{colim}_{\mu<\nu} X_\mu \to X_\nu \tag{1.3.4.1}$$

for all $\nu < \alpha$. The induced morphism $X_0 \to \operatorname{colim}_{\nu<\alpha} X_\nu$ is called the *transfinite composition* of the morphisms (1.3.4.1).

1.3.5 Exercise. Let $\mathcal{M}$ be a set of morphisms, and let $\overline{\mathcal{M}}$ be the set of morphisms expressible as transfinite compositions of morphisms in $\mathcal{M}$. Show that $\overline{\mathcal{M}}$ is closed under transfinite composition. Show that if $\mathcal{M}$ is closed under pushouts then so is $\overline{\mathcal{M}}$.

1.3.6 Exercise. Show that injections of sets are closed under pushouts and transfinite composition. Conclude the same holds for injections of simplicial sets.

⋆ **1.3.7 Definition** (Pair)**.** A *pair* of simplicial sets (X, A) is an injective map $A \to X$. A morphism of pairs $(X, A) \to (X', A')$ is commutative square. A *filtration* of a pair is a factorization as a composition of other pairs. We often (but with some necessary exceptions) identify a simplicial set X with the pair $(X, \varnothing)$.

⋆ **1.3.8 Lemma.** *Every simplicial set pair (X, A) is filtered by pushouts of simplices $(\Delta^k, \partial\Delta^k)$.*

Proof. Since $A \hookrightarrow X$ is injective, so is the pullback $A \times_X \Delta^n \to \Delta^n$ for any map $\Delta^n \to X$. Choose a simplex $\Delta^n \to X$ which is not in the image of A, and choose a simplex $\Delta^k \hookrightarrow \Delta^n$ for which $A \times_X \Delta^n \times_{\Delta^n} \Delta^k = \partial\Delta^k$.

$$\begin{array}{ccc} \partial\Delta^k & \longrightarrow & A \\ \downarrow & & \downarrow \\ \Delta^k & \longrightarrow & X \end{array} \tag{1.3.8.1}$$

Let $X_0 \subseteq X$ be the union of the images of A and Δ^k. Replacing X with X_0, the diagram remains a pullback and becomes a pushout (1.1.56). We have thus factored $A \hookrightarrow X$ into the composition of $A \hookrightarrow X_0$ (which is a pushout of $(\Delta^k, \partial\Delta^k)$) and a new injection $X_0 \hookrightarrow X$. By transfinite induction, we produce the desired filtration of (X, A). □

Lifting properties

A fundamental concept in categorical homotopy theory is lifting properties.

1.3.9 Definition (Lifting property)**.** A *lift* for a commuting diagram of solid arrows

$$\begin{tikzcd} A \arrow[r] \arrow[d] & X \arrow[d] \\ B \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd} \tag{1.3.9.1}$$

is a dotted arrow making the diagram commute. A morphism $X \to Y$ is said to satisfy the *right lifting property* with respect to a morphism $A \to B$ (and $A \to B$ satisfies the *left lifting property* with respect to $X \to Y$) when every such diagram with these given vertical arrows has a lift. The right lifting property in the special case $X \to *$ will be called the *extension property*: X satisfies the extension property with respect to a given map $A \to B$ when every map $A \to X$ factors as $A \to B \to X$.

1.3.10 Exercise. Show that the right lifting property with respect to any fixed morphism $A \to B$ is preserved under pullback and closed under op-transfinite composition.

1.3.11 Exercise. Show that the left lifting property with respect to any fixed morphism $X \to Y$ is preserved under pushout and closed under transfinite composition.

1.3.12 Definition (Kan fibration [55, 56])**.** A map of simplicial sets $X \to Y$ is called a *Kan fibration* when it has the right lifting property for every horn (Δ^n, Λ^n_i). That is, $X \to Y$ is a Kan fibration when for every commuting diagram of solid arrows

$$\begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d] & X \arrow[d] \\ \Delta^n \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd} \tag{1.3.12.1}$$

there exists a dotted arrow making the diagram commute. Kan fibrations are often indicated with the arrow $\twoheadrightarrow$.

A simplicial set X is called a *Kan complex* iff the map $X \to *$ is a Kan fibration. In other words, X is a Kan complex when it satisfies the extension property for (Δ^n, Λ^n_i), meaning every map $\Lambda^n_i \to X$ extends to Δ^n.

1.3.13 Exercise. Use the retraction property (1.3.3) to show that the singular simplicial set (**??**) of any topological space is a Kan complex.

1.3.14 Exercise (Functor π_0)**.** The set of connected components $\pi_0 X$ of a simplicial set X is the set vertices X_0 modulo the equivalence relation closure of the relation given by the edges ($x \sim x'$ iff there exists an edge $x \to x'$); this gives a functor $\pi_0 : \mathsf{sSet} \to \mathsf{Set}$. Show that if X is a Kan complex, then the edge relation is an equivalence relation.

1.3.15 Exercise. Show that every simplicial abelian group is a Kan complex by appealing to the Dold–Kan correspondence (1.2.20) and noting that $C_\bullet^{\mathrm{cell}}(\Lambda_i^n) \hookrightarrow C_\bullet^{\mathrm{cell}}(\Delta^n)$ has a retract. In fact, every simplicial group is a Kan complex (Moore [84, Théorème 3]).

1.3.16 Definition (Smash product of pairs)**.** For simplicial set pairs (X, A) and (Y, B), we term

$$(X, A) \wedge (Y, B) = (X \times Y, (X \times B) \cup_{A \times B} (A \times Y)). \tag{1.3.16.1}$$

their *smash product* (beware: like tensor product, the smash product is not the categorical product).

1.3.17 Exercise. Show that if (X', A') is a pushout of a pair (X, A), then $(X', A') \wedge (Y, B)$ is a pushout of $(X, A) \wedge (Y, B)$. Show that if (X, A) is filtered by pushouts of pairs in some collection $\mathcal{M}$, then $(X, A) \wedge (Y, B)$ is filtered by pushouts of pairs in $\mathcal{M} \wedge (Y, B)$.

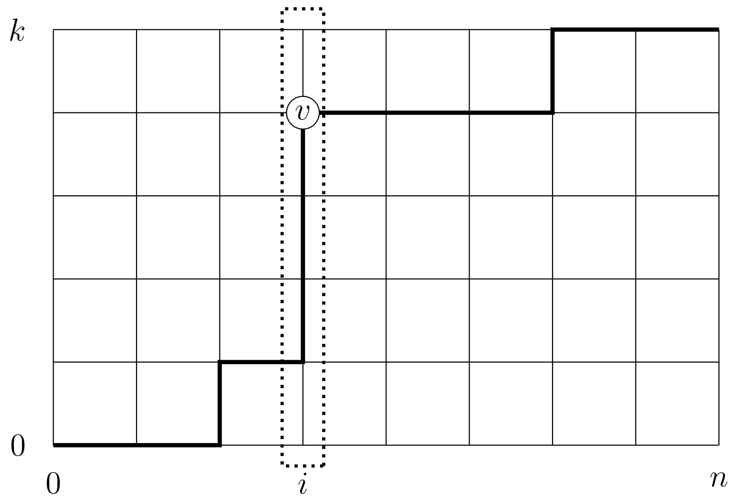
⋆ **1.3.18 Lemma.** *Every smash product $(\Delta^n, \Lambda_i^n) \wedge (\Delta^k, \partial\Delta^k)$ is filtered by pushouts of horns.*

Proof. The product $\Delta^n \times \Delta^k$ is the nerve of the category $[n] \times [k]$. Its non-degenerate $(n + k)$-simplices are thus in bijection with lattice paths from $(0, 0)$ to (n, k). This set of $(n + k)$-simplices has a natural partial order in which $\sigma \succeq \tau$ iff the path corresponding to σ lies above that corresponding to τ (when the $[n]$ coordinate is drawn horizontally and the $[k]$ coordinate vertically). We filter the pair $(\Delta^n, \Lambda_i^n) \wedge (\Delta^k, \partial\Delta^k)$ by adding the non-degenerate $(n + k)$-simplices one at a time, according to any total order refining the aforementioned partial order. It suffices to show that each simplex addition in this filtration can be realized by filling some number of horns.

Let $Q \subseteq \Delta^n \times \Delta^k$ denote the union of $(\Delta^n \times \partial\Delta^k) \cup (\Lambda_i^n \times \Delta^k)$ and any set S of non-degenerate $(n + k)$-simplices with the property that $\sigma \succeq \tau \in S$ implies $\sigma \in S$. Our aim is to show that the pair $(Q \cup \sigma, Q)$ is filtered by pushouts of horns for any $\sigma \notin S$ which is maximal among simplices not in S. Equivalently, this means filtering the pair $(\sigma, \sigma \cap Q)$ by pushouts of horns.

Let us say that the pair $(\sigma, \sigma \cap Q)$ is *coned* at a vertex v of σ iff for every simplex $\tau \subseteq \sigma \cap Q$, the cone of τ with v is also $\subseteq \sigma \cap Q$. Denoting by $\sigma_{\hat{v}} \subseteq \sigma$ the span of all vertices other than v, any filtration of $(\sigma_{\hat{v}}, \sigma_{\hat{v}} \cap Q)$ by pushouts of $(\Delta^a, \partial\Delta^a)$ determines, by coning at v, a filtration of $(\sigma, \sigma \cap Q)$ by pushouts of horns with cone point v. It thus suffices show that $(\sigma, \sigma \cap Q)$ is coned at some vertex $v \in \sigma$.

Regarding σ as a lattice path from $(0, 0)$ to (n, k), choose $v = (i, j) \in \sigma$ where i indexes the horn Λ^n_i and j is as large as possible given i.



(1.3.18.1)

Our goal is now to show that $(\sigma, \sigma \cap Q)$ is coned at v. We first describe the intersection $\sigma \cap Q$. A simplex $\tau \subseteq \sigma$ is contained in Q iff it satisfies at least one of the following conditions:

(1.3.18.2) The vertices of τ do not surject onto $[n] - \{i\}$.

(1.3.18.3) The vertices of τ do not surject onto $[k]$.

(1.3.18.4) The subset of the lattice path σ corresponding to τ misses at least one cliffbottom corner (a vertex $w \in \sigma$ for which both $w + (0,1)$ and $w - (1,0)$ are in σ).

Now suppose $\tau \subseteq \sigma$ lies in Q, and let us show that the simplex spanned by τ union v also lies in Q. The property of not surjecting onto $[n] - \{i\}$ is certainly preserved by adding v. Missing a cliffbottom corner is also preserved by adding v since v is never a cliffbottom corner. Now suppose τ does not surject onto $[k]$ but $\tau \cup v$ does. This means τ does not contain any vertex with the same second coordinate as v. If the second coordinate of v is $< k$, then τ misses a cliffbottom corner, hence so does $\tau \cup v$. If the second coordinate of v is k and $i < n$, then $\tau \cup v$ cannot surject onto $[n] - \{i\}$, since it misses everything $> i$. This completes the proof in the case $i < n$. The case $i = n$ now follows by symmetry. $\square$

Kan homotopy theory

1.3.19 Exercise. Let X and Y be simplicial sets. Use (1.3.18) (along with (1.3.17) and (1.3.8)) to show that if Y is a Kan complex then so is the simplicial mapping space $\underline{\mathrm{Hom}}(X, Y)$ (1.2.8).

1.3.20 Exercise (Homotopy category of Kan complexes hSpc)**.** For a Kan complex X and a simplicial set K, call maps $f, g : K \to X$ *homotopic* iff there exists a map $K \times \Delta^1 \to X$ whose restrictions to $K \times 0$ and $K \times 1$ coincide with f and g, respectively. Use (1.3.18) to show that homotopy is an equivalence relation on the set of maps $K \to X$. Conclude that Kan complexes and homotopy classes of maps form a category, denoted hSpc. A map of Kan complexes is called a *homotopy equivalence* iff it is an isomorphism in hSpc. A Kan complex is called *contractible* when it is homotopy equivalent to a point $*$.

1.3.21 Exercise (Transport maps of a Kan fibration)**.** Let $X \to Y$ be a Kan fibration. Associate to any edge $y \to y'$ in Y a *transport map* $X_y \to X_{y'}$ by lifting the pair $X_y \times (\Delta^1, 0)$, and show that this map is well defined up to homotopy. Show that for any 2-simplex in Y with vertices y, y', y'', the resulting triangle commutes up to homotopy. Show that the map $X_y \to X_{y'}$ associated to an edge $y \to y'$ is a homotopy equivalence (a homotopy inverse may be constructed by lifting $X_{y'} \times (\Delta^1, 1)$). Conclude that this defines a diagram $Y \to \mathsf{hSpc}_{\simeq}$.

1.3.22 Exercise. Let (X, A) be a pair of simplicial sets, and let Y be a Kan complex. Given a map $f : A \to Y$, we may consider homotopy classes of extensions of f to X relative A (that is, we consider maps $\bar{f} : X \to Y$ with $\bar{f}|_A = f$, modulo the equivalence relation $\bar{f} \sim \tilde{f}$ when there is a map $X \times \Delta^1 \to Y$ acting as f on $A \times \Delta^1$ and as $\bar{f}$ and $\tilde{f}$ over $X \times 0$ and $X \times 1$, respectively). Show this is indeed an equivalence relation. Show that a homotopy from f to $g : A \to Y$ induces a bijection between homotopy classes of extensions of f and g. Conclude that composing with a homotopy equivalence $h : Y \to Y'$ induces a bijection between homotopy classes of extensions of f and of $h \circ f$.

⋆ **1.3.23 Definition** (Trivial Kan fibration)**.** A map of simplicial sets is called a *trivial Kan fibration* iff it satisfies the right lifting property for every pair $(\Delta^n, \partial\Delta^n)$. A simplicial set is called a *trivial Kan complex* iff the map $X \to *$ is a trivial Kan fibration.

1.3.24 Exercise. Show that a trivial Kan fibration is a Kan fibration. In fact, show that a trivial Kan fibration satisfies the right lifting property for every pair (X, A) (use (1.3.8)).

1.3.25 Exercise. Show that a Kan complex is trivial iff it is contractible.

1.3.26 Exercise (Functor $\mathsf{sSet} \to \mathsf{hSpc}$)**.** Show that if $X \hookrightarrow Y$ is filtered by pushouts of horns and Z is a Kan complex, then the map $\underline{\mathrm{Hom}}(Y, Z) \to \underline{\mathrm{Hom}}(X, Z)$ is a trivial Kan fibration. Use the small object argument (**??**) to show that for every simplicial set X, there exists an inclusion $X \hookrightarrow \overline{X}$ which is filtered by pushouts of horns with $\overline{X}$ a Kan complex (call this a *Kanification* of X). Show that for any pair of such inclusions $X \hookrightarrow \overline{X}$ and $Y \hookrightarrow \overline{Y}$ and any map $X \to Y$, there exists a dotted arrow making the following diagram commute

$$\begin{tikzcd} X \arrow[r, hook] \arrow[d] & \overline{X} \arrow[d, dashed] \\ Y \arrow[r, hook] & \overline{Y} \end{tikzcd} \tag{1.3.26.1}$$

and that moreover this dotted arrow is unique up to homotopy rel X. Show that sending X to (any choice of) $\overline{X}$ and sending a map $X \to Y$ to (any choice of) extension $\overline{X} \to \overline{Y}$ gives a well defined functor $\mathsf{sSet} \to \mathsf{hSpc}$. A map of simplicial sets which is sent to an isomorphism by this functor is called a *Kan equivalence*. Show that any inclusion of simplicial sets which is filtered by pushouts of horns is a Kan equivalence.

1.3.27 Exercise. Show that a trivial Kan fibration is a Kan equivalence.

1.3.28 Exercise (Mapping path fibration)**.** Given a map of Kan complexes $X \to Y$, consider the factorization

$$\begin{tikzcd} X \arrow[r, hook, "\sim"] \arrow[dr] & X \times_Y \underline{\mathrm{Hom}}(\Delta^1, Y) \arrow[d, two heads] \\ & Y \end{tikzcd} \tag{1.3.28.1}$$

where the fiber product is via the 'evaluate at $0 \in \Delta^1$' map $\underline{\mathrm{Hom}}(\Delta^1, Y) \to Y$, the vertical map is 'evaluate at $1 \in \Delta^1$' (the roles of $0, 1 \in \Delta^1$ may also be reversed), and the horizontal map is via the 'pullback along $\Delta^1 \to *$' map $Y \to \underline{\mathrm{Hom}}(\Delta^1, Y)$. Show that the vertical map is a Kan fibration (lift (Δ^n, Λ^n_i) against it by extending $(\Delta^n, \Lambda^n_i) \to X$ and $(\Delta^n, \Lambda^n_i) \wedge (\Delta^1, \partial\Delta^1) \to Y$). Show that the evident retraction of the horizontal map (projection to X) is a trivial Kan fibration (lifting $(\Delta^n, \partial\Delta^n)$ against it amounts to extending $(\Delta^n, \partial\Delta^n) \wedge (\Delta^1, 0) \to Y$). The vertical map in (1.3.28.1) is called the *mapping path fibration* associated to the map $X \to Y$.

1.3.29 Lemma. *A Kan fibration of Kan complexes is trivial iff it is a homotopy equivalence.*

Proof. Let $\pi : X \to Y$ be a Kan fibration of Kan complexes which is a homotopy equivalence.

First, we argue that π has a section s. Begin with a map $g : Y \to X$ and a homotopy $H : Y \times \Delta^1 \to Y$ from $H(-, 0) = \pi g$ to $H(-, 1) = 1_Y$ (which exists since π is a homotopy equivalence).

$$\begin{tikzcd} Y \arrow[r, "g"] \arrow[d, "\times 0"] & X \arrow[d, "\pi"] \\ Y \times \Delta^1 \arrow[r, "H"] & Y \end{tikzcd} \tag{1.3.29.1}$$

By solving the lifting problem for $Y \times (\Delta^1, 0)$ against π, we produce at $Y \times 1$ a section $s : Y \to X$.

Now let us argue that $s\pi$ and 1_X are homotopic over Y. Consider homotopy classes of homotopies between $s\pi$ and 1_X (that is, homotopy classes of extensions of $(s\pi \sqcup 1_X) : X \times \partial\Delta^1 \to X$ to $X \times \Delta^1$). Since π is a homotopy equivalence, composition with π induces a bijection of this set with the set of homotopies between $\pi s \pi$ and π (1.3.22), which has a distinguished class given by the constant homotopy $\pi s \pi = \pi$. Fix a homotopy $H : X \times \Delta^1 \to X$ from $H(-, 0) = s\pi$ to $H(-, 1) = 1_X$ whose composition with π is homotopic to the distinguished homotopy. By lifting the homotopy between πH and the constant homotopy against π as above, we produce the desired homotopy $H' : X \times \Delta^1 \to X$ from $H(-, 0) = s\pi$ to $H(-, 1) = 1_X$ whose composition with π is constant.

Now we have shown that $\pi : X \to Y$ is a homotopy equivalence over Y, which implies π is a trivial Kan fibration. To be explicit, consider a lifting problem for $(\Delta^n, \partial\Delta^n)$ against π. Composing the map $\Delta^n \to Y$ with the section s and composing the map $\partial\Delta^n \to X$ with the homotopy from $s\pi$ to 1_X over Y, we obtain a lifting problem for $(\Delta^n, \partial\Delta^n) \wedge (\Delta^1, 0)$ against π whose restriction to $\Delta^n \times 1$ is our original problem. This new lifting problem has a solution since π is a Kan fibration and $(\Delta^n, \partial\Delta^n) \wedge (\Delta^1, 0)$ is filtered by pushouts of horns (1.3.18). $\square$

1.3.30 Lemma. *A Kan fibration is trivial iff its fibers are trivial.*

Proof. The set of diagrams of solid arrows

$$\begin{tikzcd} \partial\Delta^k \arrow[r] \arrow[d] & X \arrow[d] \\ \Delta^k \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd} \tag{1.3.30.1}$$

is the set of vertices of $\underline{\mathrm{Hom}}(\partial\Delta^k, X) \times_{\underline{\mathrm{Hom}}(\partial\Delta^k, Y)} \underline{\mathrm{Hom}}(\Delta^k, Y)$, and the set of such diagrams equipped with a lift is the set of vertices of $\underline{\mathrm{Hom}}(\Delta^k, X)$. The forgetful map

$$\underline{\mathrm{Hom}}(\Delta^k, X) \to \underline{\mathrm{Hom}}(\partial\Delta^k, X) \times_{\underline{\mathrm{Hom}}(\partial\Delta^k, Y)} \underline{\mathrm{Hom}}(\Delta^k, Y) \tag{1.3.30.2}$$

is a Kan fibration: lifting a pair (Δ^n, Λ^n_i) against this map is the same as lifting the smash product $(\Delta^n, \Lambda^n_i) \wedge (\Delta^k, \partial\Delta^k)$ against $X \to Y$. Since (1.3.30.2) is a Kan fibration, its image is a union of connected components (1.3.14) of the target. It thus suffices to show that every connected component of $\underline{\mathrm{Hom}}(\partial\Delta^k, X) \times_{\underline{\mathrm{Hom}}(\partial\Delta^k, Y)} \underline{\mathrm{Hom}}(\Delta^k, Y)$ contains a vertex whose constituent map $\Delta^k \to Y$ is constant. The forgetful map $\underline{\mathrm{Hom}}(\partial\Delta^k, X) \times_{\underline{\mathrm{Hom}}(\partial\Delta^k, Y)} \underline{\mathrm{Hom}}(\Delta^k, Y) \to \underline{\mathrm{Hom}}(\Delta^k, Y)$ is a Kan fibration since it is a pullback of $\underline{\mathrm{Hom}}(\partial\Delta^k, X) \to \underline{\mathrm{Hom}}(\partial\Delta^k, Y)$ which is a Kan fibration since $X \to Y$ is. It thus suffices to show that every connected component of $\underline{\mathrm{Hom}}(\Delta^k, Y)$ contains a vertex whose associated map $\Delta^k \to Y$ is constant. It suffices to treat the 'universal' case of $Y = \Delta^k$ and the connected component of the identity, which contains the constant map $\Delta^k \to k \to \Delta^k$ by virtue of the evident homotopy $\Delta^k \times \Delta^1 \to \Delta^k$ which is the identity on $\Delta^k \times 0$ and sends $\Delta^k \times 1$ to $k \in \Delta^k$. □

⋆ **1.3.31 Definition** (Homotopy sets π_n)**.** Let X be a Kan complex with basepoint $x \in X$. The nth homotopy set $\pi_n(X, x)$ (for integer $n \geq 0$) is the set of homotopy classes of maps $(\Delta^n, \partial\Delta^n) \to (X, x)$ (that is, $\pi_n(X, x)$ is the set of connected components π_0 (1.3.14) of the Kan complex $\underline{\mathrm{Hom}}((\Delta^n, \partial\Delta^n), (X, x))$). The homotopy sets are evidently functors $\pi_n : \mathsf{sSet}_*^{\mathsf{Kan}} \to \mathsf{Set}_*$ from based Kan complexes to based sets (the basepoint of $\pi_n(X, x)$ is the class of the constant map).

Note that $\pi_0(X, x)$ (1.3.31) coincides with $\pi_0(X)$ (1.3.14) equipped with the basepoint given by the class of x.

1.3.32 Exercise. Show that homotopic maps in $\mathsf{sSet}_*^{\mathsf{Kan}}$ induce the same map on homotopy sets, so the homotopy set functors descend to the homotopy category $\pi_n : \mathsf{hSpc}_* \to \mathsf{Set}_*$ (where hSpc_* denotes the category whose objects are based Kan complexes and whose morphisms are homotopy classes of based maps $\mathrm{Hom}_{\mathsf{hSpc}_*}((X, x), (Y, y)) = \pi_0\underline{\mathrm{Hom}}((X, x), (Y, y))$).

⋆ **1.3.33 Lemma** (Obstruction theory for maps to a Kan complex)**.** *Let X be a Kan complex. A map $f : \partial\Delta^n \to X$ extends to Δ^n iff it is null-homotopic (homotopic to a constant map). A choice of homotopy between f and the constant map to a point $x \in X$ induces a bijection between the set of homotopy classes of extensions of f and the homotopy set $\pi_n(X, x)$.*

Proof. The map $\underline{\mathrm{Hom}}(\Delta^n, X) \to \underline{\mathrm{Hom}}(\partial\Delta^n, X)$ is a Kan fibration (1.3.18)(1.3.19), so a homotopy $\Delta^1 \to \underline{\mathrm{Hom}}(\partial\Delta^n, X)$ induces a bijection between path components π_0 of the fibers over its endpoints (1.3.21). Conversely, if a map $\partial\Delta^n \to X$ extends to Δ^n, then it is certainly null-homotopic (use the homotopy $\Delta^n \times \Delta^1 \to \Delta^n$ from the identity map $\Delta^n \times 0 \to \Delta^n$ to the constant map $\Delta^n \times 1 \to \{n\} \subseteq \Delta^n$). □

⋆ **1.3.34 Whitehead's Theorem** ([110, 57])**.** *A map in* hSpc *(the homotopy category of Kan complexes* (1.3.20)*) is an isomorphism iff it induces isomorphisms on all homotopy sets* π_k (1.3.31) *for all* $k \geq 0$*.*

Proof. This is a straightforward application of obstruction theory (1.3.33).

Let $f : X \to Y$ be a map of Kan complexes which induces an isomorphism on all homotopy sets. It suffices to construct a map $g : Y \to X$ and a homotopy $H : Y \times \Delta^1 \to Y$ between $\mathbf{1}_Y$ and $Y \xrightarrow{g} X \xrightarrow{f} Y$ (1.1.50). We will construct the pair (g, H) by induction along a filtration of Y by pushouts of $(\Delta^n, \partial\Delta^n)$.

To solve the extension problem of g across $(\Delta^n, \partial\Delta^n)$, we need to know that $\partial\Delta^n \to Y \xrightarrow{g} X$ is null-homotopic. Since f is an isomorphism on homotopy sets, it is equivalent to show that $\partial\Delta^n \to Y \xrightarrow{g} X \xrightarrow{f} Y$ is null-homotopic. Using the homotopy H between $Y \xrightarrow{g} X \xrightarrow{f} Y$ and $\mathbf{1}_Y$, this is equivalent to $\partial\Delta^n \to Y$ being null-homotopic, which is true by definition. Thus the extension problem for g over $(\Delta^n, \partial\Delta^n)$ has solutions.

Given a choice of extension of g over $(\Delta^n, \partial\Delta^n)$, the extension problem for H (over this same simplex) takes the form of an extension $(\Delta^n, \partial\Delta^n) \wedge (\Delta^1, \partial\Delta^1) \to Y$. Such an extension problem has a solution iff the associated element of $\pi_n(Y)$ is trivial, or, equivalently, iff the elements of $\pi_n(Y)$ classifying (in the sense of obstruction theory (1.3.33)) the extensions $\Delta^n \times 0$ and $\Delta^n \times 1$ of the two copies $\partial\Delta^n \times \partial\Delta^1$ coincide (as identified by the homotopy $\partial\Delta^n \times \Delta^1$). This tells us what the homotopy class of extension $(\Delta^n, \partial\Delta^n) \to Y \xrightarrow{g} X \xrightarrow{f} Y$ should be. Since f is an isomorphism on homotopy sets, this determines a unique homotopy class of extension $(\Delta^n, \partial\Delta^n) \to Y \xrightarrow{g} X$. □

1.4 ∞-categories

An ∞-category is a generalization of a category. In an ∞-category, the morphisms from one object to another form a *space*, and composition is associative *up to coherent homotopy*. There are various different ways of turning this slogan into a precise mathematical definition; such a definition is termed a 'model' for the theory of ∞-categories. We use here the model known as *quasi-categories*. Quasi-categories were introduced by Bordman–Vogt [13], and their development into a working model of ∞-categories is due to Joyal [47, 48, 49] and Lurie [74].

In this section, we develop the basics of the theory of ∞-categories in elementary, intentionally unsophistocated, terms. References include Lurie [74, 76], Riehl–Verity [96], and Land [64]. Our treatment is not merely an exposition of the existing theory, rather we also offer new simplified treatments of various foundational aspects.

Definitions and examples

1.4.1 Definition (Inner/outer/left/right horn)**.** A horn $\Lambda^n_i \subseteq \Delta^n$ (1.3.1) is called *inner* when $0 < i < n$ and *outer* when $i \in \{0, n\}$. It is called *left* (resp. *right*) when $0 \leq i < n$ (resp. $0 < i \leq n$).

⋆ **1.4.2 Definition** (∞-category)**.** An ∞-*category* is a simplicial set which has the extension property (1.3.9) for all inner horns (1.4.1).

1.4.3 Remark (∞-categories vs quasi-categories)**.** A simplicial set satisfying the extension property for all inner horns is also called a *weak Kan complex* [13] or *quasi-category* [47], while the term '∞-category' may also refer to other sorts of structures (for example Kan simplicial categories (1.4.8)) which make precise the one-sentence slogan 'definition' of an ∞-category at the beginning of this section (1.4). This terminological distinction allows one to formulate the thesis that quasi-categories are a 'model' of ∞-categories (like the Church–Turing thesis, this is not something which can be formally proven, rather only supported with evidence such as equivalences between various different reasonable models).

Let us see how categories are a special case of ∞-categories.

⋆ **1.4.4 Exercise** (Nerve of a category)**.** Let C be a category. The *nerve* of C is the simplicial set whose set of n-simplices is the set of functors the poset category $[n] = (0 \to \cdots \to n)$ to C. Show that a simplicial set is the nerve of a category iff every inner horn has a *unique* filling. In particular, conclude that the nerve of any category is an ∞-category.

We will henceforth identify a category with its nerve without further comment. Once we define equivalences of ∞-categories, it will become evident that this identification respects the principle of equivalence (note that the set of n-simplices of the nerve of a category is evidently *not* invariant under equivalence).

⋆ **1.4.5 Definition** (Objects, morphisms, and composition in an ∞-category)**.** An *object* x of an ∞-category C is a vertex of C, and a *morphism* $f : x \to y$ is an edge in C. The *identity morphism* $\mathbf{1}_x$ of an object $x \in \mathsf{C}$ is the degenerate edge over x. A 2-simplex in C with boundary

$$\begin{tikzcd} & y \arrow[dr, "g"] & \\ x \arrow[ur, "f"] \arrow[rr, "h"] & & z \end{tikzcd} \tag{1.4.5.1}$$

should be thought of as a *homotopy* between the 'composition of f and g' (which is not itself a morphism in C since it is not an edge) and h. A given horn Λ^2_1 typically has many different fillings to Δ^2, so we cannot call h 'the' composition of f and g (merely 'a' composition). The higher horn filling conditions do imply, however, that extending a given map $\Lambda^2_1 \to \mathsf{C}$ to Δ^2 is a contractible choice (precisely, $\operatorname{Hom}(\Delta^2, \mathsf{C}) \to \operatorname{Hom}(\Lambda^2_1, \mathsf{C})$ is a trivial Kan fibration (1.4.21)). They also encode the data to guarantee that composition is, in a certain sense, associative up to coherent homotopy.

1.4.6 Definition (Opposite ∞-category)**.** Given an ∞-category C, its opposite C^{op} is the opposite simplicial set (i.e. its pre-composition with $\mathsf{op} : \boldsymbol{\Delta} \to \boldsymbol{\Delta}$).

1.4.7 Definition (Full subcategory)**.** A *full subcategory* of an ∞-category C is a subcomplex $\mathsf{A} \subseteq \mathsf{C}$ with the property that a simplex $\Delta^n \to \mathsf{C}$ belongs to A iff all of its vertices belong to A. Full subcategories of C are evidently in bijection with subsets of C_0.

Here are some common constructions of ∞-categories.

1.4.8 Definition (Kan simplicial category)**.** A *simplicial category* is a category C enriched (1.1.138) in simplicial sets. A simplicial category is called *Kan* when its morphism simplicial sets are Kan complexes (1.3.12).

⋆ **1.4.9 Definition** (Nerve of a Kan simplicial category; Cordier [16][74, 1.1.5])**.** The *(simplicial) nerve* of a Kan simplicial category C is the simplicial set in which an n-simplex is a tuple of objects $X_0, \ldots, X_n \in \mathsf{C}$ along with maps $f_{ij} : (\Delta^1)^{\{i+1,\ldots,j-1\}} \to \mathsf{C}(X_i, X_j)$ satisfying $f_{ik}|_{\{t_j=1\}} = f_{ij} \times f_{jk}$, which we may express as commutativity of the following diagram:

$$\begin{tikzcd} (\Delta^1)^{\{i+1,\ldots,j-1\}} \times (\Delta^1)^{\{j+1,\ldots,k-1\}} \arrow[r, "f_{ij} \times f_{jk}"] \arrow[d, "\times\{1\}^{\{j\}}"] & \mathsf{C}(X_i, X_j) \times \mathsf{C}(X_j, X_k) \arrow[d] \\ (\Delta^1)^{\{i+1,\ldots,k-1\}} \arrow[r, "f_{ik}"] & \mathsf{C}(X_i, X_k) \end{tikzcd} \tag{1.4.9.1}$$

The pullback of such data along a map $s : \Delta^m \to \Delta^n$ is given by $Y_i = X_{s(i)}$ and $g_{ij} = f_{s(i)s(j)}$ pre-composed with the map $(\Delta^1)^{\{i+1,\ldots,j-1\}} \to (\Delta^1)^{\{s(i)+1,\ldots,s(j)-1\}}$ given on vertices by the formula $t_k = \max_{s(a)=k} t_a$ (interpreted to be 0 when $s^{-1}(k)$ is empty).

1.4.10 Exercise. Describe explicitly the 0-simplices (objects), 1-simplices (morphisms), and 2-simplices of the nerve of a Kan simplicial category C. Consider the subcomplex of the nerve consisting of those simplices in which every f_{ij} is constant; how is this related to C?

1.4.11 Lemma. *The simplicial nerve of a Kan simplicial category is an ∞-category.*

Proof. The extension problem for maps from an inner horn (Δ^n, Λ^n_i) to the simplicial nerve of C amounts to the extension problem for

$$f_{0n} : (\Delta^1, \partial\Delta^1)^{\{1,\ldots,i-1\}} \wedge (\Delta^1, \{1\})^i \wedge (\Delta^1, \partial\Delta^1)^{\{i+1,\ldots,n-1\}} \to \mathsf{C}(X_0, X_n). \qquad (1.4.11.1)$$

This extension problem is solvable since $\mathsf{C}(X_0, X_n)$ is Kan and the domain pair is filtered by pushouts of horns (1.3.18). □

⋆ **1.4.12 Example** (∞-category of spaces Spc)**.** The category of Kan complexes $\mathsf{sSet}^{\mathsf{Kan}}$ has a natural enrichment $\underline{\mathsf{sSet}}^{\mathsf{Kan}}$ over the category of Kan complexes (1.2.8)(1.3.19). Its simplicial nerve is called the ∞-*category of spaces*, denoted Spc.

1.4.13 Definition (Differential graded category)**.** A $\mathbb{Z}$-linear *differential graded category* (or *dg-category*) C is a category enriched over complexes of $\mathbb{Z}$-modules. In other words, it consists of a set C of objects, a *morphism complex* $\operatorname{Hom}(X, Y) \in \mathsf{Kom}(\mathsf{Mod}_{\mathbb{Z}})$ for each $X, Y \in \mathsf{C}$, and composition maps $\operatorname{Hom}(X, Y) \otimes \operatorname{Hom}(Y, Z) \to \operatorname{Hom}(X, Z)$ which are associative and unital.

1.4.14 Definition (Nerve of a differential graded category [41, A.2.1][75, 1.3.1])**.** The *(differential graded) nerve* of a ($\mathbb{Z}$-linear) dg-category C is the simplicial set in which an n-simplex is a tuple of objects $X_0, \ldots, X_n \in \mathsf{C}$ along with maps $f_{ij} : C^{\mathrm{cell}}_\bullet((\Delta^1)^{\{i+1,\ldots,j-1\}}) \to \mathsf{C}(X_i, X_j)$ satisfying $f_{ik}|_{\{t_j=1\}} = f_{ij} \times f_{jk}$, as in (1.4.9).

1.4.15 Exercise. Show that the nerve of a differential graded category is an ∞-category (compare (1.4.11)).

⋆ **1.4.16 Definition** (Functor)**.** A functor of ∞-categories $\mathsf{C} \to \mathsf{D}$ is a map of simplicial sets. Functors from C to D are the objects of an ∞-category $\mathsf{Fun}(\mathsf{C}, \mathsf{D}) = \underline{\operatorname{Hom}}(\mathsf{C}, \mathsf{D})$ (the simplicial mapping space (1.2.8)).

⋆ **1.4.17 Definition** (Diagram)**.** Let K be a simplicial set. A K-shaped diagram in an ∞-category C is a map of simplicial sets $K \to \mathsf{C}$. Such diagrams form an ∞-category $\mathsf{Fun}(K, \mathsf{C}) = \underline{\operatorname{Hom}}(K, \mathsf{C})$ (1.4.19).

1.4.18 Exercise. For any simplicial set K and any category C, show that $\mathsf{Fun}(K, \mathsf{C})$ is (the nerve of) the category of K-shaped diagrams in C from (1.1.55). In particular, conclude that for categories C and D, the category of functors $\mathsf{Fun}(\mathsf{C}, \mathsf{D})$ defined here (1.4.16) coincides with that defined earlier (1.1.22).

1.4.19 Proposition. $\mathsf{Fun}(K, \mathsf{C})$ *is an ∞-category for any ∞-category* C.

Proof. This is very similar to (1.3.19). We are to show that C satisfies the extension property for pairs $(\Delta^n, \Lambda^n_i) \wedge K$ with $0 < i < n$. By filtering K by pushouts of pairs $(\Delta^k, \partial\Delta^k)$ (1.3.8), it suffies to show that C satisfies the extension property for pairs $(\Delta^n, \Lambda^n_i) \wedge (\Delta^k, \partial\Delta^k)$ with $0 < i < n$ and $k \geq 0$. It thus suffices to show that for $0 < i < n$, the smash product $(\Delta^n, \Lambda^n_i) \wedge (\Delta^k, \partial\Delta^k)$ is filtered by pushouts of inner horns. We verify this property next (1.4.20) (stated separately for later use). □

⋆ **1.4.20 Lemma.** *The smash product* $(\Delta^n, \Lambda^n_i) \wedge (\Delta^k, \partial\Delta^k)$ *with* $0 < i < n$ *is filtered by pushouts of inner horns.*

Proof. We saw earlier that $(\Delta^n, \Lambda^n_i) \wedge (\Delta^k, \partial\Delta^k)$ is filtered by pushouts of horns (Δ^m, Λ^m_j) (1.3.18). Let us argue that all the horns (Δ^m, Λ^m_j) appearing in this filtration are inner. The cone point $j \in \Delta^m$ of every such horn is the vertex v in (1.3.18.1); in particular, it projects to the cone point $i \in \Delta^n$. The image of the map $\Delta^m \to \Delta^n$ thus both contains i and cannot be contained in Λ^n_i, which together imply that $\Delta^m \twoheadrightarrow \Delta^n$ is in fact surjective. Thus $0 < i < n$ implies $0 < j < m$. □

1.4.21 Exercise. Show that for any ∞-category C, the map $\mathsf{Fun}(\Delta^n, \mathsf{C}) \to \mathsf{Fun}(\Lambda^n_i, \mathsf{C})$ is a trivial Kan fibration for any inner horn (Δ^n, Λ^n_i).

⋆ **1.4.22 Definition** (Inner fibration)**.** A map of simplicial sets is called an *inner fibration* when it satisfies the right lifting property with respect to inner horns.

1.4.23 Exercise. Show that for any inner fibration $Q \to X$, the simplicial set of sections $\mathsf{Sec}(X, Q)$ (a map $Z \to \mathsf{Sec}(X, Q)$ being a map $Z \times X \to Q$ over X) is an ∞-category.

⋆ **1.4.24 Definition** (Kanification diagram $\underline{\mathsf{sSet}} \to \mathsf{Spc}$)**.** We now lift the functor $\mathsf{sSet} \to \mathsf{hSpc}$ (1.3.26) to a diagram

$$\wedge : \underline{\mathsf{sSet}} \to \underline{\mathsf{sSet}}^{\mathsf{Kan}} = \mathsf{Spc} \tag{1.4.24.1}$$

equipped with a natural transformation $(1 \to \wedge) : \underline{\mathsf{sSet}} \times \Delta^1 \to \underline{\mathsf{sSet}}$ which is the identity over $\underline{\mathsf{sSet}}^{\mathsf{Kan}}$. Concretely, we send a simplicial set X to an inclusion $X \hookrightarrow \hat{X}$ filtered by pushouts of horns where $\hat{X}$ is a Kan complex. Here $\underline{\mathsf{sSet}}$ denotes the nerve (1.4.9) of the simplicial category of simplicial sets and internal mapping spaces $\underline{\mathrm{Hom}}$ (1.2.8) (note that this simplicial category is not Kan, so its nerve is just a simplicial set, not an ∞-category).

The functor $\wedge$ and natural transformation $1 \to \wedge$ may be defined as follows. Fix a small model for $\underline{\mathsf{sSet}}$. To define $1 \to \wedge$ on objects of $\underline{\mathsf{sSet}}$, we choose for every $X \in \underline{\mathsf{sSet}}$ an inclusion $X \hookrightarrow \hat{X}$ filtered by pushouts of horns into a Kan complex $\hat{X}$. Proceeding by induction on a filtration of $\underline{\mathsf{sSet}}$ by simplices, we may extend $1 \to \wedge$ to a new simplex $(\Delta^n, \partial\Delta^n)$ by solving an extension problem of the form $(\Delta^1, \partial\Delta^1)^{\{1,\ldots,n-1\}} \wedge (\hat{X}_0, X_0) \to \hat{X}_n$, which is possible since $\hat{X}_n$ is Kan and the domain is filtered by pushouts of horns (1.3.18). This definition of $\wedge$ and $1 \to \wedge$ evidently depends on many choices. We will see in (1.4.140) that $\wedge$ and $1 \to \wedge$ are in fact unique up to contractible choice, using the fact that Kanification of a simplicial set K corepresents the functor $\underline{\mathrm{Hom}}(K, -)$ (if $K \hookrightarrow \hat{K}$ is filtered by pushouts of horns, then $\underline{\mathrm{Hom}}(\hat{K}, X) \to \mathrm{Hom}(K, X)$ is a trivial Kan fibration for X a Kan complex (1.3.18)).

The homotopy category

1.4.25 Definition (Homotopy category of an ∞-category)**.** Let C be an ∞-category. For objects $x, y \in \mathsf{C}$, the relation of *right homotopy* on the set of morphisms $x \to y$ is defined by

$e \sim e'$ iff there exists a 2-simplex

$$\begin{tikzcd} & y \arrow[dr, "\mathbf{1}_y"] & \\ x \arrow[ur, "e"] \arrow[rr, "e'"] & & y \end{tikzcd} \tag{1.4.25.1}$$

Right homotopy is an equivalence relation: reflexivity holds by taking a degenerate 2-simplex over e, and symmetry and transitivity follow from the following two inner horn fillings (the boxed vertex is the cone point of the horn):

$$\begin{tikzcd} & \boxed{y} \arrow[r, "\mathbf{1}_y"] \arrow[drr, "\mathbf{1}_y"] & y \arrow[dr, "\mathbf{1}_y"] & \\ x \arrow[ur, "e"] \arrow[urr, "e'"] \arrow[rrr, "e"'] & & & y \end{tikzcd} \qquad \begin{tikzcd} & y \arrow[r, "\mathbf{1}_y"] \arrow[drr, "\mathbf{1}_y"] & \boxed{y} \arrow[dr, "\mathbf{1}_y"] & \\ x \arrow[ur, "e"] \arrow[urr, "e'"] \arrow[rrr, "e''"'] & & & y \end{tikzcd} \tag{1.4.25.2}$$

There is a corresponding equivalence relation *left homotopy*. Right homotopy implies left homotopy by filling the following inner horn (so by symmetry the converse is true as well)

$$\begin{tikzcd} & x \arrow[r, "e"] \arrow[drr, "e"] & \boxed{y} \arrow[dr, "\mathbf{1}_y"] & \\ x \arrow[ur, "\mathbf{1}_x"] \arrow[urr, "e"] \arrow[rrr, "e'"'] & & & y \end{tikzcd} \tag{1.4.25.3}$$

Since right homotopy and left homotopy are the same, we may simply call this relation *homotopy* of morphisms $x \to y$. Filling the following inner horn

$$\begin{tikzcd} & \boxed{y} \arrow[r, "b"] \arrow[drr, "b'"] & z \arrow[dr, "\mathbf{1}_z"] & \\ x \arrow[ur, "a"] \arrow[urr, "c"] \arrow[rrr, "c'"'] & & & z \end{tikzcd} \tag{1.4.25.4}$$

shows that if b and b' are homotopic, then any two fillings of $x \xrightarrow{a} y \xrightarrow{b} z$ and $x \xrightarrow{a} y \xrightarrow{b'} z$ give homotopic morphisms $x \to z$. By symmetry, we conclude that composition is well-defined on homotopy classes. The *homotopy category* hC has the same objects as C (i.e. the vertices of C) and has morphisms the homotopy classes of morphisms in C. There is a tautological functor $\mathsf{C} \to \mathsf{hC}$.

1.4.26 Exercise. Show that the homotopy category of a category is itself.

1.4.27 Exercise. Show that a functor $\mathsf{C} \to \mathsf{D}$ induces a functor $\mathsf{hC} \to \mathsf{hD}$. Show that the natural map $\mathsf{h}(\mathsf{C} \times \mathsf{D}) \to \mathsf{hC} \times \mathsf{hD}$ is an isomorphism. Conclude that a natural transformation $F \to G$ of functors $\mathsf{C} \to \mathsf{D}$ induces a natural transformation $\mathsf{h}F \to \mathsf{h}G$ of functors $\mathsf{hC} \to \mathsf{hD}$.

1.4.28 Exercise. Show that the homotopy category of ∞-category Spc (1.4.12) is the category denoted hSpc in (1.3.20). More generally, describe the homotopy category of (the nerve (1.4.9) of) a Kan simplicial category.

1.4.29 Lemma. *The functor* $\mathsf{C} \to \mathsf{hC}$ *satisfies the right lifting property with respect to the pair* $(\Delta^2, \partial\Delta^2)$.

Proof. Fill the inner horn

$$\begin{tikzcd}
 & y \arrow[r, "b"] \arrow[drr, "b"'] & \boxed{z} \arrow[dr, "\mathbf{1}_z"] & \\
x \arrow[ur, "a"] \arrow[urr, "c'"'] \arrow[rrr, "c"'] & & & z
\end{tikzcd} \tag{1.4.29.1}$$

to see that a 2-simplex with edges a, b, c exists in C iff the boundary commutes in hC. ☐

⋆ **1.4.30 Definition** (Isomorphism in an ∞-category)**.** A morphism in an ∞-category C is called an isomorphism (resp. split monomorphism, split epimorphism) iff its image in the homotopy category hC is.

1.4.31 Exercise. Show that a functor of ∞-categories sends isomorphisms to isomorphisms.

1.4.32 Exercise. As a continuation of (1.4.27), show that a natural isomorphism $F \to G$ of functors $\mathsf{C} \to \mathsf{D}$ induces a natural isomorphism $\mathsf{h}F \to \mathsf{h}G$ of functors $\mathsf{hC} \to \mathsf{hD}$.

1.4.33 Definition (Property of morphisms in an ∞-category)**.** A property of morphisms in an ∞-category C is a property of morphisms in its homotopy category hC.

Joins and slices

1.4.34 Definition (Join)**.** For simplicial sets X and Y, their join $X \star Y$ is defined by the universal property that map $Z \to X \star Y$ is a map $p : Z \to \Delta^1$ and a pair of maps $p^{-1}(0) \to X$ and $p^{-1}(1) \to Y$.

1.4.35 Exercise. Show that $(X \star Y)^{\mathsf{op}} = Y^{\mathsf{op}} \star X^{\mathsf{op}}$.

1.4.36 Exercise. Show that $\Delta^n \star \Delta^m = \Delta^{n+m+1}$ for $n, m \geq -1$ (where $\Delta^{-1} = \varnothing$).

1.4.37 Exercise. Show that the set of non-degenerate or empty simplices of $X \star Y$ is the product of the sets of non-degenerate or empty simplices of X and Y.

⋆ **1.4.38 Definition** (Right and left cone)**.** The right and left cones of a simplicial set K are the joins $K^{\triangleright} = K \star \Delta^0$ and $K^{\triangleleft} = \Delta^0 \star K$, respectively.

1.4.39 Exercise. Prove that the geometric realization of $K^{\triangleright}$ is contractible for every simplicial set K.

⋆ **1.4.40 Definition** (Slice ∞-category)**.** Given a diagram $K \to \mathsf{C}$, the over-category $\mathsf{C}_{/K}$ is defined by the universal property that a map $Z \to \mathsf{C}_{/K}$ is a map $Z \star K \to \mathsf{C}$ extending the given map $K \to \mathsf{C}$. Dually, the under-category $\mathsf{C}_{K/}$ represents extensions to $K \star Z \to \mathsf{C}$.

1.4.41 Definition (Join of pairs)**.** The join of simplicial set pairs is

$$X \star (Y, B) = (X \star Y, X \star B), \tag{1.4.41.1}$$

$$(X, A) \star (Y, B) = (X \star Y, (X \star B) \cup_{A \star B} (A \star Y)). \tag{1.4.41.2}$$

Beware that join of pairs is not compatible with identifying X and $(X, \varnothing)$.

1.4.42 Exercise. Show that $(\Delta^n, \Lambda^n_i) \star (\Delta^m, \partial\Delta^m) = (\Delta^{n+m+1}, \Lambda^{n+m+1}_i)$.

⋆ **1.4.43 Definition** (Left and right fibrations)**.** A map of simplicial sets is called a *left* (resp. *right*) *fibration* when it satisfies the right lifting property with respect to left (resp. right) horns (Δ^n, Λ^n_i), namely $0 \le i < n$ (resp. $0 < i \le n$) (1.4.1).

A left fibration over a simplicial set X is 'equivalent' in a certain sense to a diagram $X \to \mathsf{Spc}$ (1.4.59)(**??**)(1.4.179). The proof of this will come quite a bit later, so for the moment we will regard it just as intuition.

1.4.44 Exercise. Show that a left fibration of ∞-categories reflects split monomorphisms, hence reflects isomorphisms (1.1.50) (note the use of (1.4.29)).

1.4.45 Exercise. Show that for any diagram $L \to \mathsf{C}$ and any monomorphism $K \to L$, the right lifting property for $\mathsf{C}_{/L} \to \mathsf{C}_{/K}$ with respect to a pair (X, A) is equivalent to the extension property for maps $(X, A) \star (L, K) \to \mathsf{C}$. Conclude from (**??**) and (1.4.42) that the restriction map $\mathsf{C}_{/L} \to \mathsf{C}_{/K}$ is a right fibration. Conclude moreover that if $K \to L$ is filtered by pushouts of left horns, then $\mathsf{C}_{/L} \to \mathsf{C}_{/K}$ is a trivial Kan fibration.

1.4.46 Exercise. Let $\mathsf{C} \to \mathsf{D}$ be a functor of ∞-categories. Let $K \to \mathsf{C}$ be a diagram, and consider the induced functors $\mathsf{C}_{/K} \to \mathsf{D}_{/K}$ and $\mathsf{C}_{K/} \to \mathsf{D}_{K/}$ (where the target is the slice category for the composition diagram $K \to \mathsf{C} \to \mathsf{D}$). Show that if $\mathsf{C} \to \mathsf{D}$ is an inner (resp. left or right) fibration, then so are both $\mathsf{C}_{/K} \to \mathsf{D}_{/K}$ and $\mathsf{C}_{K/} \to \mathsf{D}_{K/}$.

Isomorphisms

It turns out that isomorphisms in an ∞-category can be characterized by a simple extension property (1.4.47), similar to the definition of an ∞-category itself. In practice, this characterization of isomorphisms is much more useful than the characterization in terms of the homotopy category (1.4.30). Despite this variety of characterizations, inverting an isomorphism in an ∞-category remains a somewhat awkward operation (1.4.57).

⋆ **1.4.47 Proposition** (Isomorphism as an extension property; Joyal [47])**.** *A morphism e in an ∞-category is an isomorphism iff every left outer horn $\Lambda^n_0 \subseteq \Delta^n$ with* 01 *edge e can be filled.*

Proof. If $f : x \to y$ satisfies the hypothesized horn filling condition for $n = 2, 3$, then filling the following two horns produces an inverse g to f in hC.

$$\begin{tikzcd} & y \arrow[dr, dashed, "\exists g"] & \\ \boxed{x} \arrow[ur, "f"] \arrow[rr, "\mathbf{1}_x"'] & & x \end{tikzcd} \qquad \begin{tikzcd} & y \arrow[r, "g"] \arrow[drr, "\mathbf{1}_y"] & x \arrow[dr, "f"] & \\ \boxed{x} \arrow[ur, "f"] \arrow[urr, "\mathbf{1}_x"] \arrow[rrr, "f"'] & & & y \end{tikzcd} \tag{1.4.47.1}$$

Conversely, let us show that every map $\Lambda^n_0 \to \mathsf{C}$ with 01 edge an isomorphism extends to Δ^n. The extension problem for $(\Delta^n, \Lambda^n_0) \to \mathsf{C}$ is equivalent to the lifting property

$$\begin{tikzcd} \Lambda^1_0 \arrow[r] \arrow[d] & \mathsf{C}_{/\Delta^{n-2}} \arrow[d] \\ \Delta^1 \arrow[r] \arrow[ur, dashed] & \mathsf{C}_{/\partial\Delta^{n-2}} \end{tikzcd} \tag{1.4.47.2}$$

in view of the identity $(\Delta^n, \Lambda^n_0) = (\Delta^1, \Lambda^1_0) \star (\Delta^{n-2}, \partial\Delta^{n-2})$ (1.4.42). Since $\mathsf{C}_{/\partial\Delta^{n-2}} \to \mathsf{C}$ is a right fibration (1.4.45), it reflects isomorphisms (1.4.44), so the bottom edge in (1.4.47.2) is an isomorphism. Now the map $\mathsf{C}_{/\Delta^{n-2}} \to \mathsf{C}_{/\partial\Delta^{n-2}}$ is a right fibration (1.4.45), so it suffices to show that for any right fibration of ∞-categories $\mathsf{A} \to \mathsf{B}$, the lifting problem

$$\begin{tikzcd} * \arrow[r] \arrow[d, "0"'] & \mathsf{A} \arrow[d] \\ \Delta^1 \arrow[r] \arrow[ur, dashed] & \mathsf{B} \end{tikzcd} \tag{1.4.47.3}$$

has a solution provided the bottom arrow is an isomorphism in B (in fact, it need only be a split monomorphism). Since the edge $e : \Delta^1 \to \mathsf{B}$ is a split monomorphism in B, there exists by (1.4.29) a map $\Delta^2 \to \mathsf{B}$ in which the 02 edge is degenerate and the 01 edge is e. The degenerate edge certainly lifts to A, so it suffices to solve the lifting problem for the pair $(\Delta^2, 02)$, which is filtered by pushouts of right horns. □

1.4.48 Definition (∞-groupoid)**.** An *∞-groupoid* is an ∞-category in which every morphism is an isomorphism (by (1.4.47), this is equivalent to being a Kan complex).

1.4.49 Definition (Core of an ∞-category)**.** For an ∞-category C, its core is the subcomplex $\mathsf{C}_{\simeq} \subseteq \mathsf{C}$ defined as those simplices all of whose edges are isomorphisms. A functor $\mathsf{C} \to \mathsf{D}$ evidently restricts to a functor $\mathsf{C}_{\simeq} \to \mathsf{D}_{\simeq}$.

The characterization of isomorphisms in an ∞-category by an extension property (1.4.47) leads naturally to the notion of a 'marked simplicial set'.

1.4.50 Definition (Marked simplicial set)**.** A *marked simplicial set* is a pair (X, S) consisting of a simplicial set X and a set $S \subseteq X_1$ of its edges (called the 'marked edges') containing all degenerate edges. A morphism of marked simplicial sets $(X, S) \to (X', S')$ is a morphism of simplicial sets $X \to X'$ which sends every marked edge of X to a marked edge of X'. The category of marked simplicial sets is denoted sSet^+.

By default, a simplicial set X will be regarded as being equipped with the *trivial marking*, consisting of only the degenerate edges, unless specified otherwise (this defines a fully faithful functor $\mathsf{sSet} \hookrightarrow \mathsf{sSet}^+$); for emphasis, the trivial marking is also denoted $X^\flat$. We denote by $X^\sharp$ the simplicial set X with all its edges marked. Note that a limit of marked simplicial sets (X_α, S_α) is the limit underlying simplicial sets X_α equipped with the marking consisting of those edges whose image in every (X_α, S_α) is marked.

1.4.51 Definition (Marked horn)**.** The *marked horn* $(\Delta^n, \Lambda^n_i)^\sim$ is the usual horn (Δ^n, Λ^n_i) with a marking of the edge 01 if $i = 0$ and of the edge $(n-1, n)$ if $i = n$. A map of marked simplicial sets is called a *marked fibration* when it satisfies the right lifting property with respect to marked horns (and a marked left or right fibration indicates lifting left or right marked horns).

1.4.52 Example (Marking isomorphisms in an ∞-category)**.** Let C be an ∞-category. We denote by $\mathsf{C}^\natural$ the result of marking all the isomorphisms in C. Thus (1.4.47) says that $\mathsf{C}^\natural$ satisfies the extension property with respect to all marked horns. Conversely, if a marked simplicial set (X, S) satisfies the extension property with respect to all marked horns, then X is an ∞-category and every marked edge is an isomorphism (though S need not contain all isomorphisms).

1.4.53 Proposition (Isomorphisms in diagram categories)**.** *The functor* $\mathsf{Fun}(K, \mathsf{C}) \to \mathsf{Fun}(K_0, \mathsf{C}) = \prod_{k \in K} \mathsf{C}$ *reflects isomorphisms.*

Proof. We seek to show the extension property for maps $(\Delta^n, \Lambda^n_0) \to \mathsf{Fun}(K, \mathsf{C})$ in which the image of the edge 01 in $\mathsf{Fun}(K_0, \mathsf{C}) = \prod_{k \in K} \mathsf{C}$ is an isomorphism. Equivalently, this is the extension property for maps of marked simplicial sets $(\Delta^n, \Lambda^n_0)^\sim \wedge K \to \mathsf{C}^\natural$. It thus suffices to show that the smash product $(\Delta^n, \Lambda^n_0)^\sim \wedge (\Delta^k, \partial\Delta^k)$ is filtered by pushouts of marked horns. We verify this property next (1.4.54) (stated separately for later use). □

⋆ **1.4.54 Lemma.** *The smash product* $(\Delta^n, \Lambda^n_0)^\sim \wedge (\Delta^k, \partial\Delta^k)$ *is filtered by pushouts of marked left horns (whose marked edges lie over* $0 \in \Delta^k$*).*

Proof. This argument is similar to (1.4.20).

We saw earlier that $(\Delta^n, \Lambda^n_0) \wedge (\Delta^k, \partial\Delta^k)$ is filtered by pushouts of horns (Δ^m, Λ^m_j) (1.3.18). Let us argue that all the horns (Δ^m, Λ^m_j) appearing in this filtration are marked left horns. The cone point $j \in \Delta^m$ of every such horn is the vertex v in (1.3.18.1); in particular, it projects to the cone point $0 \in \Delta^n$. The image of the map $\Delta^m \to \Delta^n$ thus both contains 0 and cannot be contained in Λ^n_0, which together imply that $\Delta^m \twoheadrightarrow \Delta^n$ is in fact surjective. This implies $0 \leq j < m$.

Let us now further show that in the case $j = 0$, the edge $01 \subseteq \Delta^m$ is marked in the product $(\Delta^n, \Lambda^n_0)^\sim \wedge (\Delta^k, \partial\Delta^k)$. Property (1.3.18.3) says that $\Delta^m \to \Delta^k$ must be surjective, so if the image of j in Δ^k (i.e. the vertical coordinate of v) is > 0, then the horn (Δ^m, Λ^m_j) is inner. Thus $j = 0$ occurs precisely when $v = (0, 0)$. By definition of v, this means the lattice path in question (corresponding to σ) contains $(1,0) \in \Delta^n \times \Delta^k$. Now properties (1.3.18.2) and (1.3.18.4) together imply that $\Delta^m \subseteq \Delta^n \times \Delta^k$ must contain this point $(1,0)$. We conclude that the edge $01 \subseteq \Delta^m$ is the product of the edge $01 \subseteq \Delta^n$ and the degenerate edge over $0 \in \Delta^k$, and hence is marked in the product $(\Delta^n, \Lambda^n_0)^\sim \wedge (\Delta^k, \partial\Delta^k)$. $\square$

1.4.55 Exercise. Show that for any inner fibration $Q \to X$, the functor $\underline{\mathrm{Sec}}(X,Q) \to \prod_{x\in X} Q_x$ reflects isomorphisms.

1.4.56 Exercise (Alternative model for slice categories)**.** Let C be an ∞-category, and let $c \in \mathsf{C}$ be an object. Recall that the slice category $\mathsf{C}_{/c}$ is defined by the property that map $Z \to \mathsf{C}_{/c}$ from a simplicial set Z is the same as a map $Z^\triangleright \to \mathsf{C}$ sending the cone point to c. Define an 'alternative model' slice category $\mathsf{C}^{/c}$ by the property that a map $Z \to \mathsf{C}^{/c}$ is a map $Z \times \Delta^1 \to \mathsf{C}$ sending $Z \times 1$ to c.

Let us show that $\mathsf{C}_{/c}$ and $\mathsf{C}^{/c}$ are equivalent over C. We construct a simplicial set Q with trivial Kan fibrations $\mathsf{C}_{/c} \leftarrow Q \to \mathsf{C}^{/c}$ over C. Define Q by the property that a map $Z \to Q$ is a map $(Z \times \Delta^1)^\triangleright \to \mathsf{C}$ sending $(Z \times 1)^\triangleright$ to c. Now Q maps to $\mathsf{C}_{/c}$ and $\mathsf{C}^{/c}$ by restricting to $(Z \times 0)^\triangleright$ and $Z \times \Delta^1$, respectively.

The map $Q \to \mathsf{C}^{/c}$ being a trivial Kan fibration amounts to the extension property for maps $((\Delta^k, \partial\Delta^k) \wedge (\Delta^1, 1)) \star (*, \varnothing) \to \mathsf{C}$. This extension property holds since this pair is filtered by pushouts of inner horns (1.4.54)(1.4.42).

The map $Q \to \mathsf{C}_{/c}$ being a trivial Kan fibration amounts to the extension property for maps $((\Delta^k, \partial\Delta^k) \wedge (\Delta^1, \partial\Delta^1))^\triangleright \to \mathsf{C}$ sending $(\Delta^k \times 1)^\triangleright$ to c. This extension property holds since this pair is filtered by pushouts of right horns whose marked edge maps to c (1.3.8) (1.4.42).

1.4.57 Example (Inverting an isomorphism)**.** Given an isomorphism e in an ∞-category C, in what sense is its inverse e^{-1} defined and unique, and in what sense is $(e^{-1})^{-1} = e$? Here is one possible answer to this question.

Let Iso denote the category with two objects a and b and a single morphism between any pair of objects (thus a and b are isomorphic). Given an ∞-category C, a functor

$$\mathsf{Iso} \to \mathsf{C} \tag{1.4.57.1}$$

a describes a pair of (homotopy coherently) inverse morphisms in C. Note that this picture is symmetric via the obvious involution of the category Iso exchanging the objects a and b. Now to express mathematically the claim that an isomorphism in C has a homotopically unique inverse, let us argue that the restriction map

$$\mathsf{Fun}(\mathsf{Iso}, \mathsf{C}) \to \mathsf{Fun}(\Delta^1, \mathsf{C}) \tag{1.4.57.2}$$

is a trivial Kan fibration over the full subcategory of $\mathsf{Fun}(\Delta^1, \mathsf{C})$ spanned by the isomorphisms in C. Note that a map from $Z \in \mathsf{sSet}$ to this full subcategory is a map of marked simplicial sets $Z \times (\Delta^1)^\sharp \to \mathsf{C}^\natural$. The desired trivial Kan fibration property thus amounts to the extension property for maps $(\mathsf{Iso}^\natural, (\Delta^1)^\sharp) \wedge (\Delta^k, \partial\Delta^k) \to \mathsf{C}^\natural$. It thus suffices by (1.4.54) to filter $(\mathsf{Iso}^\natural, (\Delta^1)^\sharp)$ by pushouts of marked horns. The nerve of Iso has precisely two non-degenerate simplices of every dimension. Let $\mathsf{Iso}_k \subseteq \mathsf{Iso}$ denote the $(k-1)$-skeleton union either one of the non-degenerate k-simplices (doesn't matter which). Now the pullback of $\mathsf{Iso}_k \subseteq \mathsf{Iso}$ under the inclusion of a non-degenerate $(k+1)$-simplex into Iso is an outer horn (inspection). The pair $(\mathsf{Iso}_{k+1}, \mathsf{Iso}_k)$ is thus a pushout of an outer horn, so $(\mathsf{Iso}, \Delta^1) = (\mathsf{Iso}, \mathsf{Iso}_1)$ is filtered by pushouts of outer horns (which are moreover marked since all morphisms in Iso are isomorphisms).

Left fibrations

1.4.58 Exercise. Show that for any left fibration $E \to X$, the simplicial set $\underline{\mathrm{Sec}}(X, E)$ is a Kan complex.

1.4.59 Exercise (Transport maps of a left fibration; compare (1.3.21))**.** Let $X \to Y$ be a left fibration. Associate to any edge $y \to y'$ in Y a 'transport map' $X_y \to X_{y'}$ by lifting the pair $X_y \times (\Delta^1, 0)$, and show that this map is well defined up to homotopy. Show that for any 2-simplex in Y with vertices y, y', y'', the resulting triangle commutes up to homotopy. Conclude that this defines a diagram $Y \to \mathsf{hSpc}$.

1.4.60 Lemma (Obstruction theory for sections of a left fibration)**.** *Let $E \to \Delta^n$ be a left fibration. The inclusion of the fiber $E_n \subseteq E$ is a Kan equivalence, and there exists a retraction $q : E \to E_n$. For any section $s : \partial\Delta^n \to E$, composition with any Kan equivalence $p : E \to \hat{E}$ to a Kan complex $\hat{E}$ (such as a retraction $q : E \to E_n$) induces a bijection between homotopy classes of extensions of s and homotopy classes of extensions of $p \circ s$.*

Proof. Consider the homotopy $H : \Delta^n \times \Delta^1 \to \Delta^n$ from the identity on $\Delta^n \times 0$ to the constant map $\Delta^n \times 1 \mapsto n$. Construct a map $Q : E \times \Delta^1 \to H^*E$ over $\Delta^n \times \Delta^1$ which is the identity on $(E \times 0) \cup (E_n \times \Delta^1)$ by filtering $(E, E_n) \wedge (\Delta^1, 0)$ by left horns (1.4.54). In other words, Q is a homotopy of maps $E \to E$ over H from the identity map to a retraction $E \xrightarrow{q} E_n \subseteq E$, which implies $E_n \subseteq E$ is a Kan equivalence.

Now let us show that composition with this retraction q induces a bijection between homotopy classes of extensions of $s : \partial\Delta^n \to E$ and homotopy classes of extensions of $q \circ s$. We consider the following diagram of restriction maps between spaces of sections.

$$\begin{tikzcd}
\underline{\mathrm{Sec}}(\Delta^n, E) \arrow[d] \arrow[r, dashed, bend left, "Q"] & \underline{\mathrm{Sec}}(\Delta^n \times \Delta^1, H^*E) \arrow[l, "(-\times 0)^*"'] \arrow[r, "(-\times 1)^*"] \arrow[d] & \underline{\mathrm{Hom}}(\Delta^n, E_n) \arrow[d] \\
\underline{\mathrm{Sec}}(\partial\Delta^n, E) \arrow[r, dashed, bend left, "Q"] & \underline{\mathrm{Sec}}(\partial\Delta^n \times \Delta^1, H^*E) \arrow[l, "(-\times 0)^*"'] \arrow[r, "(-\times 1)^*"] & \underline{\mathrm{Hom}}(\partial\Delta^n, E_n)
\end{tikzcd} \tag{1.4.60.1}$$

The bottom left horizontal map is a trivial Kan fibration since $\partial\Delta^n \wedge (\Delta^1, 0)$ is filtered by pushouts of left horns (1.4.54). The map from the upper middle space to the fiber product of the left square is a trivial Kan fibration since $(\Delta^n, \partial\Delta^n) \wedge (\Delta^1, 0)$ is filtered by pushouts of left horns (1.4.54). The corresponding statements for the right square also hold since the relevant pairs are filtered by marked right horns whose marked edge is $n \times \Delta^1$ (1.4.54) over which H is constant (**??**). It follows that $(- \times 0)^*$ and $(- \times 1)^*$ induce bijections on homotopy classes of extensions (i.e. connected components of the fibers of the vertical maps). The same thus holds for the dotted section Q, hence also for the composition $(- \times 1)^* \circ Q = q$.

Finally, we recall (1.3.22) that if a Kan equivalence $p : E \to \hat{E}$ induces a bijection between homotopy classes of extensions of a map $s : \partial\Delta^n \to E$ and homotopy classes of extensions of $p \circ s$, then so does any other Kan equivalence $p' : E \to \hat{E}'$. □

1.4.61 Exercise. Let $X \to Y$ be a left fibration. Use obstruction theory for sections of left fibrations (1.4.60) to show that a lifting problem for a right horn (Δ^n, Λ^n_n) against $X \to Y$ has a solution if the map on fibers $X_{n-1} \to X_n$ associated (uniquely up to contractible choice) to the edge $(n-1) \to n$ in $\Delta^n \to Y$ is a homotopy equivalence. In particular, conclude that a left fibration $X \to Y$ is a Kan fibration iff the morphism $X_y \to X_{y'}$ associated to every edge $e : y \to y'$ in Y is a homotopy equivalence (1.4.59). In particular, conclude that a left fibration over a Kan complex is a Kan fibration.

1.4.62 Corollary. *Every left fibration over an inner horn $\Lambda^n_i \subseteq \Delta^n$ (of cardinality $\leq \kappa$) is the restriction of a left fibration over Δ^n (of cardinality $\leq \aleph_0\kappa$).*

Proof. Let $E \to \Lambda^n_i$ be a left fibration, and let us construct a left fibration $\overline{E} \to \Delta^n$ with an isomorphism $\overline{E} \times_{\Delta^n} \Lambda^n_i = E$ over Λ^n_i.

Consider the homotopy $F : \Lambda^n_i \times \Delta^1 \to \Lambda^n_i$ from the identity map to (the restriction to Λ^n_i of) the map $\Delta^n \to \Delta^n$ given on vertices by $\max(-, i)$. Consider the further homotopy $G : \Lambda^n_i \times \Delta^1 \to \Lambda^n_i$ from this latter map to the constant map to $n \in \Lambda^n_i \subseteq \Delta^n$. Gluing these together, we obtain a 'two step' homotopy $H = F \vee G : \Lambda^n_i \times (\Delta^1 \vee \Delta^1) \to \Lambda^n_i$. Define a map $Q : E \times (\Delta^1 \vee \Delta^1) \to H^*E$ which is the identity on $E \times 0$ and $E_n \times (\Delta^1 \vee \Delta^1)$ by filtering $(E, E_n) \wedge (\Delta^1 \vee \Delta^1, 0)$ by pushouts of left horns (1.4.54). Let

$$q : E \to E_n \tag{1.4.62.1}$$

denote the retraction obtained by restricting Q to $E \times 2$ (where $2 \in \Delta^1 \vee \Delta^1$ denotes the final vertex).

Now define $\overline{E} \to \Delta^n$ as follows. A map $Z \to \overline{E}$ shall be a map $Z \to \Delta^n$ together with a diagram

$$\begin{tikzcd} Z \times_{\Delta^n} \Lambda^n_i \arrow[r] \arrow[d] & E \arrow[d, "q"] \\ Z \arrow[r] & E_n \end{tikzcd} \tag{1.4.62.2}$$

where the top horizontal map is over Λ^n_i. It is evident that $\overline{E} \times_{\Delta^n} \Lambda^n_i = E$. It is also evident that the cardinality of $\overline{E}$ is at most $\kappa + \kappa^2 + \kappa^3 + \cdots$ (which equals $\aleph_0\kappa$ by (1.4.63)): an r-simplex of $\overline{E}$ is determined by a map $\Delta^r \to \Delta^n$ (finitely many possibilities), a map $\Delta^r \times_{\Delta^n} \Lambda^n_i \to E$ (a finite collection of simplices of E since the domain is finite), and a map $\Delta^r \to E_n$ (a simplex of E).

It remains to check that $\overline{E} \to \Delta^n$ is a left fibration. Lifting problems for left horns $\Lambda^r_j \to \Delta^r$ against $\overline{E} \to \Delta^n$ come in three types depending on the map $\pi : \Delta^r \to \Delta^n$:

(1.4.62.3) $\pi^{-1}(\Lambda^n_i) = \Delta^r$.
(1.4.62.4) $\pi^{-1}(\Lambda^n_i) = \Delta^{[r]-j}$.
(1.4.62.5) $\pi^{-1}(\Lambda^n_i) \subseteq \Lambda^r_j$.

In the case (1.4.62.3), the lifting problem is just a lifting problem for $E \to \Lambda^n_i$. In the case (1.4.62.5), the lifting problem is just an extension problem $(\Delta^r, \Lambda^r_j) \to E_n$. To treat the remaining case (1.4.62.4), we consider extending in two steps $\Lambda^r_j \subseteq \partial\Delta^r \subseteq \Delta^r$. The first step is a lifting problem of $(\Delta^{[r]-j}, \partial\Delta^{[r]-j})$ against $E \to \Lambda^n_i$, while the second step is an extension problem $(\Delta^r, \partial\Delta^r) \to E_n$. Homotopy classes of lifts for $(\Delta^{[r]-j}, \partial\Delta^{[r]-j})$ are in bijection with homotopy classes of extensions of $(\Delta^{[r]-j}, \partial\Delta^{[r]-j}) \to E \xrightarrow{q} E_n$ (1.4.60). We may thus choose a lift which corresponds to the homotopy class of the extension $(\Lambda^r_j, \partial\Delta^{[r]-j}) \to E \xrightarrow{q} E_n$, and this guarantees that the subsequent extension problem $(\Delta^r, \partial\Delta^r) \to E_n$ is solvable. □

1.4.63 Lemma (Zermelo [111])**.** *For any infinite cardinal κ, we have $\kappa \cdot \kappa = \kappa$.*

Proof. Let α be the smallest ordinal of cardinality κ. Regard α as a well ordered set, and equip $\alpha \times \alpha$ (the cartesian product) with the well ordering pulled back from the lexicographic order on $\alpha \times \alpha \times \alpha$ under the map $(x, y) \mapsto (\max(x, y), x, y)$. We claim that $\alpha \times \alpha$, with this order, is order isomorphic to α. The cardinality of $\alpha \times \alpha$ is at least κ, so it suffices to show that for every $(x, y) \in \alpha \times \alpha$, the set $(\alpha \times \alpha)_{\leq(x,y)}$ has cardinality $< \kappa$. This is trivial: we have $(\alpha \times \alpha)_{\leq(x,y)} \subseteq \alpha_{\leq \max(x,y)} \times \alpha_{\max(x,y)}$ by definition of the ordering on $\alpha \times \alpha$, and $|\alpha_{\leq \max(x,y)}| < \kappa$ since α is the smallest ordinal of cardinality κ, and so we may assume that $|\alpha_{\leq \max(x,y)} \times \alpha_{\leq \max(x,y)}| = \alpha_{\leq \max(x,y)}$ by transfinite induction on κ. □

⋆ **1.4.64 Definition** (Classifying ∞-category of left fibrations)**.** Denote by $(\mathsf{sSet}^{\mathsf{L}}_{/-})_{\simeq}$ the simplicial groupoid which represents the functor $\mathsf{sSet} \ni Z \mapsto (\mathsf{sSet}^{\mathsf{L}}_{/Z})_{\simeq} \in \mathsf{Grpd}$ (the groupoid of left fibrations over Z and isomorphisms thereof). This simplicial groupoid $(\mathsf{sSet}^{\mathsf{L}}_{/-})_{\simeq}$ is an ∞-category (1.4.62)(**??**) which we call the *classifying ∞-category of left fibrations*. It is the union of its essentially small full sub-∞-categories $(\mathsf{sSet}^{\mathsf{L},\kappa}_{/-})_{\simeq}$ classifying left fibrations whose pullbacks to all simplices of the base have cardinality $< \kappa$ (any uncountable cardinal κ).

1.4.65 Corollary (Obstruction theory for left fibrations)**.** *Let $E \to \partial\Delta^n$ be a left fibration. For every retraction $q : E \to E_n$, there exists a left fibration $\overline{E} \to \Delta^n$ with an isomorphism $\overline{E}|_{\partial\Delta^n} = E$ over $\partial\Delta^n$ and a retraction $\overline{q} : \overline{E} \to \overline{E}_n = E_n$ extending q.*

Proof. This is similar to the construction of extensions of left fibrations across inner horns (1.4.62). Define $\overline{E}$ by the universal property that a map $Z \to \overline{E}$ from a simplicial set Z is a map $p : Z \to \Delta^n$ along with a diagram

$$\begin{tikzcd} p^{-1}(\partial\Delta^n) \arrow[r] \arrow[d] & E \arrow[d, "q"] \\ Z \arrow[r] & E_n \end{tikzcd} \tag{1.4.65.1}$$

where the top horizontal map is over $\partial\Delta^n$. There is an evident map $\overline{E} \to \Delta^n$ (use p) and an evident isomorphism $\overline{E}|_{\partial\Delta^n} = E$ over $\partial\Delta^n$. There is also an evident map $\overline{q} : \overline{E} \to E_n$ extending q (use the bottom horizontal map above).

It remains to show that $\overline{E} \to \Delta^n$ is a left fibration. To lift a left horn (Δ^r, Λ^r_j) against $\overline{E} \to \Delta^n$, there are a few different cases according to the map $p : \Delta^r \to \Delta^n$. If $p^{-1}(\partial\Delta^n) = \Delta^r$, then it suffices to lift the left horn (Δ^r, Λ^r_j) against the left fibration $E \to \partial\Delta^n$. If $p^{-1}(\partial\Delta^n) \subseteq \Lambda^r_j$, then it suffices extend a map from the left horn (Δ^r, Λ^r_j) to the Kan complex E_n. Now consider the remaining case $p^{-1}(\partial\Delta^n) = \Delta^{[r]-j}$, which amounts to extending a section of E over $(\Delta^{[r]-j}, \partial\Delta^{[r]-j})$ followed by extending a map $(\Delta^r, \partial\Delta^r) \to E_n$. To solve it, we reverse the order. First, solve the extension problem $(\Delta^r, \Lambda^r_j) \to E_n$ using the fact that E_n is a Kan complex. Restricting to $\Delta^{[r]-j} \subseteq \Delta^r$, we have a solution to the extension problem $(\Delta^{[r]-j}, \partial\Delta^{[r]-j}) \to E_n$ which may or may not be the image under q of a solution to the lifting problem for $(\Delta^{[r]-j}, \partial\Delta^{[r]-j})$ against E. However, according to the obstruction theory for sections of left fibrations (1.4.60), the lifting problem for $(\Delta^{[r]-j}, \partial\Delta^{[r]-j})$ against E does at least have a solution whose image under q is homotopic rel boundary to the solution $(\Delta^{[r]-j}, \partial\Delta^{[r]-j}) \to E_n$ chosen earlier (note that $p(r) = n$, as otherwise $p(\Delta^r) \subseteq \Delta^{[n]-n} \subseteq \partial\Delta^n$ and we would be in the first case above). This is enough, since any homotopy rel boundary of $(\Delta^{[r]-j}, \partial\Delta^{[r]-j}) \to E_n$ extends to a homotopy rel boundary of $(\Delta^r, \Lambda^r_j) \to E_n$. □

The ∞-category of ∞-categories

⋆ **1.4.66 Definition** (∞-category of ∞-categories Cat_∞)**.** The category of ∞-categories is naturally enriched over ∞-categories (1.4.19). By replacing each ∞-category $\mathsf{Fun}(\mathsf{C}, \mathsf{D})$ with its core $\mathsf{Fun}(\mathsf{C}, \mathsf{D})_\simeq$ (1.4.49) (that is, we remember just natural *isomorphisms* between functors), we obtain an enrichment over Kan complexes. The simplicial nerve (1.4.9) of this Kan simplicial category is called the *∞-category of ∞-categories*, denoted Cat_∞.

⋆ **1.4.67 Definition** (Equivalence of ∞-categories)**.** A functor of ∞-categories $F : \mathsf{C} \to \mathsf{D}$ is called an *equivalence* when it is an isomorphism in Cat_∞, namely when there exists a functor $G : \mathsf{D} \to \mathsf{C}$ such that $G \circ F \simeq \mathbf{1}_\mathsf{C}$ and $F \circ G \simeq \mathbf{1}_\mathsf{D}$ (isomorphisms in the functor categories $\mathsf{Fun}(\mathsf{C}, \mathsf{C})$ and $\mathsf{Fun}(\mathsf{D}, \mathsf{D})$, respectively).

1.4.68 Exercise. Show that a map of Kan complexes is a homotopy equivalence (1.3.20) iff it is an equivalence of ∞-groupoids (1.4.67).

1.4.69 Exercise. Show that a trivial Kan fibration $\mathsf{C} \to \mathsf{D}$ between ∞-categories is an equivalence.

1.4.70 Exercise. Show that if pre-composition with $\mathsf{A} \to \mathsf{B}$ is an equivalence $\mathsf{Fun}(\mathsf{B}, \mathsf{E}) \to \mathsf{Fun}(\mathsf{A}, \mathsf{E})$ for every ∞-category E, then $\mathsf{A} \to \mathsf{B}$ is an equivalence (in fact, the two cases $\mathsf{E} = \mathsf{A}$ and $\mathsf{E} = \mathsf{B}$ suffice to draw this conclusion).

1.4.71 Exercise (Functor $\mathsf{sSet} \to \mathsf{hCat}_\infty$; compare (1.3.26))**.** Show that if $X \hookrightarrow Y$ is filtered by pushouts of inner horns and C is an ∞-category, then the map $\mathsf{Fun}(Y, \mathsf{C}) \to \mathsf{Fun}(X, \mathsf{C})$ is a trivial Kan fibration. Use the small object argument (**??**) to show that for every simplicial set X, there exists an inclusion $X \hookrightarrow \overline{X}$ which is filtered by pushouts of inner horns with $\overline{X}$ an ∞-category. Show that for any pair of such inclusions $X \hookrightarrow \overline{X}$ and $Y \hookrightarrow \overline{Y}$ and any map $X \to Y$, there exists a dotted arrow making the following diagram commute

$$\begin{tikzcd} X \arrow[r, hook] \arrow[d] & \overline{X} \arrow[d, dashed] \\ Y \arrow[r, hook] & \overline{Y} \end{tikzcd} \tag{1.4.71.1}$$

and that moreover this dotted arrow is unique up to isomorphism in $\mathsf{Fun}(\overline{X}, \overline{Y})$. Show that sending X to (any choice of) $\overline{X}$ and sending a map $X \to Y$ to (any choice of) extension $\overline{X} \to \overline{Y}$ gives a well defined functor $\mathsf{sSet} \to \mathsf{hCat}_\infty$. A map of simplicial sets which is sent to an isomorphism in hCat_∞ is called a *categorical equivalence*. Show that any inclusion of simplicial sets which is filtered by pushouts of inner horns is a categorical equivalence. Extend all of this to marked simplicial sets sSet^+ in place of sSet (consider marked horns in place of inner horns, and the target remains hCat_∞); this gives a notion of *marked categorical equivalence.*

1.4.72 Exercise. Show that a map $X \to Y$ is a categorical equivalence iff $\mathsf{Fun}(Y, \mathsf{C}) \to \mathsf{Fun}(X, \mathsf{C})$ is an equivalence of ∞-categories for every ∞-category C. Conclude that if $X \to Y$ and $X' \to Y'$ are categorical equivalences, then so is $X \times X' \to Y \times Y'$ (note that $\mathsf{Fun}(X \times X', \mathsf{C}) = \mathsf{Fun}(X, \mathsf{Fun}(X', \mathsf{C}))$).

1.4.73 Exercise. Show that a trivial Kan fibration is a categorical equivalence.

The enriched homotopy category

We now recall how the homotopy category of an ∞-category is naturally enriched (1.1.138) over the homotopy category of spaces hSpc (1.3.20).

⋆ **1.4.74 Definition** (Mapping space $\mathrm{Hom}_{\mathsf{C}}$)**.** Given objects $x, y \in \mathsf{C}$, the *mapping space* $\mathrm{Hom}_{\mathsf{C}}(x, y) \in \mathsf{hSpc}$ has a few different presentations as explicit Kan complexes.

(1.4.74.1) $\mathrm{Hom}^{\mathrm{cyl}}_{\mathsf{C}}(x, y)$ is defined by the property that a map $Z \to \mathrm{Hom}^{\mathrm{cyl}}_{\mathsf{C}}(x, y)$ is a map $Z \times \Delta^1 \to \mathsf{C}$ sending $Z \times 0$ to x and sending $Z \times 1$ to y.

(1.4.74.2) $\mathrm{Hom}^{\mathrm{R}}_{\mathsf{C}}(x, y)$ is defined by the property that a map $Z \to \mathrm{Hom}^{\mathrm{R}}_{\mathsf{C}}(x, y)$ is a map $Z^{\triangleright} \to \mathsf{C}$ sending Z to x and sending the cone point to y (and dually $\mathrm{Hom}^{\mathrm{L}}_{\mathsf{C}}(x, y)$ is defined via maps $Z^{\triangleleft} \to \mathsf{C}$).

Note that $\mathrm{Hom}^{\mathrm{R}}_{\mathsf{C}}(x, y)$ is the fiber of $\mathsf{C}_{/y} \to \mathsf{C}$ (a *right* fibration) over x, while $\mathrm{Hom}^{\mathrm{cyl}}_{\mathsf{C}}(x, y)$ is the fiber of $\mathsf{C}^{/y} \to \mathsf{C}$ (1.4.56) over x. Thus (1.4.56) provides a canonical homotopy equivalence between $\mathrm{Hom}^{\mathrm{R}}_{\mathsf{C}}(x, y)$ and $\mathrm{Hom}^{\mathrm{cyl}}_{\mathsf{C}}(x, y)$ (and, by symmetry, $\mathrm{Hom}^{\mathrm{L}}_{\mathsf{C}}(x, y)$).

1.4.75 Exercise (Enriched homotopy category)**.** For objects $x, y, z \in \mathsf{C}$, let the simplicial set $\mathrm{Hom}^{\mathrm{cyl}}_{\mathsf{C}}(x, y, z)$ represent the functor sending Z to the set of maps $Z \times \Delta^2 \to \mathsf{C}$ sending

$Z \times i$ to x, y, z for $i = 0, 1, 2$, respectively. Show that the forgetful map $\mathrm{Hom}^{\mathrm{cyl}}_{\mathsf{C}}(x, y, z) \to \mathrm{Hom}^{\mathrm{cyl}}_{\mathsf{C}}(x, y) \times \mathrm{Hom}^{\mathrm{cyl}}_{\mathsf{C}}(y, z)$ is a trivial Kan fibration. Conclude that the forgetful map $\mathrm{Hom}^{\mathrm{cyl}}_{\mathsf{C}}(x, y, z) \to \mathrm{Hom}_{\mathsf{C}}(x, z)$ defines a 'composition' morphism

$$\mathrm{Hom}_{\mathsf{C}}(x, y) \times \mathrm{Hom}_{\mathsf{C}}(y, z) \to \mathrm{Hom}_{\mathsf{C}}(x, z) \tag{1.4.75.1}$$

in hSpc. Show that composition is unital (composition with $\mathbf{1}_x$ or $\mathbf{1}_y$ gives the identity map $\mathrm{Hom}_{\mathsf{C}}(x, y) \to \mathrm{Hom}_{\mathsf{C}}(x, y)$). Define a simplicial set $\mathrm{Hom}^{\mathrm{cyl}}_{\mathsf{C}}(x, y, z, w)$ and use it to show composition is associative. Conclude that this defines an enrichment of hC over hSpc, equipped with the monoidal structure $\times$ and the functor $\pi_0 : \mathsf{hSpc} \to \mathsf{Set}$. This is called the *enriched homotopy category* and is denoted $\underline{\mathsf{hC}}$.

1.4.76 Exercise (Enrichment of functors over hSpc)**.** Show that a functor $F : \mathsf{C} \to \mathsf{D}$ induces maps $\mathrm{Hom}_{\mathsf{C}}(x, y) \to \mathrm{Hom}_{\mathsf{D}}(F(x), F(y))$ which are compatible with composition. Show that for functors $F, G : \mathsf{C} \to \mathsf{D}$ and a natural transformation $F \Rightarrow G$, the following diagram commutes

$$\begin{tikzcd} \mathrm{Hom}_{\mathsf{C}}(x, y) \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathsf{D}}(F(x), F(y)) \arrow[d] \\ \mathrm{Hom}_{\mathsf{D}}(G(x), G(y)) \arrow[r] & \mathrm{Hom}_{\mathsf{D}}(F(x), G(y)) \end{tikzcd} \tag{1.4.76.1}$$

in hSpc. Conclude that this defines a lift of the map $\mathsf{Fun}(\mathsf{C}, \mathsf{D}) \to \mathsf{Fun}(\mathsf{hC}, \mathsf{hD})$ (1.4.27) to the category of enriched functors $\underline{\mathsf{hC}} \to \underline{\mathsf{hD}}$.

⋆ **1.4.77 Definition** (Fully faithful)**.** A functor $F : \mathsf{C} \to \mathsf{D}$ is called *fully faithful* iff the induced map $\mathrm{Hom}_{\mathsf{C}}(x, y) \to \mathrm{Hom}_{\mathsf{D}}(F(x), F(y))$ is an isomorphism in hSpc for all $x, y \in \mathsf{C}$ (that is, when the induced functor $\underline{\mathsf{h}F} : \underline{\mathsf{hC}} \to \underline{\mathsf{hD}}$ of enriched homotopy categories is fully faithful).

1.4.78 Exercise. Conclude from (1.4.76) that an equivalence of ∞-categories is fully faithful.

Isofibrations

⋆ **1.4.79 Definition** (Isofibration of ∞-categories)**.** A functor of ∞-categories $F : \mathsf{C} \to \mathsf{D}$ is called an *isofibration* when the map $F^\natural : \mathsf{C}^\natural \to \mathsf{D}^\natural$ (mark the isomorphisms (1.4.52)) is a marked fibration (has the right lifting property with respect to marked horns (1.4.51)). Isofibrations are often indicated with the arrow $\twoheadrightarrow$.

1.4.80 Remark (Pseudo-isofibration of simplicial sets)**.** It is not so straightforward to generalize the notion of an isofibration to morphisms of arbitrary simplicial sets. Let us call a map of simplicial sets $X \to Y$ a *pseudo-isofibration* when it becomes a marked fibration upon marking the edges which are sent to isomorphisms by categorical equivalences $X \to \hat{X}$ and $Y \to \hat{Y}$ to ∞-categories $\hat{X}$ and $\hat{Y}$ (1.4.71) (more generally, X and Y could be marked simplicial sets). This notion is not so well behaved, though it is occasionally useful when discussing maps of simplicial sets which have not yet been shown to be ∞-categories. A better notion (which we will not appeal to) is that of a *categorical fibration* of simplicial sets [74, 2.2.5.1], which means having the right lifting property with respect to all inclusions $K \hookrightarrow L$ which are categorical equivalences (a map of ∞-categories is a categorical fibration iff it is an isofibration).

1.4.81 Exercise. Show that if $F : \mathsf{C} \twoheadrightarrow \mathsf{D}$ is an isofibration of ∞-categories, then $d \in \mathsf{D}$ is in the image of F iff it is in the essential image of F.

1.4.82 Exercise. Show that for any isofibration of ∞-categories $\mathsf{C} \to \mathsf{D}$, the induced map $\mathsf{Fun}(K, \mathsf{C}) \to \mathsf{Fun}(K, \mathsf{D})$ is an isofibration of ∞-categories (use the characterization of isomorphism in functor categories (1.4.53) and the fact that $(\Delta^k, \partial\Delta^k) \wedge (\Delta^n, \Lambda^n_i)^\sim$ is filtered by pushouts of marked horns (1.4.54)). Similarly, show that for any monomorphism $K \to L$ and any ∞-category C, the restriction map $\mathsf{Fun}(L, \mathsf{C}) \to \mathsf{Fun}(K, \mathsf{C})$ is an isofibration of ∞-categories.

1.4.83 Exercise. Show that a map from an ∞-category E to a category C is an isofibration iff every it satisfies the right lifting property with respect to the marked horns $(\Delta^1, \Lambda^1_0)^\sim$ and $(\Delta^1, \Lambda^1_1)^\sim$ (use (**??**)). In particular, conclude that if every isomorphism in C is an identity, then every map from an ∞-category E to C is an isofibration.

1.4.84 Exercise. Show that if $\mathsf{C} \to \mathsf{D}$ is an isofibration of ∞-categories, then $\mathsf{C}_\simeq \to \mathsf{D}_\simeq$ is a Kan fibration.

1.4.85 Lemma. *Let $\mathsf{E} \to \mathsf{B}$ be an isofibration of ∞-categories, let $\mathsf{A} \to \mathsf{B}$ be a functor of ∞-categories, and consider the pullback $\mathsf{F} = \mathsf{E} \times_{\mathsf{B}} \mathsf{A} \to \mathsf{A}$ of simplicial sets. This pullback F is also an ∞-category, the resulting diagram*

$$\begin{tikzcd} \mathsf{F}^\natural \arrow[r] \arrow[d] & \mathsf{E}^\natural \arrow[d] \\ \mathsf{A}^\natural \arrow[r] & \mathsf{B}^\natural \end{tikzcd} \tag{1.4.85.1}$$

is a pullback of marked simplicial sets, and (hence) $\mathsf{F} \to \mathsf{A}$ is an isofibration of ∞-categories.

Proof. Inner fibrations are preserved under pullback, so $\mathsf{F} \to \mathsf{A}$ is an inner fibration and hence F is an ∞-category.

To show that (1.4.85.1) is a pullback of marked simplicial sets, we should show that a morphism f in F whose images in A and E are both isomorphisms is itself an isomorphism (that is, $\mathsf{F} \to \mathsf{A} \times \mathsf{E}$ reflects isomorphisms). To show that f is an isomorphism, we verify the extension property for marked left outer horns $(\Delta^n, \Lambda^n_0) \to \mathsf{F}$ sending the marked edge 01 to f (1.4.47). Suppose such a map $\Lambda^n_0 \to \mathsf{F}$ to be given. The composition $\Lambda^n_0 \to \mathsf{F} \to \mathsf{A}$ extends to Δ^n since f is sent to an isomorphism in A. It thus suffices to solve the resulting lifting problem for (Δ^n, Λ^n_0) against $\mathsf{F} \to \mathsf{A}$, or equivalently against $\mathsf{E} \to \mathsf{B}$. This lifting problem against $\mathsf{E} \to \mathsf{B}$ is solvable since the image of f in E is an isomorphism and $\mathsf{E} \to \mathsf{B}$ is an isofibration of ∞-categories.

Since (1.4.85.1) is a pullback of marked simplicial sets, the lifting property for marked horns against $\mathsf{E}^\natural \to \mathsf{B}^\natural$ implies the same for $\mathsf{F}^\natural \to \mathsf{A}^\natural$, so $\mathsf{F} \to \mathsf{A}$ is an isofibration of ∞-categories. $\square$

1.4.86 Lemma. *Every functor of ∞-categories $\mathsf{A} \to \mathsf{B}$ factors as a composition $\mathsf{A} \xhookrightarrow{\sim} \tilde{\mathsf{A}} \twoheadrightarrow \mathsf{B}$ where $\tilde{\mathsf{A}} \twoheadrightarrow \mathsf{B}$ is an isofibration of ∞-categories and $\mathsf{A} \xhookrightarrow{\sim} \tilde{\mathsf{A}}$ is an equivalence (in fact, has a retraction $\tilde{\mathsf{A}} \overset{\sim}{\twoheadrightarrow} \mathsf{A}$ which is a trivial Kan fibration).*

Proof. This will be a categorical analogue of the mapping path fibration construction (1.3.28). We define $\tilde{\mathsf{A}} = \mathsf{A} \times_{\mathsf{B}} \underline{\mathrm{Hom}}((\Delta^1)^\#, \mathsf{B})$; that is, for any simplicial set Z, a map $Z \to \tilde{\mathsf{A}}$ is a diagram

$$\begin{tikzcd} Z \arrow[r] \arrow[d, "\times 0"] & \mathsf{A} \arrow[d] \\ Z \times \Delta^1 \arrow[r] & \mathsf{B} \end{tikzcd} \tag{1.4.86.1}$$

in which the bottom arrow sends each edge $z \times \Delta^1$ to an isomorphism in B. There are evident maps $\tilde{\mathsf{A}} \to \mathsf{A}$ (take the top map), $\tilde{\mathsf{A}} \to \mathsf{B}$ (restrict the bottom map to $1 \in \Delta^1$), and $\mathsf{A} \to \tilde{\mathsf{A}}$ (take the bottom map to factor through the projection $Z \times \Delta^1 \to Z$).

Let us show that $\tilde{\mathsf{A}}$ is an ∞-category and that the functor $\tilde{\mathsf{A}} \to \mathsf{A} \times \mathsf{B}$ reflects isomorphisms. For both these questions, we are interested in extension problems of the form $(\Delta^n, \Lambda^n_i) \to \tilde{\mathsf{A}}$ for $n \geq 2$ (1.4.47). We decompose such an extension problem into a series of three extension problems $(\Delta^n, \Lambda^n_i) \to \mathsf{A}$, $(\Delta^n, \Lambda^n_i) \to \mathsf{B}$, and $(\Delta^n, \Lambda^n_i) \wedge (\Delta^1, \partial\Delta^1) \to \mathsf{B}$. For $0 < i < n$, these extension problems are solvable since A and B are ∞-categories (1.4.20). For $i = 0, n$, the first two extension problems $(\Delta^n, \Lambda^n_i) \to \mathsf{A}$ and $(\Delta^n, \Lambda^n_i) \to \mathsf{B}$ are solvable provided the marked edges are mapped to isomorphisms in A and B. The last extension problem $(\Delta^n, \Lambda^n_i) \wedge (\Delta^1, \partial\Delta^1) \to \mathsf{B}$ is solvable since the marked edges in (Δ^n, Λ^n_i) over $0, 1 \in \Delta^1$ are sent to isomorphisms in B (1.4.54)(1.4.47).

Now let us show that $\tilde{\mathsf{A}} \to \mathsf{B}$ is an isofibration. The lifting problem for $(\Delta^n, \Lambda^n_i)^\sim$ against $\tilde{\mathsf{A}}^\natural \to \mathsf{B}^\natural$ may be decomposed into extending $(\Delta^n, \Lambda^n_i)^\sim \to \mathsf{A}^\natural$ (always solvable (1.4.47)) followed by extending $(\Delta^n, \Lambda^n_i)^\sim \wedge (\Delta^1, \partial\Delta^1) \to \mathsf{B}^\natural$ so that $a \times \Delta^1$ is mapped to an isomorphism for all $a \in \Delta^n$ (this is only a nontrivial constraint when $n = 1$). This latter extension problem has a solution since $(\Delta^n, \Lambda^n_i)^\sim \wedge (\Delta^1, \partial\Delta^1)$ is filtered by pushouts of marked horns (1.4.54); when $n = 1$ and we need ensure that $a \times \Delta^1$ is an isomorphism for all $a \in \Delta^n$, note that this follows from the 2-out-of-3 property for isomorphisms.

Now finally let us show that the retraction $\tilde{\mathsf{A}} \to \mathsf{A}$ is a trivial Kan fibration. Lifting $(\Delta^k, \partial\Delta^k)$ against $\tilde{\mathsf{A}} \to \mathsf{A}$ amounts to extending $(\Delta^k, \partial\Delta^k) \wedge (\Delta^1, 0)^\# \to \mathsf{B}^\natural$ which is always possible (1.4.54). $\square$

1.4.87 Lemma. *An isofibration of ∞-categories which is fully faithful has the right lifting property with respect to $(\Delta^k, \partial\Delta^k)$ for all $k > 0$.*

Proof. Let $f : \mathsf{C} \to \mathsf{D}$ be an isofibration of ∞-categories which is fully faithful.

Fix a lifting problem for $(\Delta^k, \partial\Delta^k)$ against $\mathsf{C} \to \mathsf{D}$ with $k > 0$. If the map $\partial\Delta^k \to \mathsf{C}$ is constant (factors through a map $* \to \mathsf{C}$) over $\Delta^{[k]-k} \subseteq \partial\Delta^k$, then this lifting problem is equivalent to a lifting problem for $(\Delta^{[k]-k}, \partial\Delta^{[k]-k})$ against $\mathrm{Hom}^{\mathrm{R}}_{\mathsf{C}}(x, y) \to \mathrm{Hom}^{\mathrm{R}}_{\mathsf{D}}(f(x), f(y))$. This latter lifting problem has a solution: $\mathrm{Hom}^{\mathrm{R}}_{\mathsf{C}} \to \mathrm{Hom}^{\mathrm{R}}_{\mathsf{D}}$ is a trivial Kan fibration since it is a Kan fibration (since $\mathsf{C} \to \mathsf{D}$ is an isofibration (**??**)) and a homotopy equivalence (since

$\mathsf{C} \to \mathsf{D}$ is fully faithful) (1.3.29). It thus suffices to argue that a general lifting problem for $(\Delta^k, \partial\Delta^k)$ against $\mathsf{C} \to \mathsf{D}$ may be reduced to one in which the map $\partial\Delta^k \to \mathsf{C}$ is constant over $\Delta^{[k]-k} \subseteq \partial\Delta^k$.

Fix a lifting problem for $(\Delta^k, \partial\Delta^k)$ against $\mathsf{C} \to \mathsf{D}$, and let us reduce it to one in which the map $\partial\Delta^k \to \mathsf{C}$ is constant over $\Delta^{[k]-k} \subseteq \partial\Delta^k$. Consider the map $H : \Delta^k \times \Delta^1 \to \Delta^k$ which is the identity on $\Delta^k \times 1$ and which on $\Delta^k \times 0$ sends $\Delta^{[k]-k}$ to 0 and sends k to k. By pulling back our lifting problem under H, we obtain a lifting problem for

$$(\Delta^k \times \Delta^1, (\partial\Delta^k \times 1) \cup ((\Delta^{[k]-k} \sqcup \{k\}) \times \Delta^1)) \tag{1.4.87.1}$$

against $\mathsf{C} \to \mathsf{D}$. Note that the edges $\{0, k\} \times \Delta^1 \to \mathsf{C}$ are constant (since H is constant over them), in particular isomorphisms. Restricting to $\partial\Delta^k \times \Delta^1$, we have a lifting problem for $(\partial\Delta^k, \Delta^{[k]-k} \cup \{k\}) \wedge (\Delta^1, 1)$ against $\mathsf{C} \to \mathsf{D}$, and this is solvable since the domain is filtered by pushouts of right horns (1.4.54) with marked edge $\{k\} \times \Delta^1$, which is sent to an isomorphism in C. This defines a lifting problem for $(\Delta^k, \partial\Delta^k) \times \Delta^1$ against $\mathsf{C} \to \mathsf{D}$ whose restriction to $1 \in \Delta^1$ is our original lifting problem and whose restriction to $0 \in \Delta^1$ is constant over $\Delta^{[k]-k} \subseteq \Delta^k$. Now solvability of the restriction to $0 \in \Delta^1$ implies solvability for the restriction to $1 \in \Delta^1$: indeed, a solution at $0 \in \Delta^1$ leaves us with a lifting problem for $(\Delta^k, \partial\Delta^k) \wedge (\Delta^1, 0)$ against $\mathsf{C} \to \mathsf{D}$, which has a solution since the edge $\{0\} \times \Delta^1 \to \mathsf{C}$ is an isomorphism (1.4.54). □

1.4.88 Corollary. *An isofibration of ∞-categories which is fully faithful and essentially surjective is a trivial Kan fibration.*

Proof. Combine (1.4.81) and (1.4.87). □

⋆ **1.4.89 Corollary.** *A functor of ∞-categories which is fully faithful and essentially surjective is an equivalence.*

Proof. Factor a functor $\mathsf{A} \to \mathsf{B}$ into an equivalence $\mathsf{A} \to \tilde{\mathsf{A}}$ and an isofibration of ∞-categories $\tilde{\mathsf{A}} \to \mathsf{B}$ (1.4.86) and apply (1.4.88). □

1.4.90 Exercise. Show that injective categorical equivalences are preserved under pushouts of simplicial sets (use the fact that categorical equivalences are detected by $\mathsf{Fun}(-, \mathsf{E})$ for ∞-categories E (1.4.72), the fact that restriction of functors along an injection of simplicial sets is an isofibration (1.4.82), and the fact that an isofibration of ∞-categories is an equivalence iff it is a trivial Kan fibration (1.4.88)(1.4.73)).

⋆ **1.4.91 Definition** (Categorical fiber)**.** Let $F : \mathsf{C} \to \mathsf{D}$ be a functor of ∞-categories. The *(categorical) fiber* $F^{-1}(d)$ over an object $d \in \mathsf{D}$ is the full subcategory of $\mathsf{C}_{F(\cdot)/d}$ spanned by isomorphisms $F(c) \to d$.

1.4.92 Lemma. *The notion of categorical fiber is compatible with passing to opposites.*

Proof. Fix $F : \mathsf{C} \to \mathsf{D}$ and $d \in \mathsf{D}$. We are to show that the full subcategories of $\mathsf{C}_{F(\cdot)/d}$ and $\mathsf{C}_{d/F(\cdot)}$ spanned by isomorphisms are equivalent. We may (and shall) use the alternative model slice categories $\mathsf{C}^{F(\cdot)/d}$ (1.4.56) (which are equivalent to the usual model), whose full subcategory spanned by isomorphisms represents the functor of diagrams

$$\begin{tikzcd} Z \arrow[r] \arrow[d, "\times 0"] & \mathsf{C} \arrow[d, "F"] \\ Z \times (\Delta^1)^\# \arrow[r] & \mathsf{D}^\natural \end{tikzcd} \tag{1.4.92.1}$$

sending $Z \times 1 \mapsto d$.

Consider now the simplicial set representing the functor sending Z to the set of diagrams

$$\begin{tikzcd} Z \times \Delta^1 \arrow[r] \arrow[d, "02"] & Z \arrow[r] & \mathsf{C} \arrow[d, "F"] \\ Z \times (\Delta^2)^\# \arrow[rr] & & \mathsf{D}^\natural \end{tikzcd} \tag{1.4.92.2}$$

sending $Z \times \Delta^2 \supseteq Z \times 1 \mapsto d$. Restricting to 01 or 12 inside Δ^2 defines maps from this simplicial set to the full subcategories of $\mathsf{C}^{F(\cdot)/d}$ and $\mathsf{C}^{d/F(\cdot)}$ spanned by isomorphisms, respectively. These maps are both trivial Kan fibrations since $(\Delta^k, \partial\Delta^k) \wedge (\Delta^2, \Lambda^2_0)^\sim$ and $(\Delta^k, \partial\Delta^k) \wedge (\Delta^2, \Lambda^2_2)^\sim$ are filtered by pushouts of marked horns (1.4.54). □

⋆ **1.4.93 Lemma.** *Let $F : \mathsf{C} \to \mathsf{D}$ be an isofibration of ∞-categories, and let $d \in \mathsf{D}$ be an object. The tautological map from the fiber of F over d to the categorical fiber of F over d is an equivalence.*

Proof. Denote by $F^{-1}(d)$ the literal fiber, and denote by $(\mathsf{C} \downarrow^F_\sim d)$ the categorical fiber. Thus a map $Z \to (\mathsf{C} \downarrow^F_\sim d)$ is a diagram

$$\begin{tikzcd} Z \arrow[r] \arrow[d] & \mathsf{C} \arrow[d, "F"] \\ Z \times (\Delta^1)^\# \arrow[r] & \mathsf{D}^\natural \end{tikzcd} \tag{1.4.93.1}$$

sending $Z \times 1 \mapsto d$, while a map $Z \to F^{-1}(d)$ is such a diagram sending $Z \times \Delta^1 \mapsto d$.

Consider the simplicial set representing the functor of diagrams

$$\begin{tikzcd} Z \times (\Delta^1)^\# \arrow[r] \arrow[d, "01"] & \mathsf{C}^\natural \arrow[d, "F"] \\ Z \times (\Delta^2)^\# \arrow[r] & \mathsf{D}^\natural \end{tikzcd} \tag{1.4.93.2}$$

in which the bottom arrow sends $Z \times 12 \mapsto d$. Restricting to $02 \subseteq \Delta^2$ defines a forgetful map from this simplicial set to $(\mathsf{C} \downarrow^F_\sim d)$, while restricting to 12 defines a forgetful map from it to $F^{-1}(d)$. Restriction to 12 is a trivial Kan fibration: first solve the extension problem $(\Delta^k, \partial\Delta^k) \wedge (\Delta^1, 1)^\# \to \mathsf{C}^\natural$ (filtered by pushouts of marked right horns (1.4.54)) and then solve the extension problem $(\Delta^k, \partial\Delta^k) \wedge (\Delta^2, \Lambda^2_1)^\# \to \mathsf{D}^\natural$ (filtered by pushouts of inner horns (1.4.20)). Restriction to 02 is also a trivial Kan fibration: first solve the extension problem $(\Delta^k, \partial\Delta^k) \wedge (\Delta^2, \Lambda^2_2)^\# \to \mathsf{D}^\natural$ (filtered by pushouts of marked right horns (1.4.54)) and then solve the lifting problem for $(\Delta^k, \partial\Delta^k) \wedge (\Delta^1, 0)^\#$ (filtered by pushouts of marked left horns (1.4.54)) against $\mathsf{C}^\natural \to \mathsf{D}^\natural$ (which has solutions since $\mathsf{C} \to \mathsf{D}$ is an isofibration of ∞-categories). □

Final and initial objects

⋆ **1.4.94 Definition** (Final object). An object $c \in \mathsf{C}$ is called a *final object* iff the extension property holds for maps $(\Delta^n, \partial\Delta^n) \to \mathsf{C}$ which send the final vertex $n \in \Delta^n$ to c (for $n \geq 1$). Dually, an initial object in C is a final object in C^{op}.

1.4.95 Exercise. Show that the full subcategory of C spanned by final objects is either a trivial Kan complex or empty.

1.4.96 Exercise. Show that $c \in \mathsf{C}$ is a final object iff $\mathsf{C}_{/c} \to \mathsf{C}$ is a trivial Kan fibration. Conclude that every diagram $K \to \mathsf{C}$ extends to a diagram $K^\triangleright \to \mathsf{C}$ sending the final vertex to c.

1.4.97 Exercise. Show that if C has a final object, then C is Kan contractible (for example, construct a map $\mathsf{C} \times \Delta^1 \to \mathsf{C}$ which is the identity on $\mathsf{C} \times 0$ and sends $\mathsf{C} \times 1$ via the constant map to a final object).

1.4.98 Exercise. Show that if $c \in \mathsf{C}$ is final, then so is its image in hC. Show that if C is (the nerve of) a category, then an object is final iff it is final in the ∞-categorical sense.

1.4.99 Exercise. Show that any object isomorphic to a final object is final.

1.4.100 Exercise. Show that $(c \xrightarrow{\mathbf{1}_c} c) \in \mathsf{C}_{/c}$ is a final object.

1.4.101 Exercise. Use obstruction theory (1.4.60) to show that an object $c \in \mathsf{C}$ is final iff $\operatorname{Hom}_{\mathsf{C}}(x, c)$ is contractible for all $x \in \mathsf{C}$. Conclude that an equivalence of ∞-categories preserves final objects.

1.4.102 Exercise. Let $F : \mathsf{C} \to \mathsf{D}$ be a map of ∞-categories. Consider the lifting property for diagrams

$$\begin{tikzcd} \partial\Delta^r \arrow[r] \arrow[d] & \mathsf{C} \arrow[d, "F"] \\ \Delta^r \arrow[r] & \mathsf{D} \end{tikzcd} \tag{1.4.102.1}$$

where the map $\Delta^r \to \mathsf{D}$ sends the final vertex $r \in \Delta^r$ to a final object of D. Show that if this lifting property holds for all $r \geq 1$, then F reflects final objects. Show that if this lifting property holds for all $r \geq 0$, then F reflects and lifts final objects.

1.4.103 Exercise. Show that an object of $\prod_i \mathsf{C}_i$ is final iff its image in every C_i is final.

1.4.104 Proposition (Final objects in diagram categories)**.** *The functor* $\mathsf{Fun}(K,\mathsf{C}) \to \mathsf{Fun}(K_0,\mathsf{C})$ *reflects and lifts final objects.*

Proof. It suffices to show that $\mathsf{Fun}(K,\mathsf{C}) \to \mathsf{Fun}(K_0,\mathsf{C})$ satisfies the right lifting property with respect to maps from pairs $(\Delta^r, \partial\Delta^r)$ which send the final vertex $r \in \Delta^r$ to a final object of $\mathsf{Fun}(K_0,\mathsf{C})$ (1.4.102). By filtering the pair (K, K_0) by pushouts of pairs $(\Delta^k, \partial\Delta^k)$ with $k \geq 1$, we reduce to the extension property for maps

$$(\Delta^r, \partial\Delta^r) \wedge (\Delta^k, \partial\Delta^k) \to \mathsf{C} \tag{1.4.104.1}$$

whose specialization to every vertex lying over $r \in \Delta^r$ is final. The smash product $(\Delta^r, \partial\Delta^r) \wedge (\Delta^k, \partial\Delta^k)$ is filtered by pushouts of pairs $(\Delta^a, \partial\Delta^a)$. Each map $\Delta^a \to \Delta^r \times \Delta^k$ appearing in this filtration must send the final vertex $a \in \Delta^a$ to the final vertex $r \in \Delta^r$ (otherwise $\Delta^a \subseteq \partial\Delta^r \times \Delta^k$). We are thus reduced to the extension problem $(\Delta^a, \partial\Delta^a) \to \mathsf{C}$ for maps sending the final vertex $a \in \Delta^a$ to a final object, which is solvable for $a \geq 1$ (which is guaranteed by $k \geq 1$). □

⋆ **1.4.105 Definition** (Representable)**.** Let C be an ∞-category, and let $\mathsf{E} \to \mathsf{C}$ be a right fibration. An object $e \in \mathsf{E}$ (or, by abuse of language, its image in C) is said to *represent* E when it is a final object in E (equivalently, when $\mathsf{E}_{/e} \to \mathsf{E}$ is a trivial Kan fibration (1.4.96)). If E has a representing object, then said representing object is unique up to contractible choice (1.4.95) and we say that E is *representable.*

⋆ **1.4.106 Lemma** (Checking representability in the enriched homotopy category)**.** *Let* C *be an* ∞*-category, and let* $\mathsf{E} \to \mathsf{C}$ *be a right fibration. A pair* $(x \in \mathsf{C}, \xi \in \mathsf{E}_x)$ *represents* E *iff for every* $c \in \mathsf{C}$*, the induced map* $\operatorname{Hom}_{\mathsf{C}}(c,x) \to \mathsf{E}_c$ (**??**) *is a homotopy equivalence.*

Proof. The map $\mathsf{E}_{/\xi} \to \mathsf{C}_{/x}$ is a trivial Kan fibration since $\mathsf{E} \to \mathsf{C}$ is a right fibration (1.4.42). The correspondence

$$\mathsf{C}_{/x} \xleftarrow{\sim} \mathsf{E}_{/\xi} \to \mathsf{E} \tag{1.4.106.1}$$

of right fibrations over C thus induces a natural transformation of functors $\operatorname{Hom}_{\mathsf{C}}(-,x) \to \mathsf{E}(-) : \mathsf{C}^{\mathsf{op}} \to \mathsf{hSpc}$ (1.4.59). We leave it as an exercise to show that specializing this map to $c \in \mathsf{C}$ yields the map $\operatorname{Hom}_{\mathsf{C}}(c,x) \to \mathsf{E}_c$ from (**??**).

If $\xi \in \mathsf{E}$ is final, then $\mathsf{E}_{/\xi} \to \mathsf{E}$ is also a trivial Kan fibration, hence the induced map $\operatorname{Hom}_{\mathsf{C}}(c,x) \to \mathsf{E}_c$ is a homotopy equivalence for all $c \in \mathsf{C}$.

Now let us show the converse, namely that if $\operatorname{Hom}_{\mathsf{C}}(c,x) \to \mathsf{E}_c$ is a homotopy equivalence for all $c \in \mathsf{C}$, then $\mathsf{E}_{/\xi} \to \mathsf{E}$ is a trivial Kan fibration (that is, $\xi \in \mathsf{E}$ is a final object). The hypothesis means that $\mathsf{E}_{/\xi} \to \mathsf{E}$ (a map of right fibrations over C) restricts to a homotopy equivalence $(\mathsf{E}_{/\xi})_c \to \mathsf{E}_c$ of fibers over any object $c \in \mathsf{C}$. Now $\mathsf{E}_{/\xi} \to \mathsf{E}$ (hence also $(\mathsf{E}_{/\xi})_c \to \mathsf{E}_c$) is a right fibration and E_c is a Kan complex, so $(\mathsf{E}_{/\xi})_c \to \mathsf{E}_c$ is a Kan fibration (1.4.61). Thus it being a homotopy equivalence means it is a trivial Kan fibration (1.3.29). In particular, the fibers of $\mathsf{E}_{/\xi} \to \mathsf{E}$ are trivial Kan complexes, which implies that $\xi \in \mathsf{E}$ is final (1.4.101). □

Limits and colimits

⋆ **1.4.107 Definition** (Limit). A *limit diagram* is a diagram $K^{\triangleleft} \to \mathsf{C}$ which is a final object in $\mathsf{C}_{/K}$. The *limit* of a diagram $p : K \to \mathsf{C}$ is the image $\lim_K p \in \mathsf{C}$ of a final object in $\mathsf{C}_{/K}$ (if one exists; otherwise the limit is not defined). In other words, the limit of $K \to \mathsf{C}$ is the representing object of the right fibration $\mathsf{C}_{/K} \to \mathsf{C}$.

1.4.108 Exercise (Limits in slice ∞-categories). Show that a diagram $K^{\triangleleft} \to \mathsf{C}_{/L}$ is a limit diagram iff the corresponding diagram $K^{\triangleleft} \star L \to \mathsf{C}$ is a limit diagram.

1.4.109 Exercise. Let $p : L \to \mathsf{C}$ be a diagram, and let $K \subseteq L$ be a subcomplex for which (L, K) is filtered by pushouts of horns (Δ^r, Λ^r_j) with the property that if $j = r$ (right outer horn) then p sends the marked edge $(r-1, r) \subseteq \Delta^r$ to an isomorphism in C. Show that the forgetful map $\mathsf{C}_{/L} \to \mathsf{C}_{/K}$ is a trivial Kan fibration, and hence that the map $\lim_L p \to \lim_K p$ is an isomorphism.

1.4.110 Exercise. Let $p : L \to \mathsf{C}$ be a diagram, and let $K \subseteq L$ be a subcomplex containing all simplices $\Delta^r \to L$ whose final vertex $r \in \Delta^r$ is *not* mapped to the terminal object of C. Show that the forgetful map $\mathsf{C}_{/L} \to \mathsf{C}_{/K}$ is a trivial Kan fibration, and hence that the map $\lim_L p \to \lim_K p$ is an isomorphism.

⋆ **1.4.111 Lemma** (Limits in a Kan simplicial category). *Let C be a Kan simplicial category. A diagram $K^{\triangleleft} \to \mathsf{C}$ is a limit diagram if its composition with $\operatorname{Hom}(Z, -) : \mathsf{C} \to \mathsf{Spc}$ (note that this is a functor of Kan simplicial categories) is a limit diagram for every $Z \in \mathsf{C}$.*

Proof. Fix a diagram $\partial\Delta^n \star K \to \mathsf{C}$ with $n > 0$, and let us show that it extends to $\Delta^n \star K$ if the composition $n \times K \subseteq \partial\Delta^n \star K \to \mathsf{C} \xrightarrow{\operatorname{Hom}(Z,-)} \mathsf{Spc}$ is a limit diagram, where Z denotes the image of $0 \in \partial\Delta^n$ in C. Consider the composition $\partial\Delta^n \star K \to \mathsf{C} \xrightarrow{\operatorname{Hom}(Z,-)} \mathsf{Spc}$. This composition sends $0 \in \Delta^n$ to $\operatorname{Hom}(Z, Z)$, but let us modify it by restricting to the vertex $* = \mathbf{1}_Z \subseteq \operatorname{Hom}(Z, Z)$. Now extending this modified diagram $\partial\Delta^n \star K \to \mathsf{Spc}$ to $\Delta^n \star K$ is equivalent (inspection) to extending our original diagram $\partial\Delta^n \star K \to \mathsf{C}$ to $\Delta^n \star K$. $\square$

⋆ **1.4.112 Proposition** (Limits in Spc). *Given a diagram $p : K \to \underline{\mathsf{sSet}}$ (1.4.9), the presheaf on sSet associating to Z the set of extensions of $Z \sqcup p : * \sqcup K \to \underline{\mathsf{sSet}}$ to $K^{\triangleleft}$ is representable. The representing object $\lim_K^{\mathsf{sSet}} p$ (an abuse of notation since p is valued in $\underline{\mathsf{sSet}}$ not sSet) is Kan if p is valued in Kan complexes. A universal extension (representing object) $K^{\triangleleft} \to \underline{\mathsf{sSet}}^{\mathsf{Kan}}$ in this sense is also a(n ∞-categorical) limit diagram. In particular, Spc has all limits.*

Proof. Denote by $x_\sigma : p(\sigma(0)) \to \underline{\operatorname{Hom}}((\Delta^1)^{\{1,\ldots,n-1\}}, p(\sigma(n)))$ for $\sigma : \Delta^n \to K$ the structure maps (1.4.9) of the diagram $p : K \to \underline{\mathsf{sSet}}$. To extend $Z \sqcup p : * \sqcup K \to \underline{\mathsf{sSet}}$ to $K^{\triangleleft}$ amounts to choosing, for every simplex $\sigma : \Delta^n \to K$, a map $w_\sigma : Z \to \underline{\operatorname{Hom}}((\Delta^1)^{\{0,\ldots,n-1\}}, p(\sigma(n)))$, subject to the compatibility conditions stated in (1.4.9), namely that for any map $f : \Delta^m \to \Delta^n$, the following diagram commutes.

$$\begin{tikzcd}
& \underline{\mathrm{Hom}}((\Delta^1)^{\{0,\ldots,n-1\}}, p(\sigma(n))) \arrow[d, "\circ(-\times\{1\}^{\{f(m)\}})"] \\
Z \arrow[ur, "w_\sigma"] \arrow[dr, "w_{\sigma|[0\cdots f(m)]}"'] \arrow[ddr, "w_{\sigma f}"'] & \underline{\mathrm{Hom}}((\Delta^1)^{\{0,\ldots,f(m)-1\}} \times (\Delta^1)^{\{f(m)+1,\ldots,n-1\}}, p(\sigma(n))) \\
& \underline{\mathrm{Hom}}((\Delta^1)^{\{0,\ldots,f(m)-1\}}, p(\sigma(f(m)))) \arrow[u, "(x_{\sigma|[f(m)\cdots n]})\circ"'] \arrow[d, "\circ((t_i)_i\mapsto(\max_{f(i)=j} t_i)_j)"] \\
& \underline{\mathrm{Hom}}((\Delta^1)^{\{0,\ldots,m-1\}}, p(\sigma(f(m))))
\end{tikzcd} \tag{1.4.112.1}$$

Such a collection of maps, and the compatibility property, naturally pull back under any map $Z' \to Z$. Moreover, this is evidently the presheaf which defines the limit of all mapping spaces $\underline{\mathrm{Hom}}((\Delta^1)^{\{0,\ldots,n-1\}}, p(\sigma(n)))$ for $\sigma : \Delta^n \to K$ and all vertical maps in the above diagram for $f : \Delta^m \to \Delta^n$. Thus a universal extension $K^\triangleleft \to \underline{\mathsf{sSet}}$ exists since sSet has all limits.

To show that the representing object $\lim_K^{\mathsf{sSet}} p$ is Kan, we solve the extension problem $(\Delta^n, \Lambda^n_i) \to \lim_K^{\mathsf{sSet}} p$ by induction on the skeleta of K: a given simplex attachment $(\Delta^\ell, \partial\Delta^\ell)$ to K entails extending $(\Delta^n, \Lambda^n_i) \wedge (\Delta^1, \partial\Delta^1)^{\{0,\ldots,\ell-1\}} \to p(\ell)$, which is possible provided $p(\ell)$ is Kan (1.3.18).

Now consider the question of extending a diagram $p : \partial\Delta^n \star K \to \underline{\mathsf{sSet}}$ to $\Delta^n \star K$ where $n > 0$. The extension problem for $(\Delta^n, \partial\Delta^n) \subseteq (\Delta^n, \partial\Delta^n) \star K$ amounts to an extension problem of the form

$$p(0) \times (\Delta^1, \partial\Delta^1)^{\wedge\{1,\ldots,n-1\}} \to p(n). \tag{1.4.112.2}$$

The remaining extension problem $(\Delta^n, \partial\Delta^n) \star (K, \varnothing) \to \underline{\mathsf{sSet}}$ amounts to a collection of (related) extension problems of the form

$$p(0) \times (\Delta^1, \partial\Delta^1)^{\wedge\{1,\ldots,n-1\}} \wedge (\Delta^1, \partial\Delta^1)^{\{n\}} \to \underline{\mathrm{Hom}}((\Delta^1)^{\{0,\ldots,k-1\}}, p(\sigma(k))) \tag{1.4.112.3}$$

indexed by the simplices $\sigma : \Delta^k \to K$, which is (by inspection) equivalent to an extension problem

$$p(0) \times (\Delta^1, \partial\Delta^1)^{\wedge\{1,\ldots,n-1\}} \wedge (\Delta^1, \partial\Delta^1)^{\{n\}} \to \lim_K^{\mathsf{sSet}} p, \tag{1.4.112.4}$$

where the boundary condition at $1^{\{n\}}$ is the composition of the solution to the first extension problem (1.4.112.2) with the classifying map $p(n) \to \lim_K^{\mathsf{sSet}} p$. We thus conclude that the extension problem for $(\Delta^n, \partial\Delta^n) \star K \to \underline{\mathsf{sSet}}$ is equivalent to the lifting problem for $p(0) \times (\Delta^1, \partial\Delta^1)^{\wedge\{1,\ldots,n-1\}}$ against the mapping path fibration (1.3.28) (with the endpoints $0, 1 \in \Delta^1$ reversed) of the classifying map $p(n) \to \lim_K^{\mathsf{sSet}} p$. Now the mapping path fibration of a homotopy equivalence of Kan complexes is a trivial Kan fibration (1.3.28)(1.3.29), so we conclude that a diagram $p : K^\triangleleft \to \underline{\mathsf{sSet}}^{\mathsf{Kan}}$ is a limit diagram if the classifying map $p(*) \to \lim_K^{\mathsf{sSet}} p$ is a homotopy equivalence. $\square$

1.4.113 Exercise. Conclude from (1.4.112) that a diagram $p : K^{\triangleleft} \to \underline{\mathsf{sSet}}^{\mathsf{Kan}}$ is a limit diagram iff the classifying map $p(*) \to \lim_K^{\mathsf{sSet}} p$ is a homotopy equivalence.

1.4.114 Example. Let us see how the description (1.4.112) of limits in Spc works in various specific cases which often come up in practice.

(1.4.114.1) A product of Kan complexes in sSet is a product in Spc.

(1.4.114.2) A square diagram $\Delta^1 \times \Delta^1 \to \underline{\mathsf{sSet}}^{\mathsf{Kan}}$

$$\begin{tikzcd} F \arrow[r] \arrow[d] \arrow[dr] & E \arrow[d] \\ A \arrow[r] & B \end{tikzcd}$$

is a pullback in Spc iff the associated map

$$F \to A \times_B \underline{\mathrm{Hom}}(\Delta^1 \vee \Delta^1, B) \times_B E$$

is a homotopy equivalence, where $\Delta^1 \vee \Delta^1$ indicates identifying the vertex 0 in the two copies of Δ^1, and the two maps $\underline{\mathrm{Hom}}(\Delta^1 \vee \Delta^1, B) \to B$ are evaluation at the two copies of the vertex $1 \in \Delta^1$.

(1.4.114.3) A diagram of Kan complexes

$$\begin{tikzcd} & & & X_\infty \arrow[dll] \arrow[dl] \arrow[d] \\ \cdots \arrow[r] & X_2 \arrow[r] & X_1 \arrow[r] & X_0 \end{tikzcd}$$

indexed by $(\cdots \to * \to *)^{\triangleleft}$ is a limit diagram in Spc iff the induced map of Kan complexes

$$X_\infty \to \underline{\mathrm{Hom}}(\Delta^1, X_0) \times_{X_0} \underline{\mathrm{Hom}}(\Delta^1, X_1) \times_{X_1} \cdots$$

is a homotopy equivalence, where the maps $\underline{\mathrm{Hom}}(\Delta^1, X_i) \to X_i$ are given by evaluation at $1 \in \Delta^1$ and the maps $\underline{\mathrm{Hom}}(\Delta^1, X_i) \to X_{i-1}$ are given by evaluation at $0 \in \Delta^1$ followed by composition with $X_i \to X_{i-1}$.

These descriptions are reasonably concrete, yet not necessarily the most convenient. It is therefore helpful to give a few reformulations.

⋆ **1.4.115 Lemma.** *A square of Kan complexes*

$$\begin{tikzcd} F \arrow[r] \arrow[d] & E \arrow[d, two heads] \\ A \arrow[r] & B \end{tikzcd} \tag{1.4.115.1}$$

which is a pullback in sSet *is also a pullback in* Spc *if the map* $E \to B$ *is a Kan fibration.*

Proof. We should show that the map $F = A \times_B E \to A \times_B \underline{\mathrm{Hom}}(\Delta^1 \vee \Delta^1, B) \times_B E$ is a homotopy equivalence (1.4.114.2). To do this, consider the following correspondence of trivial Kan fibrations.

$$\begin{tikzcd} A \times_B E & A \times_B \underline{\mathrm{Hom}}(\Delta^1 \vee \Delta^1, E) \arrow[l, two heads, "\sim"'] \arrow[d, two heads, "\sim"] \\ & A \times_B \underline{\mathrm{Hom}}(\Delta^1 \vee \Delta^1, B) \times_B E \end{tikzcd} \tag{1.4.115.2}$$

The horizontal map is a trivial Kan fibration since $(\Delta^k, \partial\Delta^k) \wedge (\Delta^1 \vee \Delta^1, 1)$ is filtered by pushouts of horns and E is a Kan complex. The vertical map is a trivial Kan fibration since $(\Delta^k, \partial\Delta^k) \wedge (\Delta^1 \vee \Delta^1, 1)$ is filtered by pushouts of horns and $E \to B$ is a Kan fibration. Now the map in question is the composition of the vertical map with the tautological section of the horizontal map. □

⋆ **1.4.116 Lemma.** *A square of ∞-categories*

$$\begin{tikzcd} \mathsf{F} \arrow[r] \arrow[d] & \mathsf{E} \arrow[d, two heads] \\ \mathsf{A} \arrow[r] & \mathsf{B} \end{tikzcd} \tag{1.4.116.1}$$

which is a pullback in sSet *is also a pullback in* Cat_∞ *if the map* $\mathsf{E} \to \mathsf{B}$ *is an isofibration* (1.4.79).

Proof. By (1.4.111), it suffices to show that applying $\mathsf{Fun}(Z, -)_\simeq$ to this diagram produces a pullback diagram in Spc for any ∞-category Z (in fact, we will show this holds for any simplicial set Z).

Applying $\mathsf{Fun}(Z, -)$ to our diagram produces a square of ∞-categories which is a pullback in sSet (indeed, $\underline{\mathrm{Hom}}(Z, -) : \mathsf{sSet} \to \mathsf{sSet}$ preserves all limits by its universal property). Now recall that $\mathsf{Fun}(Z, -)$ preserves isofibrations of ∞-categories (1.4.82) and that $\simeq$ sends pullbacks of isofibrations of ∞-categories to pullbacks of Kan fibrations (1.4.84)(**??**). Thus applying $\mathsf{Fun}(Z, -)_\simeq$ to our diagram yields a Kan fibration pullback of Kan complexes, which is a pullback in Spc by (1.4.115). □

1.4.117 Exercise. Conclude from (1.4.115)(1.4.86)(1.4.93) that the categorical fiber (1.4.91) of a functor $\mathsf{C} \to \mathsf{D}$ over an object $d \in \mathsf{D}$ is the fiber product $\mathsf{C} \times_{\mathsf{D}} *$ in Cat_∞.

1.4.118 Corollary. *Given a diagram $p : K^\triangleright \to \underline{\mathsf{sSet}}$, its Kanification $\hat{p}$ (1.4.24) is a colimit diagram iff for every Kan complex Z, the composition $\underline{\mathrm{Hom}}(p, Z) : (K^\triangleright)^{\mathsf{op}} \to \underline{\mathsf{sSet}}^{\mathsf{Kan}}$ is a limit diagram in* Spc.

Proof. According to the criterion (1.4.111), the Kanification $\hat{p}$ is a colimit diagram in Spc iff for every Kan complex Z, its composition with $\underline{\mathrm{Hom}}(-, Z)$ is a limit diagram in Spc. Now the composition of the transformation $p \to \hat{p}$ with the functor $\underline{\mathrm{Hom}}(-, Z) : (\underline{\mathsf{sSet}})^{\mathsf{op}} \to \underline{\mathsf{sSet}}^{\mathsf{Kan}}$ is an objectwise trivial Kan fibration $\underline{\mathrm{Hom}}(\hat{p}, Z) \to \underline{\mathrm{Hom}}(p, Z)$ of diagrams $(K^\triangleright)^{\mathsf{op}} \to \underline{\mathsf{sSet}}^{\mathsf{Kan}}$, hence an isomorphism in $\mathsf{Fun}((K^\triangleright)^{\mathsf{op}}, \mathsf{Spc})$ (1.4.53). □

⋆ **1.4.119 Example.** The criterion for a diagram of simplicial sets to be a colimit diagram in Spc (after Kanification) (1.4.118) gives the following applications:

(1.4.119.1) A coproduct of simplicial sets is a coproduct in Spc (upon applying $\underline{\mathrm{Hom}}(-,Z)$, it becomes a product of Kan complexes, which is a product in Spc (1.4.114.1)).

(1.4.119.2) An injective pushout of simplicial sets

$$\begin{tikzcd}
A \arrow[r, hook] \arrow[d] & B \arrow[d] \\
X \arrow[r, hook] & Y
\end{tikzcd}$$

is a pushout in Spc (upon applying $\underline{\mathrm{Hom}}(-,Z)$, it becomes a Kan fibration pullback of Kan complexes, which is a pullback in Spc (1.4.115)).

(1.4.119.3) A sequential colimit of injections of simplicial sets is a colimit in Spc (upon applying $\underline{\mathrm{Hom}}(-,Z)$, it becomes a sequential inverse limit of Kan fibrations of Kan complexes, which is a limit in Spc (**??**).

⋆ **1.4.120 Proposition** (Colimits in Spc)**.** *Given a diagram $p : K \to \underline{\mathsf{sSet}}$ (1.4.24), the precosheaf on sSet associating to Z the set of extensions of $p \sqcup Z : K \sqcup * \to \underline{\mathsf{sSet}}$ to $K^{\triangleright}$ is corepresentable. A universal extension (corepresenting object) $K^{\triangleright} \to \underline{\mathsf{sSet}}$ becomes a colimit diagram in Spc after applying Kanification $\underline{\mathsf{sSet}} \to \mathsf{Spc}$ (1.4.24). In particular, Spc has all colimits.*

Proof. Denote by $x_\sigma : p(\sigma(0)) \times (\Delta^1)^{\{1,\ldots,n-1\}} \to p(\sigma(n))$ for $\sigma : \Delta^n \to K$ the structure maps (1.4.9) of the diagram $p : K \to \underline{\mathsf{sSet}}$. To extend $p \sqcup Z : K \sqcup * \to \underline{\mathsf{sSet}}$ to $K^{\triangleright}$ amounts to choosing, for every simplex $\sigma : \Delta^n \to K$, a map $w_\sigma : p(\sigma(0)) \times (\Delta^1)^{\{1,\ldots,n\}} \to Z$, subject to the compatibility conditions stated in (1.4.9), namely that for any map $f : \Delta^m \to \Delta^n$, the following diagram commutes.

$$\begin{tikzcd}
p(\sigma(0)) \times (\Delta^1)^{\{1,\ldots,n\}} \arrow[dr, "w_\sigma"] & \\
p(\sigma(0)) \times (\Delta^1)^{\{1,\ldots,f(0)-1\}} \times (\Delta^1)^{\{f(0)+1,\ldots,n\}} \arrow[u, "\times\{1\}^{\{f(0)\}}"'] \arrow[d, "x_{\sigma|[0\cdots f(0)]}"'] & Z \\
p(\sigma(f(0))) \times (\Delta^1)^{\{f(0)+1,\ldots,n\}} \arrow[ur, "w_{\sigma|[f(0)\cdots n]}"] & \\
p(\sigma(f(0))) \times (\Delta^1)^{\{1,\ldots,m\}} \arrow[u, "(t_i)_i \mapsto (\max_{f(i)=j} t_i)_j"'] \arrow[uur, "w_{\sigma f}"'] &
\end{tikzcd} \tag{1.4.120.1}$$

Such a collection of maps, and the compatibility property, naturally push forward along any map $Z \to Z'$. Moreover, this is evidently the precosheaf which defines the colimit of all products $p(\sigma(0)) \times (\Delta^1)^{\{1,\ldots,n\}}$ for $\sigma : \Delta^n \to K$ and all vertical maps in the above diagram for $f : \Delta^m \to \Delta^n$. Thus a universal extension $K^{\triangleright} \to \underline{\mathsf{sSet}}$ exists since sSet has all colimits.

To check that a universal extension $K^{\triangleright} \to \underline{\mathsf{sSet}}$ becomes a colimit in Spc after Kanification, it suffices to check that for every Kan complex A, its composition with $\underline{\mathrm{Hom}}(-, A) : \underline{\mathsf{sSet}}^{\mathsf{op}} \to \underline{\mathsf{sSet}}^{\mathsf{Kan}}$ is a limit diagram in Spc (1.4.118). In fact, we claim that such a composition is in fact a universal extension $(K^{\mathsf{op}})^{\triangleleft} \to \underline{\mathsf{sSet}}^{\mathsf{Kan}}$ in the sense of (1.4.112). This follows from a direct comparison of the colimit diagram defining $\operatorname{colim}_K^{\mathsf{sSet}} p$ (1.4.120.1) and the limit diagram defining $\lim_{K^{\mathsf{op}}}^{\mathsf{sSet}} p^{\mathsf{op}}$ (1.4.112.1) (noting that $\underline{\mathrm{Hom}}(-, A) : \mathsf{sSet}^{\mathsf{op}} \to \mathsf{sSet}$ sends colimits of simplicial sets to limits of simplicial sets). $\square$

1.4.121 Exercise. Suppose a continuous functor $X_\bullet : \mathsf{sSet}^{\mathsf{op}} \to \mathsf{sSet}$ has the property that $X_\bullet(\Delta^n) \twoheadrightarrow X_\bullet(\partial\Delta^n)$ is a Kan fibration for every $n \geq 0$. Regard $X_\bullet$ as a bi-simplicial set $X_{k,\ell} = X_k(\Delta^\ell)$ and consider its 'transpose' $Y_{k,\ell} = X_{\ell,k}$ for continuous $Y_\bullet : \mathsf{sSet}^{\mathsf{op}} \to \mathsf{sSet}$. Show that $Y_a \to Y_b$ is a trivial Kan fibration for every injection $[a] \hookrightarrow [b]$ (filter (Δ^b, Δ^a) by pushouts of horns). Conclude that $Y_a \to Y_b$ is a Kan equivalence for every map $[a] \to [b]$.

Final and initial functors

We now discuss final maps of simplicial sets following Lurie [74, 4.1.1].

⋆ **1.4.122 Definition** (Final; Joyal)**.** A map of simplicial sets $K \to L$ is called ∞-*final* when the pullback map $\underline{\mathrm{Sec}}(L, E) \to \underline{\mathrm{Sec}}(K, E)$ is a homotopy equivalence for every right fibration $E \to L$.

1.4.123 Exercise. Show that if $K \subseteq L$ is filtered by pushouts of right horns and $E \to L$ is a right fibration, then $\underline{\mathrm{Sec}}(L, \mathsf{E}) \to \underline{\mathrm{Sec}}(K, \mathsf{E})$ is a trivial Kan fibration, hence $K \to L$ is ∞-final.

1.4.124 Exercise. Given an ∞-final map $f : K \to L$, show that a map $g : L \to M$ is ∞-final iff $g \circ f$ is ∞-final. Show by example that ∞-final maps do not satisfy the 2-out-of-3 property (1.1.46). Show that a retract of an ∞-final map is ∞-final.

1.4.125 Lemma. *The inclusion of a final object is a final functor.*

Proof. Let C be an ∞-category, let $c \in \mathsf{C}$ be a final object, and let us show that $c : * \to \mathsf{C}$ is an ∞-final functor. The inclusion $\{c\}^{\triangleright} \hookrightarrow \mathsf{C}^{\triangleright}$ is filtered by pushouts of right horns (cone a filtration of $(\mathsf{C}, \{c\})$ by simplices $(\Delta^k, \partial\Delta^k)$ (1.4.42)), hence is ∞-final (1.4.123). Now we claim that our desired map $\{c\} \hookrightarrow \mathsf{C}$ is a retract of this map, which implies it is also ∞-final (1.4.124).

$$\begin{tikzcd} \{c\} \arrow[r, hook] \arrow[d, hook] & \{c\}^{\triangleright} \arrow[l, dashed, bend right] \arrow[d, hook] \\ \mathsf{C} \arrow[r, hook] & \mathsf{C}^{\triangleright} \arrow[l, dashed, bend right] \end{tikzcd} \tag{1.4.125.1}$$

To define the desired retraction, send the cone point to c and send the edge $\{c\}^{\triangleright}$ to the identity morphism of c. For the rest, we wish to solve the extension problem $(\mathsf{C}, \{c\}) \star (*, \varnothing) \to \mathsf{C}$ (where the cone point is sent to $c \in \mathsf{C}$), which works since $c \in \mathsf{C}$ is a final object (a filtration of $(\mathsf{C}, \{c\})$ by simplices $(\Delta^k, \partial\Delta^k)$ induces a filtration of $(\mathsf{C}, \{c\}) \star (*, \varnothing)$ by simplices (1.4.42) whose final vertex is the cone point mapping to the final object $c \in \mathsf{C}$). $\square$

1.4.126 Lemma. *An ∞-final map of simplicial sets is a homotopy equivalence.*

Proof. Let $K \to L$ be ∞-final. As a special case of (1.4.122), the pullback map $\underline{\mathrm{Hom}}(L, X) \to \underline{\mathrm{Hom}}(K, X)$ is a homotopy equivalence for every Kan complex X. In particular, it is a bijection on connected components, which implies the map $\mathrm{Hom}_{\mathsf{hSpc}}(L, -) \to \mathrm{Hom}_{\mathsf{hSpc}}(K, -)$ is an isomorphism of functors on hSpc. ☐

1.4.127 Lemma. *A product of ∞-final maps is ∞-final.*

Proof. It suffices to show that if $K' \to K$ is ∞-final, then so is $K' \times L \to K \times L$ for any simplicial set L. Given a right fibration $E \to K \times L$, we can form its 'pushforward' to K, which is the right fibration $G \to K$ defined by the universal property that a map from a simplicial set Z to G is a map $Z \to K$ together with a lift of $Z \times L \to K \times L$ to E. The right lifting property for $G \to K$ with respect to (Δ^n, Λ^n_i) follows from the right lifting property for $E \to K \times L$ with respect to pairs $(\Delta^n, \Lambda^n_i) \times L$, so $G \to K$ is indeed a right fibration (1.4.54). Since $K' \to K$ is ∞-final, the pullback map

$$\underline{\mathrm{Sec}}(K \times L, E) = \underline{\mathrm{Sec}}(K, G) \to \underline{\mathrm{Sec}}(K', G) = \underline{\mathrm{Sec}}(K' \times L, E) \tag{1.4.127.1}$$

is a homotopy equivalence. ☐

1.4.128 Lemma. *A right fibration is ∞-final iff it is a trivial Kan fibration.*

Proof. Let $K \to L$ be a right fibration which is ∞-final. The trick is to apply the definition of ∞-finality of $K \to L$ to the right fibration $K \to L$ itself. That is, we consider the pullback map $\underline{\mathrm{Sec}}(L, K) \to \underline{\mathrm{Sec}}(K, K) = \underline{\mathrm{Sec}}(K, K \times_L K)$, which is a Kan equivalence since $K \to L$ is ∞-final. In particular, there exists a vertex of $\underline{\mathrm{Sec}}(L, K)$ (in other words, a section $L \to K$), whose image in $\underline{\mathrm{Sec}}(K, K)$ (namely, for which the composition $K \to L \to K$) has an edge to the identity map 1_K (namely, is homotopic over L to the identity map of K). This data restricts to a deformation retraction of every fiber of $K \to L$, and a right fibration with contractible fibers is a trivial Kan fibration (1.4.60).

It remains to show that a trivial Kan fibration is ∞-final, and this is straightforward. Let $\pi : K \to L$ be a trivial Kan fibration, and let $E \to L$ be a left fibration. A choice of section $s : L \to K$ determines a retraction s^* of the pullback map $\pi^* : \underline{\mathrm{Sec}}(L, E) \to \underline{\mathrm{Sec}}(K, E)$. It is enough to show this retraction is a deformation retraction, that is to produce a homotopy $\underline{\mathrm{Sec}}(K, E) \times \Delta^1 \to \underline{\mathrm{Sec}}(K, E)$ between the identity map and $(s\pi)^*$. Such a homotopy is equivalent to the data of a map $\underline{\mathrm{Sec}}(K, E) \to \underline{\mathrm{Sec}}(K \times \Delta^1, E)$ whose restriction to $K \times 0$ is the identity and whose restriction to $K \times 1$ is $(s\pi)^*$. Pullback under any homotopy $K \times \Delta^1 \to K$ over L between 1_K and $s\pi$ (which exists since π is a trivial Kan fibration) defines such a map. ☐

1.4.129 Exercise. Conclude from the characterization of ∞-final right fibrations (1.4.128) and the small object argument (**??**) that a map is ∞-final iff it factors as the composition of a map filtered by pushouts of right horns followed by a trivial Kan fibration.

1.4.130 Lemma. *If $K \to L$ is ∞-final and $E \to L$ is a left fibration, then the map $K \times_L E \to E$ is a Kan equivalence.*

Proof. It suffices to consider two cases: $K \to L$ filtered by pushouts of right horns and $K \to L$ a trivial Kan fibration (1.4.129).

If $K \to L$ is a trivial Kan fibration, then its pullback $K \times_L E \to E$ is as well, hence in particular is a Kan equivalence.

If $K \to L$ is filtered by pushouts of right horns, then $K \times_L E \to E$ is filtered by pushouts of pairs of the form $\mathsf{E} \times_{\Delta^k} (\Delta^k, \Lambda^k_i)$ where $\mathsf{E} \to \Delta^k$ is a left fibration and $0 < i \leq k$. Now E deformation retracts down to the fiber E_k over $k \in \Delta^k$ (1.4.60) as does $\mathsf{E} \times_{\Delta^k} \Lambda^k_i$ (1.4.62.1) (although that construction is made under the assumption that $0 < i < k$, it works just as well for $i = k$), so $\mathsf{E} \times_{\Delta^k} (\Delta^k, \Lambda^k_i)$ is a Kan equivalence. An injective pushout of a Kan equivalence is Kan equivalence (1.4.119.2), so we conclude that $K \times_L E \to E$ is filtered by Kan equivalences. □

1.4.131 Lemma. *Let $K \to L$ be a map of simplicial sets. The morphism $\underline{\mathrm{Sec}}(L, E) \to \underline{\mathrm{Sec}}(K, E)$ in hSpc depends only on the isomorphism class of the right fibration $E \to L$ in the functor ∞-category $\mathsf{Fun}(L, (\mathsf{sSet}^{\mathsf{R}}_{/-})_{\simeq})$ (1.4.64).*

Proof. Suppose $E_0, E_1 \to L$ are right fibrations isomorphic in $\mathsf{Fun}(L, (\mathsf{sSet}^{\mathsf{R}}_{/-})_{\simeq})$. A morphism from E_0 to E_1 in $\mathsf{Fun}(L, (\mathsf{sSet}^{\mathsf{R}}_{/-})_{\simeq})$ is a right fibration $E \to L \times \Delta^1$ with $E|_{L \times i} = E_i$ for $i = 0, 1$. If this morphism is an isomorphism, then the map $(E_0)_\ell \to (E_1)_\ell$ associated (up to contractible choice) by the restriction of E to the edge $\ell \times \Delta^1$ will be a Kan equivalence for every vertex $\ell \in L$ (consider the functor $(\mathsf{sSet}^{\mathsf{R}}_{/-})_{\simeq} \to \mathsf{hSpc}$ (1.4.59)). Now the pullback maps for E_0, E_1, and E are related by the following diagram.

$$\begin{tikzcd} \underline{\mathrm{Sec}}(L, E_0) \arrow[d] & \underline{\mathrm{Sec}}(L \times \Delta^1, E) \arrow[l, two heads, "\sim"'] \arrow[r, two heads, "\sim"] \arrow[d] & \underline{\mathrm{Sec}}(L, E_1) \arrow[d] \\ \underline{\mathrm{Sec}}(K, E_0) & \underline{\mathrm{Sec}}(K \times \Delta^1, E) \arrow[l, two heads, "\sim"'] \arrow[r, two heads, "\sim"] & \underline{\mathrm{Sec}}(K, E_1) \end{tikzcd} \tag{1.4.131.1}$$

It suffices to show that the (wlog top) horizontal arrows are trivial Kan fibrations. That is, we should show the extension property for sections of E over $L \times (\Delta^r, \partial\Delta^r) \wedge (\Delta^1, i)$ for $i = 0, 1$. Filter L by simplices $(\Delta^n, \partial\Delta^n)$, filter $(\Delta^n, \partial\Delta^n) \wedge (\Delta^r, \partial\Delta^r) \wedge (\Delta^1, i)$ by left/right horns (for $i = 0, 1$, respectively) (1.4.54), and appeal to fact that left horns lift against right fibrations which send the marked edge (which in this case is $\ell \times \Delta^1$) to a Kan equivalence between fibers (1.4.61). □

1.4.132 Exercise. Fix morphisms of simplicial sets $K \xrightarrow{f} L \xrightarrow{g} M$. Recall that if f is ∞-final, then ∞-finality of g is equivalent to ∞-finality of gf, but that it is not true in general that g and gf being ∞-final implies that f is ∞-final (1.4.124). Conclude from (1.4.131) that this 'missing' case of the 2-out-of-3 property for ∞-final morphisms does hold under the additional assumption that the pullback functor on ∞-categories of right fibrations $g^* : \mathsf{Fun}(M, (\mathsf{sSet}^{\mathsf{R}}_{/-})_{\simeq}) \to \mathsf{Fun}(L, (\mathsf{sSet}^{\mathsf{R}}_{/-})_{\simeq})$ is essentially surjective (for example, if g is a categorical equivalence).

⋆ **1.4.133 Proposition.** *∞-finality is a property of morphisms in* hCat_∞.

Proof. We are to show that whether or not a map of simplicial sets $K \to L$ is ∞-final depends only on its image in $\mathsf{Fun}(\Delta^1, \mathsf{hCat}_\infty)$ (1.4.71).

Let us begin by reducing to the case that K and L are both ∞-categories. In other words, we will show that every map of simplicial sets $K \to L$ is isomorphic in $\mathsf{Fun}(\Delta^1, \mathsf{hCat}_\infty)$ to a map of ∞-categories $K' \to L'$ which is ∞-final iff $K \to L$ is ∞-final. Choose a map $L \hookrightarrow L'$ filtered by pushouts of inner horns where L' is an ∞-category (**??**). Every right fibration over L extends to L' (1.4.62), and $L \hookrightarrow L'$ is ∞-final since it is filtered by pushouts of inner horns (1.4.123), so we conclude that $K \to L$ is ∞-final iff $K \to L'$ is ∞-final. Now choose a map $K \hookrightarrow K'$ filtered by pushuts of inner horns where K' is an ∞-category. Since L' is an ∞-category, the map $K \to L'$ factors as $K \hookrightarrow K' \to L'$. The map $K \hookrightarrow K'$ is ∞-final since it is filtered by pushout of inner horns (1.4.123), so we conclude that $K \to L'$ is ∞-final iff $K' \to L'$ is ∞-final. We are thus reduced to showing that ∞-finality of a functor of ∞-categories depends only on its isomorphism class in $\mathsf{Fun}(\Delta^1, \mathsf{hCat}_\infty)$.

Now let us show that a categorical equivalence is ∞-final. By the previous paragraph, it suffices to show that an equivalence of ∞-categories is ∞-final. Any functor of ∞-categories $\mathsf{A} \to \mathsf{B}$ may be factored into a functor $\mathsf{A} \hookrightarrow \tilde{\mathsf{A}}$ which has a trivial Kan fibration retraction $\tilde{\mathsf{A}} \xrightarrow{\sim} \mathsf{A}$ followed by an isofibration $\tilde{\mathsf{A}} \to \mathsf{B}$. If $\mathsf{A} \to \mathsf{B}$ is an equivalence, then so is $\tilde{\mathsf{A}} \to \mathsf{B}$, hence being an isofibration it is a trivial Kan fibration (1.4.88) thus ∞-final (1.4.128). To show that the functor $\mathsf{A} \hookrightarrow \tilde{\mathsf{A}}$ is ∞-final, consider the composition $\mathsf{A} \to \tilde{\mathsf{A}} \to \mathsf{A}$: the identity $\mathsf{A} \to \mathsf{A}$ is certainly ∞-final, and the second map $\tilde{\mathsf{A}} \to \mathsf{A}$ is ∞-final since it is a trivial Kan fibration (1.4.128), which implies the first map $\mathsf{A} \to \tilde{\mathsf{A}}$ is ∞-final since the second map is a categorical equivalence (1.4.132).

Now it remains to show that if two functors of ∞-categories $F : \mathsf{C} \to \mathsf{D}$ and $F' : \mathsf{C}' \to \mathsf{D}'$ become isomorphic in $\mathsf{Fun}(\Delta^1, \mathsf{hCat}_\infty)$, then F is ∞-final iff F' is ∞-final. The hypothesis that F and F' are isomorphic in $\mathsf{Fun}(\Delta^1, \mathsf{hCat}_\infty)$ means that there exists a diagram

$$\begin{tikzcd} \mathsf{C} \arrow[r, "\sim"] \arrow[d, "F"'] & \mathsf{C}' \arrow[d, "F'"] \\ \mathsf{D} \arrow[r, "\sim"] & \mathsf{D}' \end{tikzcd} \tag{1.4.133.1}$$

commuting up to natural isomorphism of functors, where the horizontal maps are equivalences of ∞-categories (hence ∞-final by the above). Since $\mathsf{C} \to \mathsf{C}'$ is ∞-final, we conclude that $\mathsf{C}' \to \mathsf{D}'$ is ∞-final iff the composition $\mathsf{C} \to \mathsf{C}' \to \mathsf{D}'$ is ∞-final (1.4.124). Since $\mathsf{D} \to \mathsf{D}'$ is ∞-final and an equivalence, we conclude that $\mathsf{C} \to \mathsf{D}$ is ∞-final iff the composition $\mathsf{C} \to \mathsf{D} \to \mathsf{D}'$ is ∞-final (1.4.132). We are thus faced with checking that for two isomorphic functors $\mathsf{C} \to \mathsf{D}'$, one is ∞-final iff the other is. Given a right fibration $\mathsf{E} \to \mathsf{D}'$, consider the pullback map on spaces of sections over $\mathsf{C} \rightrightarrows \mathsf{C} \times \Delta^1 \to \mathsf{D}'$. If the natural transformation $\Delta^1 \to \mathsf{Fun}(\mathsf{C}, \mathsf{D}')$ is a natural isomorphism, then the two restriction maps $\underline{\mathrm{Sec}}(\mathsf{C} \times \Delta^1, \mathsf{E}) \rightrightarrows \underline{\mathrm{Sec}}(\mathsf{C}, \mathsf{E})$ are trivial Kan fibrations as noted in (1.4.131). □

⋆ **1.4.134 Theorem** (Joyal). *A map $K \to \mathsf{D}$ from a simplicial set K to an ∞-category D is ∞-final iff the slice category $(d \downarrow K) = (d \downarrow \mathsf{D}) \times_{\mathsf{D}} K$ is Kan contractible for every $d \in \mathsf{D}$.*

Proof. We first reduce to the case that $K \to \mathsf{D}$ is a right fibration. Consider any factorization $K \hookrightarrow \tilde{K} \twoheadrightarrow \mathsf{D}$ into a map $K \hookrightarrow \tilde{K}$ filtered by pushouts of right horns and a right fibration $\tilde{K} \to \mathsf{D}$ (**??**). The map $K \hookrightarrow \tilde{K}$ is ∞-final since it is filtered by pushouts of right horns (1.4.123), so we conclude that $K \to \mathsf{D}$ is ∞-final iff $\tilde{K} \to \mathsf{D}$ is ∞-final (1.4.124). The map $(d \downarrow K) \to (d \downarrow \tilde{K})$ is a Kan equivalence since it is the pullback of the left fibration $(d \downarrow \mathsf{D}) \to \mathsf{D}$ under the ∞-final map $K \hookrightarrow \tilde{K}$ (1.4.130); in particular, $(d \downarrow K)$ is Kan contractible iff $(d \downarrow \tilde{K})$ is Kan contractible. We are thus reduced to proving the result for the right fibration $\tilde{K} \to \mathsf{D}$.

Now recall that a right fibration is ∞-final iff it is a trivial Kan fibration (1.4.128) and that a right fibration is a trivial Kan fibration iff its fibers are Kan contractible (1.4.60). It thus suffices to show that for any right fibration $\mathsf{E} \to \mathsf{D}$, the natural inclusion $\mathsf{E}_d \to (d \downarrow \mathsf{E}) = (d \downarrow \mathsf{D}) \times_{\mathsf{D}} \mathsf{E}$ is a Kan equivalence. This holds since the inclusion of the identity map object $\{1_d\} \hookrightarrow (d \downarrow \mathsf{D})$ is ∞-initial (1.4.125) and the pullback of any ∞-initial map along a right fibration is a Kan equivalence (1.4.130). □

1.4.135 Lemma. *The functor $\mathsf{C}_{/c} \to \mathsf{C}$ preserves and lifts (hence also reflects) limits over Kan contractible indexing diagrams.*

Proof. Given a diagram $K \to \mathsf{C}_{/c}$, the slice ∞-category $(\mathsf{C}_{/c})_{/K}$ governing its limit coincides with the slice ∞-category $\mathsf{C}_{/K^{\triangleright}}$ governing the limit of the corresponding diagram $K^{\triangleright} \to \mathsf{C}$. So, we must show that the functor $\mathsf{C}_{/K^{\triangleright}} \to \mathsf{C}_{/K}$ preserves and lifts final objects. To do this, it suffices to show that $K \to K^{\triangleright}$ is ∞-initial (**??**).

To show that $K \to K^{\triangleright}$ is ∞-initial, we appeal to the slice category criterion (1.4.134); note that we may assume wlog that K is an ∞-category (the statement we are trying to prove is unchanged by attaching inner horns to K). The slice $(K \downarrow a)$ for $a \in K$ is Kan contractible, while for a the cone point, the slice $(K \downarrow a)$ is K itself, which is Kan contractible by hypothesis. □

Cartesian and cocartesian fibrations

Cocartesian fibrations are a generalization of left fibrations. While left fibrations encode a diagram of spaces, cocartesian fibrations encode a diagram of ∞-categories.

⋆ **1.4.136 Definition** (Cartesian fibration)**.** A map of simplicial sets $X \to Y$ is called a *cocartesian fibration* when it is an inner fibration and every left horn (Δ^1, Λ^1_0) lifts to an edge in X which is *cocartesian over* Y. An edge e in X is called cocartesian over Y when $X \to Y$ satisfies the right lifting property with respect to left horns (Δ^n, Λ^n_0) (with $n > 1$) with 01 edge mapping to e. The term *cartesian* is dual to cocartesian: cartesian for $X \to Y$ means cocartesian for $X^{\mathsf{op}} \to Y^{\mathsf{op}}$.

1.4.137 Exercise. Show that $X \to Y$ is a cocartesian fibration iff there exists a marking X^+ of X in the sense of (1.4.50) such that $X^+ \to Y^\#$ is a marked left fibration (1.4.51) (satisfies the right lifting property with respect to marked left horns), and that in this case the marked edges are cocartesian (though cocartesian edges need not be marked). We denote by $X^{\text{cocart}} \to Y^{\#}$ (or $X^{\text{cocart}/Y} \to Y^{\#}$ to be precise) the marking of X by all edges which are cocartesian over Y.

1.4.138 Exercise (Transport maps of a (co)cartesian fibration)**.** Generalize (1.4.59) and construct for any cocartesian fibration $X \to Y$ a functor $Y \to \mathsf{hCat}_\infty$.

1.4.139 Exercise (Cartesian edges as representing objects)**.** Let $\mathsf{E} \to \Delta^1$ be an inner fibration with fibers C and D over $0, 1 \in \Delta^1$, respectively. Show that an edge $e : c \to d$ in E is cartesian over Δ^1 iff it represents the right fibration $\mathsf{C}_{/d} = \mathsf{C} \times_{\mathsf{E}} \mathsf{E}_{/d}$ over C (1.4.105) (note that both conditions are equivalent to $\mathsf{C}_{/e} \to \mathsf{C}_{/d}$ being a trivial Kan fibration).

1.4.140 Exercise (Kanification diagram $\underline{\mathsf{sSet}} \to \mathsf{Spc}$)**.** A Kanification functor $\wedge : \underline{\mathsf{sSet}} \to \mathsf{Spc}$ was defined in (1.4.24). We now argue that this functor (together with the natural transformation $1 \to \wedge$) is well defined up to contractible choice.

Consider the simplicial category $\Delta^1 \times \underline{\mathsf{sSet}}$ and its full subcategory E consisting of pairs $(i \in \{0, 1\}, X \in \underline{\mathsf{sSet}})$ where if $i = 1$ then X is Kan. The map $\mathsf{E} \to \Delta^1$ now has fibers $\mathsf{E}_0 = \underline{\mathsf{sSet}}$ and $\mathsf{E}_1 = \underline{\mathsf{sSet}}^{\mathsf{Kan}} = \mathsf{Spc}$ and satisfies the right lifting property for inner horns whose image in Δ^1 contains the vertex $1 \in \Delta^1$. Show that an edge $(0, X) \to (1, Y)$ of E is cocartesian if $X \to Y$ is a Kan equivalence (note that it is equivalent to show that $X \to Y$ corepresents the left fibration over $\underline{\mathsf{sSet}}^{\mathsf{Kan}}$ associated to $\underline{\mathrm{Hom}}(X, -)$ (1.4.139), which can be checked at the level of enriched homotopy categories (1.4.106), and that taking simplicial nerve preserves the enriched homotopy category (**??**)). Conclude that a cocartesian transport map $\underline{\mathsf{sSet}} \times \Delta^1 \to \mathsf{E}$ over Δ^1 (which is unique up to contractible choice) is the same thing as a functor (on simplicial nerves) $\wedge : \underline{\mathsf{sSet}} \to \underline{\mathsf{sSet}}^{\mathsf{Kan}}$ together with a natural transformation $1 \to \wedge$ which specialized to any simplicial set K is a Kan equivalence from K to a Kan complex.

1.4.141 Exercise. Let $K \hookrightarrow L$ be an injection of simplicial sets which admits a filtration by simplices $(\Delta^r, \partial\Delta^r)$ whose initial vertices $0 \in \Delta^r$ do not lie in K (for example, $K \hookrightarrow K^{\triangleleft}$ has this property). Show that for any ∞-category C, the restriction map $\underline{\mathrm{Hom}}(L, \mathsf{C}) \to \underline{\mathrm{Hom}}(K, \mathsf{C})$ is a cocartesian fibration, where the cocartesian edges are the maps $L \times \Delta^1 \to \mathsf{C}$ which send edges $\ell \times \Delta^1$ to isomorphisms in C for vertices $\ell \in L \setminus K$. More generally, consider a cocartesian fibration $X \to Y$, and show that $\underline{\mathrm{Hom}}(L, X) \to \underline{\mathrm{Hom}}(L, Y) \times_{\underline{\mathrm{Hom}}(K,Y)} \underline{\mathrm{Hom}}(K, X)$ is a cocartesian fibration, where an edge $L \times \Delta^1 \to X$ is cocartesian when each edge $\ell \times \Delta^1 \to X$ is cocartesian over Y for $\ell \in L \setminus K$.

1.4.142 Lemma. *Let $f : \mathsf{C} \to \mathsf{D}$ be a cocartesian fibration of ∞-categories. The map $\mathrm{Hom}_{\mathsf{C}}(x, y) \to \mathrm{Hom}_{\mathsf{D}}(f(x), f(y))$ is modelled by a Kan fibration whose fibers are canonically homotopy equivalent to morphism spaces $\mathrm{Hom}_{f^{-1}(f(y))}(-, y)$ in the fiber of f over $f(y)$. In particular, a final object in the fiber over a final object of D is a final object in C* (1.4.101).

Proof. We consider the map $\mathrm{Hom}^{\mathsf{L}}_{\mathsf{C}}(x, y) \to \mathrm{Hom}^{\mathsf{L}}_{\mathsf{D}}(f(x), f(y))$. Recall that a map $Z \to \mathrm{Hom}^{\mathsf{L}}_{\mathsf{C}}(x, y)$ is a map $Z^{\triangleleft} \to \mathsf{C}$ sending $* \mapsto x$ and $Z \mapsto y$. Lifting a horn (Δ^n, Λ^n_i) against $\mathrm{Hom}^{\mathsf{L}}_{\mathsf{C}}(x, y) \to \mathrm{Hom}^{\mathsf{L}}_{\mathsf{D}}(f(x), f(y))$ amounts to lifting $(*, \varnothing) \star (\Delta^n, \Lambda^n_i)$ against $\mathsf{C} \to \mathsf{D}$. For any left horn (Δ^n, Λ^n_i), the join $(*, \varnothing) \star (\Delta^n, \Lambda^n_i)$ is an inner horn (1.4.42), hence lifts. Thus

$\mathrm{Hom}^{\mathsf{L}}_{\mathsf{C}}(x,y) \to \mathrm{Hom}^{\mathsf{L}}_{\mathsf{D}}(f(x),f(y))$ is a left fibration (using just the fact that $\mathsf{C} \to \mathsf{D}$ is an inner fibration). Since the base is a Kan complex (1.4.47), this map is in fact a Kan fibration (1.4.61).

Now let us identity the fibers of $\mathrm{Hom}^{\mathsf{L}}_{\mathsf{C}}(x,y) \to \mathrm{Hom}^{\mathsf{L}}_{\mathsf{D}}(f(x),f(y))$. Choose a point of $\mathrm{Hom}^{\mathsf{L}}_{\mathsf{D}}(f(x),f(y))$, namely an edge $f(x) \to f(y)$ in D, and denote by $e : x \to \bar{x}$ its cocartesian lift with initial vertex x. Now a map from Z to the fiber over our chosen point $f(e) \in \mathrm{Hom}^{\mathsf{L}}_{\mathsf{D}}(f(x),f(y))$ is a diagram of the following shape.

$$\begin{tikzcd} Z^{\triangleleft} \arrow[r, "{\substack{*\mapsto x \\ Z\mapsto y}}"] \arrow[d, "{\substack{*\mapsto 0 \\ Z\mapsto 1}}"'] & \mathsf{C} \arrow[d, "f"] \\ \Delta^1 \arrow[r, "f(e)"] & \mathsf{D} \end{tikzcd} \tag{1.4.142.1}$$

Consider the simplicial set parameterizing diagrams of the following shape.

$$\begin{tikzcd} (Z^{\triangleleft})^{\triangleleft} \arrow[r, "{\substack{*^{\triangleleft}\mapsto e \\ Z\mapsto y}}"] \arrow[d, "{\substack{*\mapsto 0 \\ Z^{\triangleleft}\mapsto 1}}"'] & \mathsf{C} \arrow[d, "f"] \\ \Delta^1 \arrow[r, "f(e)"] & \mathsf{D} \end{tikzcd} \tag{1.4.142.2}$$

Forgetting down to $(Z \subseteq Z^{\triangleleft})^{\triangleleft}$ defines a map to the fiber we are trying to compute, and this map is a trivial Kan fibration since $(\Delta^1, 0) \star (\Delta^k, \partial\Delta^k)$ is $(\Delta^{k+2}, \Lambda^{k+2}_0)$ with marked edge mapping to e, which is cocartesian. Forgetting down to (the other) $Z^{\triangleleft} \subseteq (Z^{\triangleleft})^{\triangleleft}$ defines a map to $\mathrm{Hom}^{\mathsf{L}}_{f^{-1}(f(y))}(e(1), y)$. This map is a trivial Kan fibration since $(\Delta^1, 1) \star (\Delta^k, \partial\Delta^k)$ is an inner horn $(\Delta^{k+2}, \Lambda^{k+2}_1)$. □

1.4.143 Corollary. *Let $F : \mathsf{C} \to \mathsf{D}$ be a cocartesian fibration of ∞-categories. If $F(c) \in \mathsf{D}$ is final and $c \in F^{-1}(F(c))$ is final, then $c \in \mathsf{C}$ is final.*

Proof. Let $x \in \mathsf{C}$. The map $\mathrm{Hom}_{\mathsf{C}}(x,c) \to \mathrm{Hom}_{\mathsf{D}}(F(x),F(c))$ is a Kan fibration whose fibers are morphism spaces $\mathrm{Hom}_{F^{-1}(F(c))}(-, F(c))$ (1.4.142), and an object is final iff the space of morphisms to it is contractible (1.4.101). □

1.4.144 Corollary. *Let $\mathsf{E} \to \mathsf{C}$ and $\mathsf{E}' \to \mathsf{C}$ be cocartesian fibrations over an ∞-category C. A map $\mathsf{E} \to \mathsf{E}'$ over C which sends cocartesian edges to cocartesian edges and is an equivalence on each fiber is an equivalence.*

Proof. It suffices to show that $\mathsf{E} \to \mathsf{E}'$ is fully faithful and essentially surjective (1.4.89). Essential surjectivity follows immediately from the fact that $\mathsf{E} \to \mathsf{E}'$ is an equivalence on fibers. To show full faithfulness, note that $\mathrm{Hom}_{\mathsf{E}}(x,y)$ fibers over $\mathrm{Hom}_{\mathsf{C}}(x,y)$ with fibers given by morphism spaces in the fibers of $\mathsf{E} \to \mathsf{C}$ (1.4.142). The identification of the fibers of $\mathrm{Hom}_{\mathsf{E}} \to \mathrm{Hom}_{\mathsf{C}}$ and $\mathrm{Hom}_{\mathsf{E}'} \to \mathrm{Hom}_{\mathsf{C}}$ with morphism spaces in fibers of $\mathsf{E} \to \mathsf{C}$ and $\mathsf{E}' \to \mathsf{C}$ is compatible with the map $\mathsf{E} \to \mathsf{E}'$ since this map sends cocartesian edges to cocartesian edges. It follows that $\mathrm{Hom}_{\mathsf{E}}(x,y) \to \mathrm{Hom}_{\mathsf{E}'}(x',y')$ is a map of Kan fibrations over the Kan complex $\mathrm{Hom}_{\mathsf{C}}(x,y)$ which is a fiberwise homotopy equivalence, hence it is a homotopy equivalence (**??**). □

1.4.145 Exercise. Let $F : \mathsf{C} \to \mathsf{D}$ be a functor, and consider the simplicial set $(\mathsf{C} \downarrow^F \mathsf{D})$ representing the functor sending $Z \in \mathsf{sSet}$ to the set of diagrams:

$$\begin{tikzcd} Z \arrow[r] \arrow[d, "\times 0"] & \mathsf{C} \arrow[d, "F"] \\ Z \times \Delta^1 \arrow[r] & \mathsf{D} \end{tikzcd} \tag{1.4.145.1}$$

Show that the evident (evaluate at $1 \in \Delta^1$) projection $(\mathsf{C} \downarrow^F \mathsf{D}) \to \mathsf{D}$ is a cocartesian fibration, and show that the projection $(\mathsf{C} \downarrow^F \mathsf{D}) \to \mathsf{C}$ is a cartesian fibration (use (1.4.47) and (1.4.54)).

1.4.146 Lemma (Obstruction theory for sections of a cocartesian fibration)**.** *Let $E \to \Delta^n$ be a cocartesian fibration. The inclusion of the fiber $E_n \subseteq E^{\mathrm{cocart}}$ is a marked categorical equivalence, and there exists a retraction $q : E^{\mathrm{cocart}} \to E_n^\natural$. For any section $s : \partial\Delta^n \to E$, composition with any marked categorical equivalence from E^{cocart} to an ∞-category (such as a retraction $q : E \to E_n$ sending cocartesian edges to isomorphisms) induces a bijection between isomorphism classes of extensions of s and isomorphism classes of extensions of $p \circ s$.*

Proof. The special case of left fibrations was proven in (1.4.60). To adapt that argument to cocartesian fibrations requires little more than keeping track of markings appropriately. □

1.4.147 Lemma. *If $\mathsf{E} \to \Delta^n$ is a cocartesian fibration, then $\mathsf{E} \times_{\Delta^n} (\Delta^n, \Lambda^n_n)$ is a categorical equivalence if we mark the cocartesian edges over $(n-1, n) \subseteq \Delta^n$.*

Proof. The pair $(\Lambda^n_n, \Delta^{[n]\setminus\{n-1\}} \vee \Delta^{\{n-1,n\}})$ is filtered by pushouts of right horns of dimension $< n$ with boundary edge $(n-1, n)$ (right cone a filtration of $(\Delta^{[n]\setminus n}, \Delta^{[n]\setminus\{n-1,n\}} \sqcup \{n-1\})$ by simplices), so by induction it suffices to show that $\mathsf{E} \times_{\Delta^n} (\Delta^n, \Delta^{[n]\setminus\{n-1\}} \vee \Delta^{\{n-1,n\}})$ is a categorical equivalence when we mark the cocartesian edges over $(n-1, n)$. It suffices to show that $\mathsf{E} \times_{\Delta^n} (\Delta^n, \Delta^{[n]\setminus\{n-1\}})$ and $\mathsf{E} \times_{\Delta^n} (\Delta^{[n]\setminus\{n-1\}} \vee \Delta^{\{n-1,n\}}, \Delta^{[n]\setminus\{n-1\}})$ are both categorical equivalences (with these same marked edges). Now these pairs both admit marked deformation retractions (**??**) obtained as in (1.4.146) by cocartesian transport over the deformation retraction $\Delta^n \times \Delta^1 \to \Delta^n$ from the identity 1_{Δ^n} at $0 \in \Delta^1$ to the retraction $\Delta^n \to \Delta^{[n]\setminus\{n-1\}}$ at $1 \in \Delta^1$. □

1.4.148 Definition (Cartesian functor)**.** A functor $F : \mathsf{C} \to \mathsf{D}$ is called *cartesian* iff for every object $c \in \mathsf{C}$ and every morphism $d \to F(c)$ in D, the right fibration $\mathsf{C}_{/c} \times_{\mathsf{D}_{/F(c)}} \mathsf{D}_{/(d \to F(c))} \to \mathsf{C}$ (1.4.149) is representable and the map from the image in D of its representing object to d is an isomorphism.

1.4.149 Exercise. Use (1.4.45) to show that $\mathsf{C}_{/c} \times_{\mathsf{D}_{/F(c)}} \mathsf{D}_{/(d \to F(c))} \to \mathsf{C}$ is a right fibration.

1.4.150 Lemma. *A morphism $c \to c'$ in C is cartesian with respect to $F : \mathsf{C} \to \mathsf{D}$ iff the diagram*

$$\begin{tikzcd} c' \arrow[r] \arrow[d] & c \arrow[d] \\ F^*F(c') \arrow[r] & F^*F(c) \end{tikzcd} \tag{1.4.150.1}$$

is a pullback square in $\mathsf{P}(\mathsf{C})$*, where* $F^* : \mathsf{P}(\mathsf{D}) \to \mathsf{P}(\mathsf{C})$ *denotes pullback of presheaves and we implicitly apply Yoneda functors.*

Proof. □

1.4.151 Lemma. *Let* $F : \mathsf{C} \to \mathsf{D}$ *be a functor, and fix a diagram*

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & Y' \arrow[d] \\ X \arrow[r] & Y \end{tikzcd} \tag{1.4.151.1}$$

in C *whose image under* F *is a pullback and whose bottom arrow* $X \to Y$ *is cartesian. In this case, the diagram* (1.4.151.1) *is a pullback iff* $X' \to Y'$ *is cartesian.*

Proof. Consider the diagrams (1.4.150.1) associated to the morphisms $X \to Y$ and $X' \to Y'$, which fit together into a cube.

$$\begin{tikzcd} X' \arrow[rr] \arrow[dr] \arrow[dd] & & Y' \arrow[dr] \arrow[dd] & \\ & X \arrow[rr] \arrow[dd] & & Y \arrow[dd] \\ F^*F(X') \arrow[rr] \arrow[dr] & & F^*F(Y') \arrow[dr] & \\ & F^*F(X) \arrow[rr] & & F^*F(Y) \end{tikzcd} \tag{1.4.151.2}$$

The assumptions that $X \to Y$ is cartesian and that F(1.4.151.1) is a pullback imply that two faces of this cube are pullbacks. By cancellation for fiber products (1.1.57), $X' \to Y'$ being cartesian and (1.4.151.1) being a pullback are both equivalent to the composite square

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & Y' \arrow[d] \\ F^*F(X) \arrow[r] & F^*F(Y) \end{tikzcd} \tag{1.4.151.3}$$

being a pullback. □

1.4.152 Lemma. *Let* $F : \mathsf{C} \to \mathsf{D}$ *be cartesian and suppose* D *has pullbacks. Then cartesian morphisms are preserved under pullback, and* F *sends pullbacks of cartesian morphisms to pullbacks in* D*.*

Proof. Fix a cartesian morphism $X \to Y$ and an arbitrary morphism $Y' \to Y$. Define a cartesian morphism $X' \to Y'$ as the cartesian lift of $F(X) \times_{F(Y)} F(Y') \to F(Y)$. There is a morphism $X' \to X$ completing the diagram (1.4.151.1) since $X \to Y$ is cartesian. Now by construction $X' \to Y'$ is cartesian and F(1.4.151.1) is a pullback, so (1.4.151.1) is a pullback (1.4.151). By construction, the morphism $X' \to Y'$ is cartesian and the image F(1.4.151.1) is a pullback. □

Relative limits and colimits

1.4.153 Definition (Relative limit)**.** Let $X \to Y$ be an inner fibration. Given a simplicial set K and a diagram of solid arrows

$$\begin{tikzcd} K \arrow[r] \arrow[d] & X \arrow[d] \\ K^{\triangleleft} \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd} \tag{1.4.153.1}$$

we can consider the simplicial set of dotted lifts. This is an ∞-category since $(K^{\triangleleft}, K) \wedge (\Delta^n, \Lambda^n_i)$ is filtered by pushouts of inner horns when $0 < i < n$ (1.4.20). A final object in this ∞-category is called the *relative limit* of the diagram.

1.4.154 Exercise (Relative limits as limits in fibers)**.** Let $X \to Y$ be a cartesian fibration, and fix a relative limit problem.

$$\begin{tikzcd} K \arrow[r] \arrow[d] & X \arrow[d] \\ K^{\triangleleft} \arrow[r] & Y \end{tikzcd} \tag{1.4.154.1}$$

Now pull back the bottom map under the retraction $(K \times \Delta^1)^{\triangleleft} \to (K \times 1)^{\triangleleft} = K^{\triangleleft}$ which sends $(K \times 0)^{\triangleleft}$ to the cone point.

$$\begin{tikzcd} K \times 1 \arrow[r] \arrow[d] & X \arrow[d] \\ (K \times \Delta^1)^{\triangleleft} \arrow[r] & Y \end{tikzcd} \tag{1.4.154.2}$$

Fix a lift over $K \times \Delta^1$ which sends edges $k \times \Delta^1$ to cartesian edges ($K \wedge (\Delta^1, 1)^\sharp$ is filtered by pushouts of right marked horns (1.4.54)).

$$\begin{tikzcd} K \times \Delta^1 \arrow[r] \arrow[d] & X \arrow[d] \\ (K \times \Delta^1)^{\triangleleft} \arrow[r] & Y \end{tikzcd} \tag{1.4.154.3}$$

This diagram (or, by abuse of terminology, the map $K \to X_y$ obtained by restricting its top map to $K = K \times 0 \subseteq K \times \Delta^1$, where $y \in Y$ is the image of the basepoint $* \in K^{\triangleleft}$ in Y) is called the *cartesian transport* (show that it is unique up to contractible choice) of the input diagram (1.4.154.1).

Now let us argue that the relative limit of (1.4.154.1) coincides (up to contractible choice) with the limit of its cartesian transport (in particular, if one exists, then the other does). Consider the ∞-categories of lifts of the transport (1.4.154.3) to $(K \times 0)^{\triangleleft}$, $(K \times 1)^{\triangleleft}$, and $(K \times \Delta^1)^{\triangleleft}$, which are related by forgetful functors. Show that these forgetful functors are trivial Kan fibrations (filter the pairs $(*, \varnothing) \star (K \times (\Delta^1, 0))$ and $(*, \varnothing) \star (K \times (\Delta^1, 1))$), and draw the desired conclusion.

1.4.155 Lemma (Transitivity of relative limits)**.** *Let $X \to Y$ be a cocartesian fibration, let $Y \to Z$ be an inner fibration, and consider a diagram of the following shape.*

$$\begin{tikzcd} K \arrow[r] \arrow[dd] & X \arrow[d] \\ & Y \arrow[d] \\ K^{\triangleleft} \arrow[r, "h"] \arrow[ur, "g"] \arrow[uur, "f"] & Z \end{tikzcd} \tag{1.4.155.1}$$

If f is a limit relative g and g is a limit relative h, then f is a limit relative h.

Proof. The map from the ∞-category of lifts to X to the ∞-category of lifts to Y is a cocartesian fibration (1.4.141). Now apply (1.4.143). $\square$

1.4.156 Definition (Relative functor category)**.** Let $X \to Y$ be a map of simplicial sets, and let C be an ∞-category. The *relative functor category* $\mathsf{Fun}_Y(X, \mathsf{C})$ is the simplicial set defined by the universal property that a map $Z \to \mathsf{Fun}_Y(X, \mathsf{C})$ is a pair of maps $Z \to Y$ and $X \times_Y Z \to \mathsf{C}$.

Formation of the relative functor category is compatible with pullback: if $X' \to Y'$ is a pullback of $X \to Y$, then the natural map $\mathsf{Fun}_{Y'}(X', \mathsf{C}) \to \mathsf{Fun}_Y(X, \mathsf{C}) \times_Y Y'$ is an isomorphism. In the case $Y = *$, the relative functor category reduces to the usual functor category $\mathsf{Fun}(X, \mathsf{C})$. The fiber of the map $\mathsf{Fun}_Y(X, \mathsf{C}) \to Y$ over a point $y \in Y$ is thus the functor category $\mathsf{Fun}(X_y, \mathsf{C})$.

1.4.157 Lemma (Lurie [74, 3.2.2.12])**.** *If $Q \to B$ is a cocartesian fibration and E is an ∞-category, then $\mathsf{Fun}_B(Q, \mathsf{E}) \to B$ is a cartesian fibration, and an edge $\Delta^1 \to \mathsf{Fun}_B(Q, \mathsf{E})$ is cartesian iff it sends cocartesian edges of $Q \times_B \Delta^1 \to \Delta^1$ to isomorphisms in E.*

Proof. A lifting problem for a horn (Δ^n, Λ^n_i) against $\mathsf{Fun}_B(Q, \mathsf{E}) \to B$ amounts to an extension problem for maps $Q \times_B (\Delta^n, \Lambda^n_i) \to \mathsf{E}$. Since the restriction functor $\mathsf{Fun}(Q \times_B \Delta^n, \mathsf{E}) \to \mathsf{Fun}(Q \times_B \Lambda^n_i, \mathsf{E})$ is an isofibration of ∞-categories (1.4.82), a map $Q \times_B \Lambda^n_i \to \mathsf{E}$ extends to $Q \times_B \Delta^n$ iff it lies in the essential image of the restriction functor (1.4.81). If (Δ^n, Λ^n_i) is inner $(0 < i < n)$, then $Q \times_B (\Delta^n, \Lambda^n_i)$ is a categorical equivalence (1.4.170), so the restriction map is an equivalence of ∞-categories, and hence every extension problem $Q \times_B (\Delta^n, \Lambda^n_i) \to \mathsf{E}$ has a solution.

It remains to address the case of right horns (Δ^n, Λ^n_n), where we would like to show that the restriction functor $\mathsf{Fun}(Q \times_B \Delta^n, \mathsf{E})' \to \mathsf{Fun}(Q \times_B \Lambda^n_i, \mathsf{E})'$ is essentially surjective, where the superscript $'$ indicates those functors which send cocartesian edges in $Q \times_B \Delta^n$ over the edge $(n-1, n) \subseteq \Delta^n$ to isomorphisms in E. In fact, it is an equivalence since $Q \times_B (\Delta^n, \Lambda^n_n)$ becomes a categorical equivalence upon marking the cocartesian edges over $(n-1, n)$ (1.4.147). $\square$

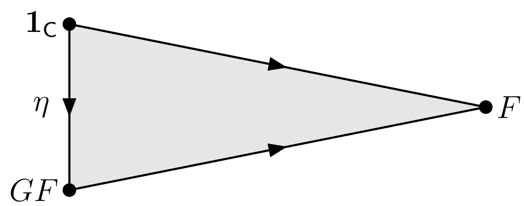
Adjoint functors

We now discuss adjoint functors.

⋆ **1.4.158 Definition** (Adjunction)**.** An *adjunction* of functors of ∞-categories is a cartesian and cocartesian fibration $\mathsf{E} \to \Delta^1$. Given an adjunction $\mathsf{E} \to \Delta^1$, we have ∞-categories $\mathsf{C} = \mathsf{E}_0$ and $\mathsf{D} = \mathsf{E}_1$ and an *adjoint pair* (F, G) of functors $F : \mathsf{C} \rightleftarrows \mathsf{D} : G$ defined (uniquely up to contractible choice) by cocartesian and cartesian lifting along $\mathsf{E} \to \Delta^1$, respectively. We may, informally, refer to an adjunction of ∞-categories by naming the adjoint pair (F, G) rather than the underlying cartesian and cocartesian fibration.

The composite ∞-category E is recoverable (up to equivalence) from either F or G (1.4.164) (**??**)(1.4.144). Hence if a functor has a left or a right adjoint, this adjoint is determined uniquely up to contractible choice.

⋆ **1.4.159 Definition** (Unit of an adjunction)**.** Let $(\mathsf{E} \to \Delta^1, F : \mathsf{C} \rightleftarrows \mathsf{D} : G)$ be an adjunction. The *unit* transformation $\eta : \mathbf{1}_{\mathsf{C}} \to GF$ is defined by mapping $\mathsf{C} \times \Delta^2 \to \mathsf{E}$ so that $\mathsf{C} \times 0$ is identity, $\mathsf{C} \times (02)$ is cocartesian, and $\mathsf{C} \times (12)$ is cartesian (such a map is unique up to contractible choice by the filtration $\mathsf{C} \times (0 \subseteq 02 \subseteq (02 \cup 12) \subseteq \Delta^2)$).



(1.4.159.1)

Dually, there is the *counit* map $\varepsilon : GF \to \mathbf{1}_{\mathsf{D}}$.

The mapping simplex

We now relate cocartesian fibrations over Δ^n with sequences of maps ∞-categories $\phi(0) \to \cdots \to \phi(n)$, using the mapping simplex (1.4.160) and categorical mapping simplex (1.4.164) constructions. We then use these to study cocartesian fibrations over a simplex (1.4.167) (1.4.168)(1.4.170). This discussion may be viewed as a preparatory setup for the Straightening Equivalence (1.4.179), although it has other uses as well. A similar discussion is found in Lurie [74, §3.2.2].

⋆ **1.4.160 Definition** (Mapping simplex)**.** Let $\phi = (\phi(0) \to \cdots \to \phi(n)) : \Delta^n \to \mathsf{sSet}$ be a diagram of simplicial sets. The *mapping simplex* $M(\phi)$ of ϕ is the simplicial set over Δ^n in which a k-simplex is a pair of maps $f : \Delta^k \to \Delta^n$ and $g : \Delta^k \to \phi(f(0))$. It is often useful to equip the mapping simplex with the marking consisting of those edges $(f, g) : \Delta^1 \to M(\phi)$ whose associated edge $g : \Delta^1 \to \phi(f(0))$ is degenerate.

1.4.161 Exercise. Show that the mapping simplex $M(A \xrightarrow{f} B)$ is the pushout $(A \times \Delta^1) \cup^f_{A \times 1} B$ (i.e. the 'mapping cylinder'). Show that the formation of mapping simplices is compatible with pullback: $M(\phi \times_{\Delta^n} \Delta^m) \xrightarrow{\sim} M(\phi) \times_{\Delta^n} \Delta^m$. Show that there is a canonical pushout diagram

$$\begin{tikzcd} \phi(0) \times \Delta^{[n]-0} \arrow[r] \arrow[d, hook] & M(\phi \times_{\Delta^n} \Delta^{[n]-0}) \arrow[d, hook] \\ \phi(0) \times \Delta^n \arrow[r] & M(\phi) \end{tikzcd} \tag{1.4.161.1}$$

and that the same holds for the marked mapping simplices when we equip Δ^n and $\Delta^{[n]-0}$ with their markings $(\Delta^n)^\#$ and $(\Delta^{[n]-0})^\#$. Conclude that the pair $M(\phi) \times_{\Delta^n} (\Delta^n, \Delta^{\{0,1\}} \vee \cdots \vee \Delta^{\{n-1,n\}})$ is filtered by pushouts of pairs $\phi(i) \times (\Delta^{\{i,\ldots,n\}}, \Delta^{\{i,i+1\}} \vee \Delta^{\{i+1,\ldots,n\}})$ each of which is filtered by pushouts of inner horns. Since the vertical maps in the diagrams above are cofibrations, these diagrams remain pushouts in Cat_∞ (**??**).

1.4.162 Lemma. *For $\phi \to \phi'$ a morphism of diagrams $\Delta^n \to \mathsf{sSet}$ in which each $\phi_i \to \phi'_i$ is a categorical equivalence, the induced map $M(\phi) \to M(\phi')$ is a categorical equivalence.*

Proof. Since $M(\phi) \times_{\Delta^n} (\Delta^n, \Delta^{\{0,1\}} \vee \cdots \vee \Delta^{\{n-1,n\}})$ is filtered by pushouts of inner horns (1.4.161), it is a categorical equivalence. It thus suffices to show that $M(\phi) \to M(\phi')$ restricts to a categorical equivalence over $\Delta^{\{0,1\}} \vee \cdots \vee \Delta^{\{n-1,n\}}$. Now an injective pushout of simplicial sets is a pushout in Cat_∞ (**??**), so an injective pushout of categorical equivalences is a categorical equivalence. For $n \geq 2$, express $\Delta^{\{0,1\}} \vee \cdots \vee \Delta^{\{n-1,n\}}$ as the pushout of $\Delta^{\{0,1\}} \vee \cdots \vee \Delta^{\{n-2,n-1\}}$ over $\{n-1\}$ with $\Delta^{\{n-1,n\}}$. Pulling back to $M(\phi) \to M(\phi')$ and applying induction, we reduce to the case $n = 1$. For $n = 1$, the mapping cylinder $M(\phi)$ is the injective pushout $(\phi_0 \times \Delta^1) \cup_{\phi_0 \times 1} \phi_1$, so we are done by appealing again to the fact that an injective pushout of categorical equivalences is a categorical equivalence. □

1.4.163 Lemma. *Let $\mathsf{E} \to \Delta^n$ be a cocartesian fibration. There exist maps $\mathsf{E}_0 \to \mathsf{E}_1 \to \cdots \to \mathsf{E}_n$ and a map $M(\mathsf{E}_0 \to \cdots \to \mathsf{E}_n) \to \mathsf{E}$ over Δ^n which is the identity on fibers and which sends marked edges to cocartesian edges.*

Proof. Construct a map $\xi_i : \mathsf{E}_i \times \Delta^1 \to \mathsf{E}$ over the $(i, i+1)$ edge of Δ^n which sends each edge $e \times \Delta^1$ to an edge of E which is cocartesian over Δ^n (that is, it is a map of marked simplicial sets $\mathsf{E}_i \times (\Delta^1)^\# \to \mathsf{E}^{\mathrm{cocart}/\Delta^n}$). Each ξ_i determines a map $M(\xi_i(\cdot, 1) : \mathsf{E}_i \to \mathsf{E}_{i+1}) \to \mathsf{E}$ over $\Delta^{\{i,i+1\}}$, and these maps fit together into a map $M(\mathsf{E}_0 \to \cdots \to \mathsf{E}_n) \to \mathsf{E}$ over $\Delta^{\{0,1\}} \vee \cdots \vee \Delta^{\{n-1,n\}} \subseteq \Delta^n$. To extend this map to all of Δ^n, it suffices to note that the pair $M(\mathsf{E}_0 \to \cdots \to \mathsf{E}_n) \times_{\Delta^n} (\Delta^n, \Delta^{\{0,1\}} \vee \cdots \vee \Delta^{\{n-1,n\}})$ is filtered by pushouts of inner horns (1.4.161), which lift against $\mathsf{E} \to \Delta^n$. The resulting map sends marked edges to cocartesian edges since cocartesian edges are closed under composition (**??**). □

It will be important to have an explicit model of the result of attaching marked left horns to a mapping simplex $M(\phi) \to \Delta^n$ to turn it into a cocartesian fibration (with marked edges becoming cocartesian). This is accomplished by introducing the 'categorical mapping simplex' (1.4.164) $\vec{M}(\phi) \to \Delta^n$ and noting that the comparison map $M(\phi) \to \vec{M}(\phi)$ is filtered by pushouts of inner horns (1.4.166) (at least when the maps comprising ϕ are all injective).

⋆ **1.4.164 Definition** (Categorical mapping simplex)**.** Let $\phi = (\phi(0) \to \cdots \to \phi(n)) : \Delta^n \to \mathsf{sSet}$ be a diagram of simplicial sets (however this construction will be most relevant when every $\phi(i)$ is an ∞-category). The *categorical mapping simplex* $\vec{M}(\phi)$ of ϕ is the simplicial set over Δ^n defined by the universal property that a map $Z \to \vec{M}(\phi)$ from an arbitrary simplicial set Z is a map $f : Z \to \Delta^n$ together with maps $f^{-1}(\Delta^{[0\cdots i]}) \to \phi(i)$ which are compatible in the sense that the following diagram commutes for all $i \leq j$:

$$\begin{tikzcd} f^{-1}(\Delta^{[0\cdots i]}) \arrow[r] \arrow[d] & \phi(i) \arrow[d] \\ f^{-1}(\Delta^{[0\cdots j]}) \arrow[r] & \phi(j) \end{tikzcd} \tag{1.4.164.1}$$

There is an evident comparison map $M(\phi) \to \vec{M}(\phi)$ from the mapping simplex to the categorical mapping simplex.

1.4.165 Exercise. Show that the formation of categorical mapping simplices is compatible with pullback: $\vec{M}(\phi \times_{\Delta^n} \Delta^m) \xrightarrow{\sim} \vec{M}(\phi) \times_{\Delta^n} \Delta^m$ for any map $\Delta^m \to \Delta^n$.

Now here is the key technical result comparing the mapping simplex (1.4.160) and the categorical mapping simplex (1.4.164). The proof is somewhat intricate and is not used later except as a black box.

1.4.166 Proposition (Pelle Steffens)**.** *Let* $A = (A_0 \hookrightarrow \cdots \hookrightarrow A_n) : \Delta^n \to \mathsf{sSet}$ *be a sequence of* injections *of simplicial sets. The comparison map* $M(A) \to \vec{M}(A)$ *is filtered by pushouts of inner horns.*

Proof. This proof was communicated to me by Pelle Steffens.

We begin with a concrete description of $M(A)$ and $\vec{M}(A)$ as subcomplexes of $\Delta^n \times A_n$. For a simplex $(g, f) : \Delta^r \to \Delta^n \times A_n$ (consisting of $f : \Delta^r \to A_n$ and $g : \Delta^r \to \Delta^n$), we have:

(1.4.166.1) (g, f) lands in $M(A)$ iff $f(\Delta^r) \subseteq A_{g(0)}$.

(1.4.166.2) (g, f) lands in $\vec{M}(A)$ iff $f(\Delta^{\{0,\ldots,j\}}) \subseteq A_{g(j)}$ for $0 \leq j \leq r$.

This concrete description will be useful at various points in the argument below.

The first step in showing that $(\vec{M}(A), M(A))$ is filtered by pushouts of inner horns is to reduce to checking a simple 'local model'. To make this reduction, we must generalize to a 'relative' situation. Consider an objectwise injection $A \hookrightarrow A'$.

$$\begin{tikzcd} A_0 \arrow[r, hook] \arrow[d, hook] & \cdots \arrow[r, hook] & A_n \arrow[d, hook] \\ A'_0 \arrow[r, hook] & \cdots \arrow[r, hook] & A'_n \end{tikzcd} \qquad \begin{tikzcd} M(A) \arrow[r, hook] \arrow[d, hook] & \vec{M}(A) \arrow[d, hook] \\ M(A') \arrow[r, hook] & \vec{M}(A') \end{tikzcd} \tag{1.4.166.3}$$

The induced map $M(A') \sqcup_{M(A)} \vec{M}(A) \to \vec{M}(A')$ is injective (equivalently, the right square above is a pullback) provided $A_i = A'_i \cap A_n$ for $0 \leq i \leq n$ (inspection using (1.4.166.1) (1.4.166.2)). We call such pairs (A', A) *nice*, and for any nice pair we define

$$(\vec{M}, M)(A', A) = (\vec{M}(A'), M(A') \sqcup_{M(A)} \vec{M}(A)). \tag{1.4.166.4}$$

Note that the pair $(A, \varnothing)$ is always nice and satisfies $(\vec{M}, M)(A, \varnothing) = (\vec{M}(A), M(A))$. Now for nice pairs (A'', A') and (A', A), the composition (A'', A) is nice and $(\vec{M}, M)(A'', A)$ is the composition of $(\vec{M}, M)(A'', A')$ and a pushout of $(\vec{M}, M)(A', A)$. It follows that the set of nice pairs (A', A) for which the pair $(\vec{M}, M)(A', A)$ is filtered by pushouts of inner horns is closed under transfinite composition. Now every nice pair admits a transfinite filtration by *single simplex attachments* (1.3.8), by which we mean (necessarily nice) pairs (A', A) where $A_i = A'_i$ for $i < d$ and the maps $A_i \to A'_i$ for $i \geq d$ are pushouts of the 'same' simplex $(\Delta^a, \partial\Delta^a)$ (any $0 \leq d \leq n$) in the sense that we have the following diagram of pushouts.

$$\begin{tikzcd}
\partial\Delta^a \arrow[r] \arrow[d] & A_d \arrow[r, hook] \arrow[d, hook] & \cdots \arrow[r, hook] & A_n \arrow[d, hook] \\
\Delta^a \arrow[r] & A'_d \arrow[r, hook] & \cdots \arrow[r, hook] & A'_n
\end{tikzcd} \tag{1.4.166.5}$$

It thus suffices to show that $(\vec{M}, M)(A', A)$ is filtered by pushouts of inner horns for any single simplex attachment (A', A).

In fact, it suffices to treat single simplex attachments of the following simple form.

$$\begin{tikzcd}
A_0 \arrow[r, hook] \arrow[d, equal] & \cdots \arrow[r, hook] & A_{d-1} \arrow[r, hook] \arrow[d, equal] & \partial\Delta^a \arrow[r, equal] \arrow[d, hook] & \cdots \arrow[r, equal] & \partial\Delta^a \arrow[d, hook] \\
A'_0 \arrow[r, hook] & \cdots \arrow[r, hook] & A'_{d-1} \arrow[r, hook] & \Delta^a \arrow[r, equal] & \cdots \arrow[r, equal] & \Delta^a
\end{tikzcd} \tag{1.4.166.6}$$

Indeed, for any single simplex attachment (A', A), the pair $(\vec{M}, M)(A', A)$ is a pushout of $(\vec{M}, M)(B', B)$ where (B', B) is the single simplex attachment obtained from $A \hookrightarrow A'$ by pulling back under $\Delta^a \to A'_n$. Here is a sketch of why $(\vec{M}, M)(A', A)$ is a pushout of $(\vec{M}, M)(B', B)$: the square (1.4.166.3) (right) for (B', B) is the pullback of that for (A', A) under the map $\Delta^a \to A'_n$ (inspection of (1.4.166.1)(1.4.166.2)), at which point we show $(\vec{M}, M)(A', A)$ is a pushout of $(\vec{M}, M)(B', B)$ by appealing to (1.1.56), whose hypothesis is satisfied since $\vec{M}(B') \sqcup \vec{M}(A) \to \vec{M}(A')$ is surjective.

Let us now further simplify the situation by reducing a general problem (1.4.166.6) to one in which $A_i = \Delta^{\{0,\ldots,k_i\}}$ for all $i < d$ (some $k_i < d$). Given a general problem (A', A) (1.4.166.6), we consider the maximal prefix $B_i = B'_i = \Delta^{\{0,\ldots,k_i\}} \subseteq A_i = A'_i \subseteq \Delta^a$ for $i < d$ (set $B_i = A_i = \partial\Delta^a$ and $B'_i = A'_i = \Delta^a$ for $i \geq d$). It suffices to show that the map

$$(\vec{M}, M)(B', B) \to (\vec{M}, M)(A', A) \tag{1.4.166.7}$$

is a pushout. Appealing to the pushout criterion (1.1.56), it suffices to show that (1.4.166.7) is a pullback and that $\vec{M}(B') \cup \vec{M}(A) \to \vec{M}(A')$ is surjective. To show that $\vec{M}(B') \cup \vec{M}(A) \to \vec{M}(A')$ is surjective, we note that if $(g, f) : \Delta^r \to \Delta^n \times \Delta^a$ lies in $\vec{M}(A')$ but not in $\vec{M}(A)$, then $f : \Delta^r \to \Delta^a$ must be surjective (1.4.166.2), so $f(\Delta^{\{0,\ldots,j\}}) = \Delta^{\{0,\ldots,f(j)\}}$, and thus the condition (1.4.166.2) that (g, f) be contained in $\vec{M}(A')$ is equivalent to $f(j) \leq k_{g(j)}$, whence (g, f) lies in $\vec{M}(B')$. To show that $(\vec{M}, M)(B', B) \to (\vec{M}, M)(A', A)$ is a pullback, we should show that for $(g, f) \in \vec{M}(B')$, if $(g, f) \in \vec{M}(A) \cup M(A')$ then $(g, f) \in \vec{M}(B) \cup M(B')$.

If $(g,f) \in \vec{M}(A)$, then $f(\Delta^r) \subseteq \partial\Delta^a$, whence $(g,f) \in \vec{M}(B')$ implies $(g,f) \in \vec{M}(B)$. If $(g,f) \in M(A')$ and $(g,f) \notin \vec{M}(A)$, then $f(\Delta^r) = \Delta^a$, whence $(g,f) \in M(A')$ implies $(g,f) \in M(B')$ (1.4.166.2).

We have thus reduced to showing that $(\vec{M}, M)(A', A)$ is filtered by pushouts of inner horns for $A \hookrightarrow A'$ of the following form.

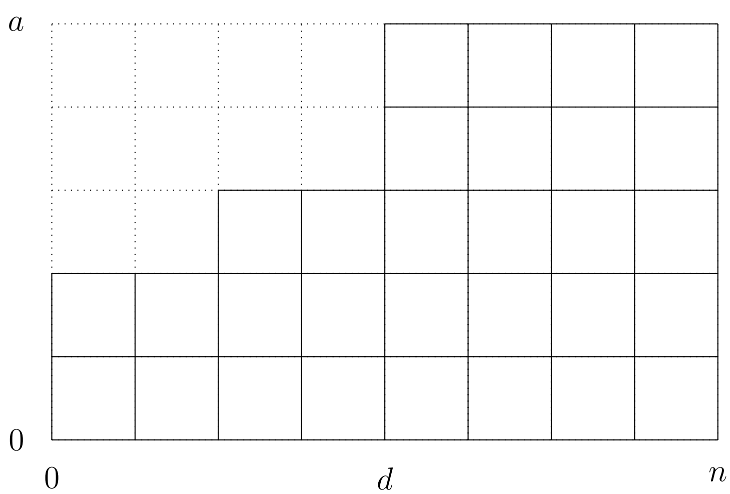
$$\begin{tikzcd}
\Delta^{\{0,\ldots,k_0\}} \arrow[r, hook] \arrow[d, equal] & \cdots \arrow[r, hook] & \Delta^{\{0,\ldots,k_{d-1}\}} \arrow[r, hook] \arrow[d, equal] & \partial\Delta^a \arrow[r, equal] \arrow[d, hook] & \cdots \arrow[r, equal] & \partial\Delta^a \arrow[d, hook] \\
\Delta^{\{0,\ldots,k_0\}} \arrow[r, hook] & \cdots \arrow[r, hook] & \Delta^{\{0,\ldots,k_{d-1}\}} \arrow[r, hook] & \Delta^a \arrow[r, equal] & \cdots \arrow[r, equal] & \Delta^a
\end{tikzcd} \tag{1.4.166.8}$$

Now the pullback of this problem along any simplicial map $\Delta^m \to \Delta^n$ has again the same form, so by filtering Δ^n by simplices $(\Delta^b, \partial\Delta^b)$, we may further reduce to showing that

$$(\vec{M}(A'), M(A') \cup \vec{M}(A) \cup (\vec{M}(A') \times_{\Delta^n} \partial\Delta^n)) \tag{1.4.166.9}$$

is filtered by pushouts of inner horns. Note that we may assume that $d > 0$, since otherwise $M(A') = \vec{M}(A')$ and there is nothing to prove.

We will now define an explicit filtration of (1.4.166.9) by pushouts of inner horns. To begin, let us describe this pair inside $\Delta^n \times \Delta^a$. The category $\vec{M}(A')$ is the full subcategory of $\Delta^n \times \Delta^a$ spanned by pairs (i, c) with $c \leq k_i$ if $i < d$.

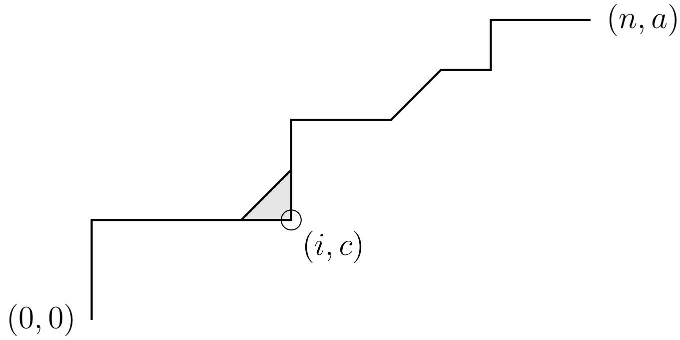


(1.4.166.10)

Now we claim that a simplex of $\vec{M}(A')$ does not lie in $M(A') \cup \vec{M}(A) \cup (\vec{M}(A') \times_{\Delta^n} \partial\Delta^n)$ precisely when it surjects onto both $[n]$ and $[a]$. Being contained in $(\vec{M}(A') \times_{\Delta^n} \partial\Delta^n)$ is equivalent to not surjecting onto $[n]$, and being contained in $\vec{M}(A)$ is equivalent to not surjecting onto $[a]$. As for being contained in $M(A')$, this is a somewhat complicated condition, but at least for simplices which surject onto both $[n]$ and $[a]$, it is equivalent to $\Delta^a \subseteq A'_0$, which is never the case unless $d = 0$ (which we can ignore since in this case $M(A') = \vec{M}(A')$ so there is nothing to prove). Having described $\vec{M}(A')$ and its subset $M(A') \cup \vec{M}(A) \cup (\vec{M}(A') \times_{\Delta^n} \partial\Delta^n)$, we may now define the desired filtration by pushouts of inner horns.

A filtration of a pair (X, B) by pushouts of inner horns induces a pairing of the non-degenerate simplices of X not contained in B (each inner horn attachment adds precisely two new non-degenerate simplices). We call the larger simplex of such a pair the *primary* simplex and the other the *secondary* one. To define our filtration of $(\vec{M}(A'), M(A') \cup \vec{M}(A) \cup (\vec{M}(A') \times_{\Delta^n} \partial\Delta^n))$ by pushouts of inner horns, we begin by specifying the primary and secondary simplices and the bijection between them.

A non-degenerate simplex of $\vec{M}(A')$ not lying in $M(A') \cup \vec{M}(A) \cup (\vec{M}(A') \times_{\Delta^n} \partial\Delta^n)$ is, by our characterization just above, a lattice path from $(0,0)$ to (n,a) with edges of the form $(0,1)$, $(1,0)$, or $(1,1)$ (indeed, anything not of this form will fail to surject onto $[n]$ or $[a]$). Given such a lattice path, define its *critical segment* to be the first (starting from $(0,0)$) occurence of a $(1,1)$ edge or of a $(1,0)$ edge followed by a $(0,1)$ edge (note that every lattice path in question has a critical segment since $d > 0$). We declare a simplex to be primary when its critical segment is $(1,0)$–$(0,1)$ and to be secondary when its critical segment is $(1,1)$. Given a primary simplex, we may delete the middle vertex (i,c) of its critical segment (call this the *critical vertex*) to obtain a secondary simplex, and this defines a bijection between the primary and secondary simplices.



(1.4.166.11)

Note that the vertex deleted from a primary simplex to obtain its associated secondary simplex is indeed always an inner vertex.

We now lift this bijection between primary and secondary simplices to a filtration by pushouts of inner horns. It suffices to prescribe an ordering of the primary/secondary pairs with the property that for every primary simplex, removing any vertex other than its critical vertex yields a simplex which either fails to surject onto $[n]$ or $[a]$ or is contained in an earlier primary simplex. Removing a vertex which comes before the critical vertex (i,c) yields something which fails to surject onto $[n]$ or $[a]$, except in the case that $i > 0$ and $c > 0$ and we remove $(0,c)$. Removing $(0,c)$ yields a secondary simplex associated to a primary simplex with critical vertex $(1, c-1)$, which has strictly smaller second (vertical) coordinate than ours' critical vertex (i,c). Now consider removing a vertex which comes after the critical vertex (i,c). If the vertex removed is anything other than $(i, c+1)$, then the result is a primary simplex of strictly smaller dimension (or fails to surject onto $[n]$ or $[a]$). In the final situation of removing $(i, c+1)$, there are a few cases to consider. If the next vertex is $(i, c+2)$ or $(i+1, c+2)$, then removing $(i, c+1)$ destroys surjectivity onto $[a]$. If the next vertex is $(i+1, c+1)$, then removing $(i, c+1)$ yields a secondary simplex associated to a primary simplex with critical vertex $(i+1,c)$, which has the same second (vertical) coordinate and strictly larger first (horizontal) coordinate in comparison with (i,c). There is now an evident acceptable ordering of the primary/secondary pairs: first order according to dimension, then order according to second (vertical) coordinate of the critical vertex, and finally order (in reverse) by first (horizontal) coordinate of the critical vertex. $\square$

⋆ **1.4.167 Corollary.** *Let $\phi = (\phi(0) \to \cdots \to \phi(n))$ be a diagram of ∞-categories, and let $\mathsf{E} \to \Delta^n$ be a cocartesian fibration. A map $M(\phi) \to \mathsf{E}$ over Δ^n sending marked edges to cocartesian edges is a categorical equivalence iff it is a categorical equivalence on fibers. In particular, $M(\phi) \to \vec{M}(\phi)$ is a categorical equivalence.*

Note that maps satisfying the hypothesis of (1.4.167) are easy to produce (1.4.163).

Proof. Chose injections $\phi' = (\phi'_0 \hookrightarrow \cdots \hookrightarrow \phi'_n)$ mapping to ϕ via trivial Kan fibrations (argue by induction). The map $M(\phi') \to M(\phi)$ is a categorical equivalence (1.4.162), so we are reduced to the case that the maps $\phi_i \to \phi_{i+1}$ are injective.

Since ϕ is now a sequence of injections, we can choose a factorization $M(\phi) \to \vec{M}(\phi) \to \mathsf{E}$ using the fact that $M(\phi) \to \vec{M}(\phi)$ is filtered by pushouts of inner horns (1.4.166) (which also implies that $M(\phi) \to \vec{M}(\phi)$ is a categorical equivalence). Now $\vec{M}(\phi) \to \mathsf{E}$ is a map of cocartesian fibrations over Δ^n which sends cocartesian edges to cocartesian edges, hence is an equivalence iff it is a fiberwise equivalence (1.4.144). $\square$

⋆ **1.4.168 Corollary.** *For any cocartesian fibration $\mathsf{E} \to \Delta^n$, the pair $\mathsf{E} \times_{\Delta^n} (\Delta^n, \Delta^{\{0,1\}} \vee \Delta^{\{1,2\}} \vee \cdots \vee \Delta^{\{n-1,n\}})$ is a categorical equivalence.*

Proof. The idea is to reduce to the case of a mapping simplex, which can then be checked explicitly. Choose a map $M(\phi) \to \mathsf{E}$ over Δ^n which is a fiberwise categorical equivalence (1.4.163) hence a categorical equivalence (1.4.167). It suffices to show that $M(\phi) \times_{\Delta^n} (\Delta^n, \Delta^{\{0,1\}} \vee \cdots \vee \Delta^{\{n-1,n\}})$ and $(M(\phi) \to \mathsf{E}) \times_{\Delta^n} (\Delta^{\{0,1\}} \vee \cdots \vee \Delta^{\{n-1,n\}})$ are both categorical equivalences.

The pair $M(\phi) \times_{\Delta^n} (\Delta^n, \Delta^{\{0,1\}} \vee \cdots \vee \Delta^{\{n-1,n\}})$ admits an explicit filtration by pushouts of inner horns (1.4.161) hence is a categorical equivalence.

The map $M(\phi) \to \mathsf{E}$ pulls back to a categorical equivalence under any map $\Delta^m \to \Delta^n$ (1.4.167). In particular, its restricts to a categorical equivalence over every edge and every vertex of Δ^n, which implies its restriction to $\Delta^{\{0,1\}} \vee \cdots \vee \Delta^{\{n-1,n\}}$ is a categorical equivalence by induction as in (1.4.162). $\square$

1.4.169 Exercise. Show that $(\Lambda^n_i, \Delta^{\{0,1\}} \vee \cdots \vee \Delta^{\{n-1,n\}})$ filtered by pushouts of inner horns for $0 < i < n$ (recall from (1.4.161) that $(\Delta^k, \Delta^{\{0,1\}} \vee \cdots \vee \Delta^{\{k-1,k\}})$ is filtered by pushouts of inner horns, apply this to $\Delta^{[0\cdots i]} \subseteq \Lambda^n_i$ and $\Delta^{[i\cdots n]} \subseteq \Lambda^n_i$, and then filter $(\Lambda^n_i, \Delta^{[0\cdots i]} \vee \Delta^{[i\cdots n]})$ by coning a filtration of $(\partial\Delta^{[n]-i}, \Delta^{[0\cdots(i-1)]} \sqcup \Delta^{[(i+1)\cdots n]})$).

⋆ **1.4.170 Corollary.** *If $\mathsf{E} \to \Delta^n$ is a cocartesian fibration, then $\mathsf{E} \times_{\Delta^n} (\Delta^n, \Lambda^n_i)$ is a categorical equivalence for $0 < i < n$.*

Proof. Since $\mathsf{E}\times_{\Delta^n}(\Delta^n,\Delta^{\{0,1\}}\vee\cdots\vee\Delta^{\{n-1,n\}})$ is a categorical equivalence (1.4.168), it suffices to show that $\mathsf{E}\times_{\Delta^n}(\Lambda^n_i,\Delta^{\{0,1\}}\vee\cdots\vee\Delta^{\{n-1,n\}})$ is a categorical equivalence. This is an iterated pushout of $\mathsf{E}\times_{\Delta^n}(\Delta^r,\Lambda^r_j)$ for various inner horns (Δ^r,Λ^r_j) with $r<n$ (1.4.169), which we can assume are categorical equivalences by induction on n. Now recall that injective categorical equivalences are preserved under pushout (**??**). $\square$

Kan extensions

1.4.171 Definition (Weak Kan extension)**.** Let $f:\mathsf{A}\to\mathsf{B}$ be a functor. Given a functor $G:\mathsf{A}\to\mathsf{E}$, a *weak left Kan extension* of G along f is a functor $f_!G:\mathsf{B}\to\mathsf{E}$ and a natural transformation $\eta:G\to f_!G\circ f$ such that the pair $(f_!G,\eta)$ is an initial object in the category $\mathsf{Fun}(\mathsf{B},\mathsf{E})_{G/f^*(\cdot)}$.

$$\begin{tikzcd} \mathsf{A} \arrow[r, "f"] \arrow[d, "G"'] \arrow[dr, phantom, "\Rightarrow"] & \mathsf{B} \arrow[dl, dashed, "f_!G"] \\ \mathsf{E} & \end{tikzcd} \tag{1.4.171.1}$$

When every G has a weak left Kan extension, the resulting left adjoint to f^* is denoted $f_!:\mathsf{Fun}(\mathsf{A},\mathsf{E})\to\mathsf{Fun}(\mathsf{B},\mathsf{E})$. The dual notion is called weak right Kan extension and is denoted f_*, which is right adjoint to f^*.

The category of functors $H:\mathsf{B}\to\mathsf{E}$ equipped with a natural transformation $G\to H\circ f$ is the category of maps

$$\mathrm{MC}(f)=(\mathsf{A}\times\Delta^1)\cup^f_{\mathsf{A}\times 1}\mathsf{B}\to\mathsf{E} \tag{1.4.171.2}$$

whose restriction to A equals G. We would like to replace the 'mapping cylinder' $\mathrm{MC}(f)$ in this situation with an ∞-category.

1.4.172 Exercise (Semi-orthogonal gluing)**.** Given a functor of ∞-categories $f:\mathsf{A}\to\mathsf{B}$, let $\langle\mathsf{A},\mathsf{B}\rangle_f$ denote the simplicial set defined by the property that a map $Z\to\langle\mathsf{A},\mathsf{B}\rangle_f$ is a map $p:Z\to\Delta^1$ and a diagram

$$\begin{tikzcd} p^{-1}(0) \arrow[r] \arrow[d] & \mathsf{A} \arrow[d, "f"] \\ Z \arrow[r] & \mathsf{B} \end{tikzcd} \tag{1.4.172.1}$$

Show that $\langle\mathsf{A},\mathsf{B}\rangle_f$ is an ∞-category.

For any functor of ∞-categories $f:\mathsf{A}\to\mathsf{B}$, the tautological map $\mathrm{MC}(f)=(\mathsf{A}\times\Delta^1)\cup^f_{\mathsf{A}\times 1}\mathsf{B}\to\langle\mathsf{A},\mathsf{B}\rangle_f$ from the mapping cylinder to the semi-orthogonal gluing is a categorical equivalence (1.4.167).

$\star$ **1.4.173 Definition** (Kan extension)**.** Fix functors $f:\mathsf{A}\to\mathsf{B}$ and $G:\mathsf{A}\to\mathsf{E}$. An extension of G to $\langle\mathsf{A},\mathsf{B}\rangle_f$ is called a *(pointwise) left Kan extension* when its pre-composition with the tautological map

$$(\mathsf{A}_{f(\cdot)/b})^{\triangleright}\to\langle\mathsf{A},\mathsf{B}\rangle_f \tag{1.4.173.1}$$

is a colimit diagram in E for every $b\in\mathsf{B}$.

⋆ **1.4.174 Proposition** (Existence of Kan extensions)**.** *A left Kan extension of $G : \mathsf{A} \to \mathsf{E}$ along $f : \mathsf{A} \to \mathsf{B}$ exists iff $\operatorname{colim}_{\mathsf{A}_{f(\cdot)/b}} G$ exists for all $b \in \mathsf{B}$. A Kan extension is a weak Kan extension.*

Proof. A weak left Kan extension of G along f is an initial object in the ∞-category of functors $\mathrm{MC}(f) \to \mathsf{E}$ whose restriction to A coincides with G. That is, it is an intial object in the fiber of $\mathsf{Fun}(\mathrm{MC}(f), \mathsf{E}) \to \mathsf{Fun}(\mathsf{A}, \mathsf{E})$ over G. Alternatively, it is an initial object in the fiber of $\mathsf{Fun}(\langle \mathsf{A}, \mathsf{B} \rangle_f, \mathsf{E}) \to \mathsf{Fun}(\mathsf{A}, \mathsf{E})$ over G. Indeed, $\mathrm{MC}(f) \to \langle \mathsf{A}, \mathsf{B} \rangle_f$ is a categorical equivalence (1.4.167), so $\mathsf{Fun}(\langle \mathsf{A}, \mathsf{B} \rangle_f, \mathsf{E}) \to \mathsf{Fun}(\mathrm{MC}(f), \mathsf{E})$ is an equivalence. Both $\mathsf{Fun}(\mathrm{MC}(f), \mathsf{E}) \to \mathsf{Fun}(\mathsf{A}, \mathsf{E})$ and $\mathsf{Fun}(\langle \mathsf{A}, \mathsf{B} \rangle_f, \mathsf{E}) \to \mathsf{Fun}(\mathsf{A}, \mathsf{E})$ are isofibrations of ∞-categories (1.4.82), so the inclusion functors from their fibers into their categorical fibers are equivalences (1.4.93), and the latter (categorical fibers) are invariant under equivalence. □

1.4.175 Lemma. *If $f : \mathsf{A} \to \mathsf{B}$ is fully faithful and $g : \mathsf{A} \to \mathsf{E}$ has a left Kan extension $f_! g$, then the natural transformation $g \to f^* f_! g$ is an isomorphism.*

Proof. Consider the functor $\langle \mathsf{A}, \mathsf{B} \rangle_f \to \mathsf{E}$ underlying the left Kan extension $f_! g$. For $a \in \mathsf{A}$, consider the following tautological diagram.

$$\begin{tikzcd}
\Delta^1 \arrow[r, "\mathbf{1}_{f(a)}:f(a)\to f(a)"] \arrow[d, "a\times"] & (\mathsf{A}_{f(\cdot)/f(a)})^{\triangleright} \arrow[d] \arrow[dr] & \\
(\mathsf{A} \times \Delta^1) \cup^f_{\mathsf{A}\times 1} \mathsf{B} \arrow[r] & \langle \mathsf{A}, \mathsf{B} \rangle_f \arrow[r] & \mathsf{E}
\end{tikzcd} \tag{1.4.175.1}$$

The diagonal arrow is a colimit diagram by definition of Kan extension, and the point $\mathbf{1}_{f(a)} \in \mathsf{A}_{f(\cdot)/f(a)}$ is a final object since f is fully faithful, so we conclude that the composition $\Delta^1 \to \mathsf{E}$ is an isomorphism. Considering now the composition $\Delta^1 \to \mathsf{E}$ through the rest of the diagram, this means precisely that the unit transformation $g(a) \to (f_! g)(f(a))$ is an isomorphism. □

1.4.176 Corollary (Kan extension and full faithfulness)**.** *If $f : \mathsf{A} \to \mathsf{B}$ is fully faithful, then the left Kan extension functor $f_! : \mathsf{Fun}(\mathsf{A}, \mathsf{E}) \to \mathsf{Fun}(\mathsf{B}, \mathsf{E})$ is fully faithful on its domain of definition.*

Proof. Combine (1.4.175) with (1.1.83). □

1.4.177 Corollary. *Let $f : \mathsf{A} \to \mathsf{B}$ be fully faithful. A functor $\mathsf{B} \to \mathsf{E}$ is a left Kan extension along f iff the composition $(\mathsf{A}_{f(\cdot)/b})^{\triangleright} \to \mathsf{B} \to \mathsf{E}$ is a colimit diagram for every $b \in \mathsf{B}$.*

Proof. Since the unit transformation $g \to f^* f_! g$ is an isomorphism for every g (1.4.175), the functor $(\mathsf{A} \times \Delta^1) \cup^f_{\mathsf{A}\times 1} \mathsf{B} \to \mathsf{E}$ underlying a left Kan extension factors through the projection $(\mathsf{A} \times \Delta^1) \cup^f_{\mathsf{A}\times 1} \mathsf{B} \to \langle \mathsf{A}, \mathsf{B} \rangle_f \to \mathsf{B}$. □

The Straightening Equivalence: Proof

The Straightening Equivalence (1.4.179) is one of the technical cornerstones of the theory of ∞-categories. It asserts that the Kan simplicial category $\underline{\mathsf{sSet}}^{+\mathsf{cocart}}_{/K^\#}$ (1.4.178) of cocartesian fibrations over a simplicial set K is equivalent to the ∞-category of functors $\mathsf{Fun}(K, \mathsf{Cat}_\infty)$ (restricting, in particular, to an equivalence between the Kan simplicial category of left fibrations $\underline{\mathsf{sSet}}^{\mathsf{L}}_{/K}$ over K and the ∞-category of diagrams $\mathsf{Fun}(K, \mathsf{Spc})$). The significance of this result is that it gives a concrete model for $\mathsf{Fun}(K, \mathsf{Cat}_\infty)$ which is more direct than the other available models (such as the ∞-category of functors from K to the Kan simplicial category of ∞-categories (1.4.66) or to the classifying ∞-category of cocartesian fibrations (**??**)). The Straightening Equivalence is an indispensable tool for working with such diagram ∞-categories $\mathsf{Fun}(K, \mathsf{Cat}_\infty)$ and $\mathsf{Fun}(K, \mathsf{Spc})$.

The Straightening Equivalence is due originally to Lurie [74, §3.2]. The proof given here is an adaptation of one proposed to me by Pelle Steffens. Despite the existence of a number of different proofs of it, there is no (known) simple explicit way of writing down the equivalence, even in the simplest non-trivial case $K = \Delta^1$(!). The basic idea of the functor from cocartesian (or left) fibrations over K to diagrams $K \to \mathsf{Cat}_\infty$ (or $K \to \mathsf{Spc}$) is quite simply to consider the transport maps (1.4.59)(1.4.138), which are well defined up to contractible choice; however, it turns out to be more complicated to turn this into an actual definition than one might expect at first glance (and we will proceed differently).

1.4.178 Definition (∞-category of left fibrations and cocartesian fibrations)**.** Given a simplicial set K, we denote by $\underline{\mathsf{sSet}}^{\mathsf{L}}_{/K}$ the Kan simplicial category whose objects are left fibrations $X \to K$ and in which the morphism complex from X to Y is the simplicial set representing the functor sending $Z \in \mathsf{sSet}^{\mathsf{op}}$ to the set of maps $X \times Z \to Y$ over K. Note that this morphism complex satisfies the extension property with respect to left horns (1.4.20)(1.4.54) hence is Kan (1.4.47) (alternatively, extension for right horns follows from obstruction theory for sections of left fibrations (1.4.54)(1.4.61)).

More generally, we denote by $\underline{\mathsf{sSet}}^{+\mathsf{cocart}}_{/K^\#}$ the Kan simplicial category whose objects are cocartesian fibrations $X \to K$ and in which the morphism complex from X to Y is the simplicial set representing the functor sending $Z \in \mathsf{sSet}^{\mathsf{op}}$ to the set of maps of marked simplicial sets $X^{\mathrm{cocart}/K} \times Z^\# \to Y^{\mathrm{cocart}/K}$ over $K^\#$. The same reasoning as before shows that these morphism complexes are Kan.

To view $K \mapsto \underline{\mathsf{sSet}}^{\mathsf{L}}_{/K}$ and $K \mapsto \underline{\mathsf{sSet}}^{+\mathsf{cocart}}_{/K^\#}$ as functors $\mathsf{sSet}^{\mathsf{op}} \to \mathsf{Cat}_\infty$ (indeed, functors $\mathsf{sSet}^{\mathsf{op}} \to \mathsf{sSet}^{\mathsf{qcat}}$), we declare their objects to be left (resp. cocartesian) fibrations $E \to K$ in which every fiber of $E_n \to K_n$ is (identified with) an object of some particular fixed small model of the category Set.

1.4.179 Straightening Equivalence (proved in (**??**))**.** *There is a canonical equivalence of ∞-categories*

$$\mathsf{Fun}(K, \mathsf{Cat}_\infty) = \underline{\mathsf{sSet}}^{+\mathsf{cocart}}_{/K^\#} \tag{1.4.179.1}$$

for every simplicial set K. This equivalence is natural in K (extends to an isomorphism of functors $\mathsf{sSet} \to \mathsf{Cat}_\infty$) and is the identity for $K = *$. It identifies the full subcategories $\mathsf{Fun}(K, \mathsf{Spc}) = \underline{\mathsf{sSet}}^{\mathsf{L}}_{/K}$ of both sides.

The proof of the Straightening Equivalence does not provide (in any reasonable way) an explicit equivalence between the two sides. Rather, it proceeds by arguing that they both satisfy certain formal properties which characterize them uniquely. The argument begins with the case $K = \Delta^1$, which is treated by giving an intrinsic characterization of the functor $\mathsf{C} \to \mathsf{Fun}(\Delta^1, \mathsf{C})$ for any ∞-category C with a final and an initial object (in our case, $\mathsf{C} = \mathsf{Cat}_\infty$) and showing that this intrinsic characterization is satisfied by $\mathsf{Cat}_\infty \to \mathsf{sSet}^{+\mathsf{cocart}}_{/\Delta^1}$. By gluing together copies of Δ^1, we may deduce the case of $K = \Delta^{\{0,1\}} \vee \cdots \vee \Delta^{\{n-1,n\}} \subseteq \Delta^n$ and from this the case $K = \Delta^n$. Next, we introduce a gadget we call 'interpolation diagrams' which provides a *functorial* equivalence for all simplices $K \in \mathbf{\Delta}^{\mathsf{op}}$. Finally, we deduce from this the general case by (morally) arguing that both sides are continuous functors $\mathsf{sSet}^{\mathsf{op}} \to \mathsf{Cat}_\infty$ hence are characterized uniquely by their restriction to $\mathbf{\Delta}^{\mathsf{op}} \subseteq \mathsf{sSet}^{\mathsf{op}}$.

Now let us start with straightening over Δ^1.

1.4.180 Exercise. Let C be an ∞-category. Note that the functors $0, 1 : \Delta^0 \rightleftarrows \Delta^1 : p$ are adjoint $(0, p, 1)$, which provides adjunctions $(1^*, p^*, 0^*)$ of functors $p^* : \mathsf{C} \to \mathsf{Fun}(\Delta^1, \mathsf{C}) : 0^*, 1^*$ (1.1.104). If C has an initial (resp. final) object, show that the left (resp. right) Kan extension $1_!$ (resp. 0_*) exists and is given by $c \mapsto (\varnothing \to c)$ (resp. $c \mapsto (c \to *)$). Finally, show that for any object $(x \to y) \in \mathsf{Fun}(\Delta^1, \mathsf{C})$, the diagram

$$\begin{tikzcd} (\varnothing \to x) \arrow[r] \arrow[d] & (x \to x) \arrow[d] \\ (\varnothing \to y) \arrow[r] & (x \to y) \end{tikzcd} \tag{1.4.180.1}$$

is a pushout (**??**).

1.4.181 Exercise (Intrinsic characterization of $\mathsf{C} \to \mathsf{Fun}(\Delta^1, \mathsf{C})$)**.** Let C be an ∞-category with a final and an initial object, and let $\bar{p}^* : \mathsf{C} \hookrightarrow \overline{\mathsf{C}}$ be a fully faithful functor with adjoints $(\bar{1}_!, \bar{1}^*, \bar{p}^*, \bar{0}^*, \bar{0}_*)$. Consider the composition $\bar{p}^*\bar{0}^* \to 1 \to \bar{p}^*\bar{1}^*$ (of the unit of $(\bar{1}^*, \bar{p}^*)$ with the counit of $(\bar{p}^*, \bar{0}^*)$), which is a natural transformation of functors landing inside the essential image of $\bar{p}^*$, hence may be regarded as a natural transformation $\bar{0}^* \to \bar{1}^*$, that is a functor $\overline{\mathsf{C}} \to \mathsf{Fun}(\Delta^1, \mathsf{C})$. Note that the composition $(\bar{0}^* \to \bar{1}^*)\bar{p}^*$ is canonically isomorphic to p^* (use the triangle identities (1.1.130.6) and full faithfulness of $\bar{p}^*$ (1.1.83) to argue that both maps $\bar{p}^*\bar{0}^*\bar{p}^* \to \bar{p}^* \to \bar{p}^*\bar{1}^*\bar{p}^*$ are isomorphisms).

$$\begin{tikzcd} \mathsf{C} \arrow[r, hook, "\bar{p}^*"] \arrow[dr, "p^*"'] & \overline{\mathsf{C}} \arrow[d, "(\bar{0}^* \to \bar{1}^*)"] \\ & \mathsf{Fun}(\Delta^1, \mathsf{C}) \end{tikzcd} \tag{1.4.181.1}$$

Now we may ask whether the vertical arrow $(\bar{0}^* \to \bar{1}^*) : \overline{\mathsf{C}} \to \mathsf{Fun}(\Delta^1, \mathsf{C})$ is fully faithful. Note that this functor preserves colimits (according to (**??**), it is enough to show that its compositions with 0^* and 1^* preserve colimits, and these are nothing other than $\bar{0}^*$ and $\bar{1}^*$, which are both left adjoints). It therefore suffices, in checking full faithfulness of this functor, to restrict attention to pairs of objects of $\overline{\mathsf{C}}$ whose first factor is required to lie in some set of objects of $\overline{\mathsf{C}}$ which generate it under colimits.

Show that the map

$$\mathrm{Hom}_{\overline{\mathsf{C}}}(\bar{p}^*a, b) \xrightarrow{(\bar{0}^*\to\bar{1}^*)} \mathrm{Hom}_{\mathsf{Fun}(\Delta^1,\mathsf{C})}((\bar{0}^* \to \bar{1}^*)\bar{p}^*a, (\bar{0}^* \to \bar{1}^*)b) \tag{1.4.181.2}$$

is an isomorphism as follows. Use the adjunction $(\bar{p}^*, \bar{0}^*)$ to write it as the composition $\mathrm{Hom}_{\mathsf{C}}(a, \bar{0}^*b) \xrightarrow{p^*} \mathrm{Hom}_{\mathsf{Fun}(\Delta^1,\mathsf{C})}(p^*a, p^*\bar{0}^*b) = \mathrm{Hom}_{\mathsf{Fun}(\Delta^1,\mathsf{C})}(p^*a, (\bar{0}^* \to \bar{1}^*)\bar{p}^*\bar{0}^*b) \xrightarrow{\bar{p}^*\bar{0}^*\to 1} \mathrm{Hom}_{\mathsf{Fun}(\Delta^1,\mathsf{C})}(p^*a, (\bar{0}^* \to \bar{1}^*)b)$. Note that the first map is an isomorphism since p^* is fully faithful. For the second map, note that to show that $\mathrm{Hom}_{\mathsf{Fun}(\Delta^1,\mathsf{C})}(p^*a, -)$ sends a morphism to an isomorphism, it is enough to show that 0^* sends it to an isomorphism (by the adjunction $(p^*, 0^*)$), reducing us to showing that $\bar{0}^*(\bar{p}^*\bar{0}^* \to 1)$ is an isomorphism, which follows from the triangle identity and full faithfulness of $\bar{p}^*$.

Show that the map

$$\mathrm{Hom}_{\overline{\mathsf{C}}}(\bar{1}_!a, b) \xrightarrow{(\bar{0}^*\to\bar{1}^*)} \mathrm{Hom}_{\mathsf{Fun}(\Delta^1,\mathsf{C})}((\bar{0}^* \to \bar{1}^*)\bar{1}_!a, (\bar{0}^* \to \bar{1}^*)b) \tag{1.4.181.3}$$

is an isomorphism provided $\bar{0}^*\bar{1}_! = \varnothing$ and $\bar{1}_!$ is fully faithful. Indeed, since $\bar{0}^*\bar{1}_! = \varnothing$, the functor $(\bar{0}^* \to \bar{1}^*)\bar{1}_!$ lands inside the essential image of $1_!$, which means that applying 1^* to the target above is an isomorphism (by full faithfulness of $1_!$ and the adjunction $(1_!, 1^*)$). So, it suffices to show that the action of the composition $1^*(\bar{0}^* \to \bar{1}^*) = \bar{1}^*$, namely the map $\mathrm{Hom}_{\overline{\mathsf{C}}}(\bar{1}_!a, b) \xrightarrow{\bar{1}^*} \mathrm{Hom}_{\mathsf{C}}(\bar{1}^*\bar{1}_!a, \bar{1}^*b)$, is an isomorphism, which follows from full faithfulness of $\bar{1}_!$ and the adjunction $(\bar{1}_!, \bar{1}^*)$.

Now conclude: the functor $(\bar{0}^* \to \bar{1}^*) : \overline{\mathsf{C}} \to \mathsf{Fun}(\Delta^1, \mathsf{C})$ is fully faithful provided $\bar{1}_!$ is fully faithful, $\bar{0}^*\bar{1}_! = \varnothing$, and $\overline{\mathsf{C}}$ is generated under colimits by the images of $\bar{p}^*$ and $\bar{1}_!$.

1.4.182 Corollary (Straightening over a 1-simplex)**.** *Consider the following functors relating* $\mathsf{Cat}_\infty = \underline{\mathsf{sSet}}^{+\mathsf{cocart}}_{/(\Delta^0)^\#}$ *and* $\underline{\mathsf{sSet}}^{+\mathsf{cocart}}_{/(\Delta^1)^\#}$*.*

$$\bar{p}^*\mathsf{C} = (\mathsf{C} \times \Delta^1 \to \Delta^1) \tag{1.4.182.1}$$

$$\bar{0}_*\mathsf{C} = (\mathsf{C}^\triangleright \to \Delta^1) \qquad \bar{0}^*\mathsf{E} = \mathsf{E} \times_{\Delta^1} 0 \tag{1.4.182.2}$$

$$\bar{1}_!\mathsf{C} = (\mathsf{C} \xrightarrow{1} \Delta^1) \qquad \bar{1}^*\mathsf{E} = \mathsf{E} \times_{\Delta^1} 1 \tag{1.4.182.3}$$

There are canonical adjunctions $(\bar{1}_!, \bar{1}^*, \bar{p}^*, \bar{0}^*, \bar{0}_*)$*, and this ensemble is equivalent to the adjunctions* $(1_!, 1^*, p^*, p^*, 0^*, 0_*)$ *of functors* $\mathsf{Cat}_\infty \rightleftarrows \mathsf{Fun}(\Delta^1, \mathsf{Cat}_\infty)$ (1.4.180)*.*

Proof. □

Now let us deduce the straightening equivalence over Δ^n from the case $n = 1$.

1.4.183 Proposition (From straightening over Δ^1 to straightening over Δ^n)**.** *Suppose that for $n = 1$, there is an equivalence of ∞-categories*

$$\mathsf{Fun}(\Delta^n, \mathsf{Cat}_\infty) = \underline{\mathsf{sSet}}^{+\mathsf{cocart}}_{/(\Delta^n)^\#} \tag{1.4.183.1}$$

respecting the evaluation at $i \in \Delta^n$ functors from both sides to Cat_∞. There is then such an equivalence for all $n \geq 0$.

Proof. □

Now let us make the straightening equivalence over Δ^n functorial.

1.4.184 Definition (Interpolation diagram)**.** Let $X : \Delta^n \to \underline{\mathsf{sSet}}^{+\mathsf{cocart}}_{/(\Delta^n)^\#}$ be a functor. For each pair $i, j \in \Delta^n$, we may consider the ∞-category $X(i)_j$ (evaluate X at $i \in \Delta^n$ to obtain a cocartesian fibration $X(i) \to \Delta^n$ and specialize to the fiber over $j \in \Delta^n$). Given (i, j) and (i', j') with $i \leq i'$ and $j \leq j'$, there is a canonical (up to contractible choice) induced functor $X(i)_j \to X(i')_{j'}$ (combine the morphism $i \to i'$ in Δ^n with the cocartesian transport map along the edge $j \to j'$ (1.4.138)). When the functor $X(i)_j \to X(i')_{j'}$ is an equivalence whenever $\max(i, j) = \max(i', j')$, we say that X is an *interpolation diagram.*

$$\begin{array}{ccccccc} A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & D \\ \downarrow & & \| & & \| & & \| \\ B & = & B & \longrightarrow & C & \longrightarrow & D \\ \downarrow & & \downarrow & & \| & & \| \\ C & = & C & = & C & \longrightarrow & D \\ \downarrow & & \downarrow & & \downarrow & & \| \\ D & = & D & = & D & = & D \end{array} \tag{1.4.184.1}$$

We denote by $\mathsf{Int}(\Delta^n, \underline{\mathsf{sSet}}^{+\mathsf{cocart}}_{/(\Delta^n)^\#}) \subseteq \mathsf{Fun}(\Delta^n, \underline{\mathsf{sSet}}^{+\mathsf{cocart}}_{/(\Delta^n)^\#})$ the full subcategory spanned by interpolation diagrams.

1.4.185 Corollary (Making straightening over a simplex functorial)**.** *Suppose that there is an equivalence* $\underline{\mathsf{sSet}}^{+\mathsf{cocart}}_{/(\Delta^n)^\#} = \mathsf{Fun}(\Delta^n, \mathsf{Cat}_\infty)$ *satisfying the following properties:*

(1.4.185.1) *The functor* $\underline{\mathsf{sSet}}^{+\mathsf{cocart}}_{/(\Delta^n)^\#} \to \mathsf{Cat}_\infty$ *given by taking the fiber over* $i \in \Delta^n$ *is isomorphic to the functor* $\mathsf{Fun}(\Delta^n, \mathsf{Cat}_\infty) \to \mathsf{Cat}_\infty$ *given by evaluation at* $i \in \Delta^n$*.*

(1.4.185.2) *For every object* $X \in \underline{\mathsf{sSet}}^{+\mathsf{cocart}}_{/(\Delta^n)^\#}$*, the cocartesian transport map* $X_i \to X_{i+1}$ *is an isomorphism iff the corresponding functor* $\Delta^n \to \mathsf{Cat}_\infty$ *sends the edge* $i \to (i+1)$ *of* Δ^n *to an isomorphism.*

In this case, the functors

$$\mathsf{Fun}(\Delta^n, \mathsf{Cat}_\infty) \leftarrow \mathsf{Int}(\Delta^n, \underline{\mathsf{sSet}}^{+\mathsf{cocart}}_{/(\Delta^n)^\#}) \to \underline{\mathsf{sSet}}^{+\mathsf{cocart}}_{/(\Delta^n)^\#} \tag{1.4.185.3}$$

given by specializing to the fiber over $0 \in \Delta^n$ *(the base of the cocartesian fibrations) and evaluation at* $0 \in \Delta^n$ *(the domain of the functors), respectively, are both equivalences.*

Proof. Applying the equivalence $\underline{\mathsf{sSet}}^{+\mathsf{cocart}}_{/(\Delta^n)^\#} = \mathsf{Fun}(\Delta^n, \mathsf{Cat}_\infty)$, the correspondence (1.4.185.3) takes the form

$$\mathsf{Fun}(\Delta^n, \mathsf{Cat}_\infty) \xleftarrow{F(\cdot,0) \hookleftarrow F} \mathsf{Int}(\Delta^n \times \Delta^n, \mathsf{Cat}_\infty) \xrightarrow{F \mapsto F(0,\cdot)} \mathsf{Fun}(\Delta^n, \mathsf{Cat}_\infty) \tag{1.4.185.4}$$

where $\mathsf{Int}(\Delta^n \times \Delta^n, \mathsf{Cat}_\infty) \subseteq \mathsf{Fun}(\Delta^n \times \Delta^n, \mathsf{Cat}_\infty)$ is the full subcategory corresponding to $\mathsf{Int}(\Delta^n, \underline{\mathsf{sSet}}^{+\mathsf{cocart}}_{/(\Delta^n)^\#}) \subseteq \mathsf{Fun}(\Delta^n, \underline{\mathsf{sSet}}^{+\mathsf{cocart}}_{/(\Delta^n)^\#})$ under the equivalence $\underline{\mathsf{sSet}}^{+\mathsf{cocart}}_{/(\Delta^n)^\#} = \mathsf{Fun}(\Delta^n, \mathsf{Cat}_\infty)$. The functor on the left has the indicated form $F \mapsto F(\cdot, 0)$ by hypothesis (1.4.185.1).

Now let us determine what the full subcategory $\mathsf{Int}(\Delta^n\times\Delta^n,\mathsf{Cat}_\infty)\subseteq\mathsf{Fun}(\Delta^n\times\Delta^n,\mathsf{Cat}_\infty)$ is. A functor $F:\Delta^n\to\underline{\mathsf{sSet}}^{+\mathsf{cocart}}_{/\Delta^n}$ is an interpolation diagram when it satisfies the following two conditions:

(1.4.185.5) $F(a)_c\to F(b)_c$ is an isomorphism for $a\leq b\leq c$.

(1.4.185.6) $F(k)_i\to F(k)_j$ is an isomorphism for $i\leq j\leq k$.

We claim that on the corresponding functor $F:\Delta^n\times\Delta^n\to\mathsf{Cat}_\infty$, these translate to the following evidently analogous conditions:

(1.4.185.7) $F(a,c)\to F(b,c)$ is an isomorphism for $a\leq b\leq c$.

(1.4.185.8) $F(k,i)\to F(k,j)$ is an isomorphism for $i\leq j\leq k$.

For the first conditions, this equivalence follows from hypothesis (1.4.185.1). For the second conditions, the equivalence follows from hypothesis (1.4.185.2) (note that these conditions are equivalent to their special case $j=i+1$).

We have thus identified $\mathsf{Int}(\Delta^n\times\Delta^n,\mathsf{Cat}_\infty)\subseteq\mathsf{Fun}(\Delta^n\times\Delta^n,\mathsf{Cat}_\infty)$ with the full subcategory of functors $\Delta^n\times\Delta^n\to\mathsf{Cat}_\infty$ sending the edges $(x,y)\to(x',y')$ with $\max(x,y)=\max(x',y')$ to isomorphisms. To conclude that the forgetful functors (1.4.185.4) are equivalences, it suffices to show that the inclusions $\Delta^n\times\{0\}\subseteq(\Delta^n\times\Delta^n,S)\supseteq\{0\}\times\Delta^n$ are categorical equivalences where S denotes the marking of these edges (1.4.186). □

1.4.186 Lemma. *The map $\Delta^n\xrightarrow{\times 0}((\Delta^n)^2,S)$ is a marked categorical equivalence* (1.4.71) *where the marking S of $(\Delta^n)^2$ consists of those edges $(i,j)\to(i',j')$ with $\max(i,j)=\max(i',j')$.*

Proof. This should be apparent from a visual inspection of (1.4.184.1), but we must of course also give a precise argument. Recall that $(\Delta^n,\Delta^{\{0,1\}}\vee\cdots\vee\Delta^{\{n-1,n\}})$ is filtered by pushouts of inner horns (1.4.161). It follows that $((\Delta^n)^2,(\Delta^{\{0,1\}}\vee\cdots\vee\Delta^{\{n-1,n\}})^2)$ is filtered by pushouts of inner horns (1.4.20). We may thus consider the map

$$\Delta^{\{0,1\}}\vee\cdots\vee\Delta^{\{n-1,n\}}\xrightarrow{\times 0}(\Delta^{\{0,1\}}\vee\cdots\vee\Delta^{\{n-1,n\}})^2 \tag{1.4.186.1}$$

in place of $\Delta^n\xrightarrow{\times 0}(\Delta^n)^2$ (note that the marked edges in $(\Delta^n)^2$ are 'generated' by those marked in $(\Delta^{\{0,1\}}\vee\cdots\vee\Delta^{\{n-1,n\}})^2$). Now this map admits an explicit filtration by pushouts of $(\Delta^1,0)^\#$ and $(\Delta^1,1)\wedge(\Delta^1,0)$ (with either $(0,0)\to(0,1)$ or $(0,1)\to(1,1)$ marked), all of which are filtered by pushouts of marked horns. □

The results so far combine to give the Straightening Equivalence (1.4.179) over simplices:

1.4.187 Corollary. *There is a correspondence*

$$\mathsf{Fun}(-,\mathsf{Cat}_\infty)\xleftarrow{\sim}\mathsf{M}\xrightarrow{\sim}\underline{\mathsf{sSet}}^{+\mathsf{cocart}}_{/(-)^\#} \tag{1.4.187.1}$$

of functors $\mathbf{\Delta}^{\mathsf{op}}\to\mathsf{sSet}^{\mathsf{qcat}}$, whose evaluation at any $[n]\in\mathbf{\Delta}^{\mathsf{op}}$ is a pair of equivalences of ∞-categories and whose evaluation at $[0]\in\mathbf{\Delta}^{\mathsf{op}}$ is the identity correspondence between Cat_∞ and itself.

Proof. Combine (1.4.182)(1.4.183)(1.4.185) and let $\mathsf{M}(\Delta^n)=\mathsf{Int}(\Delta^n,\underline{\mathsf{sSet}}^{+\mathsf{cocart}}_{/(\Delta^n)^\#})$. □

The Straightening Equivalence: Applications

1.4.188 Definition (Last vertex map)**.** For any simplicial set K, there is a canonical map $\mathbf{\Delta}_{/K} \to K$ called the *last vertex map*. It sends a simplex $[n_0] \to \cdots \to [n_k] \to K$ of $\mathbf{\Delta}_{/K}$ to the simplex of K given by pre-composing the final map $[n_k] \to K$ with the map $\Delta^k \to [n_k]$ which sends $i \in \Delta^k$ to the image of $n_i \in [n_i] \to [n_k]$.

1.4.189 Lemma (Commutation of colimit and pullback for fibrations of diagrams)**.** *Let $p \to q$ be a morphism in $\mathsf{Fun}(K, \mathsf{Spc})$. If the map $p \to q$ sends edges in K to pullback squares in Spc, then the diagram*

$$\begin{tikzcd} p(v) \arrow[r] \arrow[d] & \operatorname*{colim}_K p \arrow[d] \\ q(v) \arrow[r] & \operatorname*{colim}_K q \end{tikzcd} \tag{1.4.189.1}$$

is a pullback in Spc for every vertex $v \in K$.

Proof. Using the small object argument (**??**), represent $p \to q$ as a composition of left fibrations $P \to Q \to K$. Consider the diagram

$$\begin{tikzcd} P_k \arrow[r] \arrow[d] & P_{k'} \arrow[d] \\ Q_k \arrow[r] & Q_{k'} \end{tikzcd} \tag{1.4.189.2}$$

associated to an edge $k \to k'$ in K. Given an edge $q \to q'$ in Q lying over $k \to k'$, there is a map $P_q \to P_{q'}$ induced by the fact that $P \to Q$ is a left fibration. Equivalently, this is the map from the homotopy fiber of $P_k \to Q_k$ over q to the homotopy fiber of $P_{k'} \to Q_{k'}$ over q' induced by the edge $q \to q'$. Since the square (1.4.189.2) is a pullback, this map is a homotopy equivalence. Now any left fibration $P \to Q$ for which the maps $P_q \to P_{q'}$ associated to edges $q \to q'$ in Q are homotopy equivalences is a Kan fibration (1.4.61).

Now the desired diagram (1.4.189.1) may be realized concretely as the pullback of simplicial sets

$$\begin{tikzcd} P \times_K v \arrow[r] \arrow[d] & P \arrow[d] \\ Q \times_K v \arrow[r] & Q \end{tikzcd} \tag{1.4.189.3}$$

which remains a pullback in Spc since $P \to Q$ is a Kan fibration. □

1.4.190 Lemma. *Let $X_\alpha \to Y_\alpha$ be a morphism of filtered diagrams in Spc (that is, we have a filtered simplicial set K and a diagram $K \times \Delta^1 \to \mathsf{Spc}$). The following are equivalent:*

(1.4.190.1) *The induced map $\varinjlim_\alpha X_\alpha \to \varinjlim_\alpha Y_\alpha$ is an isomorphism.*

(1.4.190.2) *For every vertex $\beta \in K$, there exists a map $Y_\beta \to \varinjlim_\alpha X_\alpha$ such that the following diagram commutes up to homotopy.*

$$\begin{tikzcd} X_\beta \arrow[r] \arrow[d] & \varinjlim_\alpha X_\alpha \arrow[d] \\ Y_\beta \arrow[r] \arrow[ur, dashed] & \varinjlim_\alpha Y_\alpha \end{tikzcd}$$

Proof. If $\varinjlim_\alpha X_\alpha \to \varinjlim_\alpha Y_\alpha$ is an isomorphism (1.4.190.1), then composing its inverse with $Y_\beta \to \varinjlim_\alpha Y_\alpha$ gives a map of the desired shape (1.4.190.2). For the converse, note that the existence of pointwise lifts up to homotopy (1.4.190.2) implies that the map $\varinjlim_\alpha X_\alpha \to \varinjlim_\alpha Y_\alpha$ induces isomorphisms on all homotopy sets (note that the homotopy set functors commute with filtered colimits, then check surjectivity using the bottom right triangle and injectivity using the top left triangle) and apply Whitehead (1.3.34). $\square$

Yoneda Lemma

1.4.191 Yoneda Lemma. *The functor* $\mathsf{Fun}(\mathsf{C}^{\mathsf{op}}, \mathsf{Spc}) \to \mathsf{Spc}$ *given by evaluation at an object* $c \in \mathsf{C}$ *is corepresented by the object* $\mathsf{C}_{/c} \to \mathsf{C}$ *in* $\underline{\mathsf{sSet}}^{\mathsf{R}}_{/\mathsf{C}} = \mathsf{Fun}(\mathsf{C}^{\mathsf{op}}, \mathsf{Spc})$ (1.4.179) *and the point* 1_c *in its fiber over* c.

Proof. Representability may be checked at the level of the enriched homotopy category (1.4.106), and taking simplicial nerve preserves the enriched homotopy category (**??**). We are therefore tasked with showing that for every right fibration $\mathsf{E} \to \mathsf{C}$, the map

$$\underline{\mathrm{Hom}}_{\mathsf{C}}(\mathsf{C}_{/c}, \mathsf{E}) \xrightarrow{1_c} \mathsf{E}_c \tag{1.4.191.1}$$

given by evaluation at $1_c \in (\mathsf{C}_{/c})_c$ is a Kan equivalence. Now $\underline{\mathrm{Hom}}_{\mathsf{C}}(\mathsf{C}_{/c}, \mathsf{E})$ is the space of sections of the pullback of $\mathsf{E} \to \mathsf{C}$ to $\mathsf{C}_{/c}$, and $1_c \in \mathsf{C}_{/c}$ is a final object (1.4.100), hence its inclusion is ∞-final (1.4.125), which implies the map (1.4.191.1) is a Kan equivalence by definition of ∞-finality (1.4.122). $\square$

Presheaves and formally adjoining colimits

1.4.192 Proposition. *For a functor $f : \mathsf{C} \to \mathsf{D}$, the following are equivalent:*

(1.4.192.1) *The identity functor 1_{D} is the left Kan extension of f along f.*

(1.4.192.2) *The composition $\mathsf{D} \xrightarrow{\mathcal{Y}_{\mathsf{C}}} \mathsf{P}(\mathsf{D}) \xrightarrow{f^*} \mathsf{P}(\mathsf{C})$ is fully faithful.*

Proof. The identity functor 1_{D} is the left Kan extension of f along f iff for every object $d \in \mathsf{D}$, the tautological diagram $(\mathsf{C} \downarrow d)^{\triangleright} \to \mathsf{D}$ is a colimit diagram (1.4.173). For every $d' \in \mathsf{D}$, the functor $\mathrm{Hom}_{\mathsf{D}}(-, d') : \mathsf{D} \to \mathsf{Spc}^{\mathsf{op}}$ preserves colimits (**??**), hence sends left Kan extensions of functors valued in D to left Kan extensions of functors valued in $\mathsf{Spc}^{\mathsf{op}}$. Conversely, a diagram $K^{\triangleright} \to \mathsf{D}$ is a colimit if its composition with every such functor $\mathrm{Hom}_{\mathsf{D}}(-, d')$ is a colimit (**??**).

It follows that 1_{D} is the left Kan extension of f along f iff for every $d' \in \mathsf{D}$, the functor $\mathrm{Hom}_{\mathsf{D}}(-,d')$ is the left Kan extension along f of the composition $\mathsf{C} \xrightarrow{f} \mathsf{D} \xrightarrow{\mathrm{Hom}_{\mathsf{D}}(-,d')} \mathsf{Spc}^{\mathsf{op}}$.

$$\begin{tikzcd} \mathsf{C} \arrow[r, "f"] \arrow[d, "f"'] & \mathsf{D} \arrow[r, "\mathrm{Hom}_{\mathsf{D}}(-,d')"] & \mathsf{Spc}^{\mathsf{op}} \\ \mathsf{D} \arrow[ur, "1_{\mathsf{D}}"] \arrow[urr, "\mathrm{Hom}_{\mathsf{D}}(-,d')"'] & & \end{tikzcd} \tag{1.4.192.3}$$

Now Spc has all limits (1.4.112), so the left Kan extension of $\mathrm{Hom}_{\mathsf{D}}(f(-),d')$ along f always exists, giving a comparison (counit) map

$$\mathrm{Hom}_{\mathsf{D}}(-,d') \to f_*f^*\mathrm{Hom}_{\mathsf{D}}(-,d') = f_*\mathrm{Hom}_{\mathsf{D}}(f(-),d') \tag{1.4.192.4}$$

associated to the adjunction (f^*, f_*) of functors $f^* : \mathsf{P}(\mathsf{D}) \to \mathsf{P}(\mathsf{C}) : f_*$ (we have changed notation from $\mathsf{D} \to \mathsf{Spc}^{\mathsf{op}}$ to $\mathsf{D}^{\mathsf{op}} \to \mathsf{Spc}$, so left Kan extension $f_!$ corresponds to *right* Kan extension $(f^{\mathsf{op}})_*$, which we just denote by f_* for simplicity). We conclude that 1_{D} is the left Kan extension of f along f iff the above counit map (1.4.192.4) is an isomorphism for all $d' \in \mathsf{D}$.

A morphism in $\mathsf{P}(\mathsf{D})$ (in this case, the counit map $(1 \to f_*f^*)\mathrm{Hom}_{\mathsf{D}}(-,d')$ (1.4.192.4)) is an isomorphism iff it is sent to an isomorphism by evaluation at every $d \in \mathsf{D}$ (1.4.53), which by the Yoneda Lemma (1.4.191) is the same as $\mathrm{Hom}_{\mathsf{P}(\mathsf{D})}(\mathrm{Hom}_{\mathsf{D}}(-,d),-)$. Now the adjunction (f^*, f_*) identifies $\mathrm{Hom}_{\mathsf{P}(\mathsf{D})}(F,G) \to \mathrm{Hom}_{\mathsf{P}(\mathsf{D})}(F, f_*f^*G) = \mathrm{Hom}_{\mathsf{P}(\mathsf{C})}(f^*F, f^*G)$ with the action of the functor f^* on morphism spaces (**??**). We conclude that the counit map (1.4.192.4) is an isomorphism iff for every $d \in \mathsf{D}$, the map

$$\mathrm{Hom}_{\mathsf{P}(\mathsf{D})}(\mathrm{Hom}_{\mathsf{D}}(-,d), \mathrm{Hom}_{\mathsf{D}}(-,d')) \xrightarrow{f^*} \mathrm{Hom}_{\mathsf{P}(\mathsf{C})}(\mathrm{Hom}_{\mathsf{D}}(f(-),d), \mathrm{Hom}_{\mathsf{D}}(f(-),d')) \tag{1.4.192.5}$$

is an isomorphism. This map being an isomorphism for all $d, d' \in \mathsf{D}$ is what it means for $f^* : \mathsf{P}(\mathsf{D}) \to \mathsf{P}(\mathsf{C})$ to be fully faithful on the essential image of $よ_{\mathsf{D}}$. Now $よ_{\mathsf{D}}$ is itself fully faithful (**??**), so this is in turn equivalent to full faithfulness of the composition $\mathsf{D} \xrightarrow{よ_{\mathsf{D}}} \mathsf{P}(\mathsf{D}) \xrightarrow{f^*} \mathsf{P}(\mathsf{C})$. □

⋆ **1.4.193 Corollary.** *The identity functor of* $\mathsf{P}(\mathsf{C})$ *is left Kan extended along the Yoneda functor* $\mathsf{C} \hookrightarrow \mathsf{P}(\mathsf{C})$.

Proof. According to (1.4.192), it is equivalent to show that the composition $\mathsf{P}(\mathsf{C}) \xrightarrow{よ_{\mathsf{P}(\mathsf{C})}} \mathsf{P}(\mathsf{P}(\mathsf{C})) \xrightarrow{よ_{\mathsf{C}}^*} \mathsf{P}(\mathsf{C})$ is fully faithful. The Yoneda Lemma (1.4.191)(**??**) says that this composition is in fact the identity functor. □

We now discuss universal properties of presheaf categories, which assert (in various precise senses) that passing to a presheaf category freely adjoins colimits.

1.4.194 Exercise. Let $p : K \to \mathsf{C}$ be any diagram. Apply the small object argument (**??**) to express p as the composition of a map $K \hookrightarrow \widehat{K}$ which is filtered by pushouts of right horns and a map $\hat{p} : \widehat{K} \to \mathsf{C}$ which is a right fibration. Note that $K \hookrightarrow \widehat{K}$ is final since it is filtered by pushouts of right horns (1.4.123) and hence that $\operatorname{colim} p \to \operatorname{colim} \hat{p}$ is an isomorphism (**??**). Combine this with the fact that a right fibration is its own colimit (**??**) to conclude that $\operatorname{colim}_{\mathsf{P}(\mathsf{C})} p$ is the right fibration $\hat{p} : \widehat{K} \to \mathsf{C}$.

⋆ **1.4.195 Definition** (Finite presheaf)**.** A presheaf $F \in \mathsf{P}(\mathsf{C})$ is called *finite* when it is a finite colimit of representable presheaves. The full subcategory spanned by finite presheaves is denoted $\mathsf{P}(\mathsf{C})_{\mathsf{fin}} \subseteq \mathsf{P}(\mathsf{C})$.

1.4.196 Lemma (Classification of morphisms in $\mathsf{P}(\mathsf{C})_{\mathsf{fin}}$)**.** *Let $p : K \to \mathsf{C}$ be a finite diagram. Every morphism out of p in $\mathsf{P}(\mathsf{C})_{\mathsf{fin}}$ is isomorphic in $(p \downarrow \mathsf{P}(\mathsf{C})_{\mathsf{fin}})$ to the tautological map $p \to q$ associated to a diagram $q : L \to \mathsf{C}$, an injection $K \hookrightarrow L$, and an isomorphism $q|_K = p$.*

Proof. Fix a finite diagram $q : L \to \mathsf{C}$ and an arbitrary morphism $p \to q$ in $\mathsf{P}(\mathsf{C})_{\mathsf{fin}}$. The object $q \in \mathsf{P}(\mathsf{C})_{\mathsf{fin}}$ is represented by the right fibration $\hat{q} : \widehat{L} \to \mathsf{C}$ obtained from $q : L \to \mathsf{C}$ by applying the small object argument (1.4.194). Our morphism $p \to q$ is thus induced by a map $K \to \widehat{L}$ over C (**??**). Since K is finite and right horns are finite, this morphism necessarily factors through the result $\overline{L} \subseteq \widehat{L}$ of attaching just finitely many right horns to L. The morphism $\operatorname{colim} q \to \operatorname{colim} \bar{q}$ is an isomorphism for the same reason $\operatorname{colim} q \to \operatorname{colim} \hat{q}$ is (1.4.194). Thus our morphism $p \to q$ is represented by the morphism $K \to \overline{L}$ of finite simplicial sets over C. This map may not be injective, so we may replace it with the mapping cylinder $K = K \times 0 \subseteq (K \times \Delta^1) \cup_{K \times 1} \overline{L}$. □

⋆ **1.4.197 Proposition.** *The full subcategory of finite presheaves $\mathsf{P}(\mathsf{C})_{\mathsf{fin}} \subseteq \mathsf{P}(\mathsf{C})$ is closed under finite colimits in $\mathsf{P}(\mathsf{C})$.*

Proof. It suffices to show that a pushout of finite presheaves is finite (**??**). So, consider morphisms $X \leftarrow Y \to Z$ in $\mathsf{P}(\mathsf{C})_{\mathsf{fin}}$. Represent Y by a finite diagram $p : K \to \mathsf{C}$. By the classification of morphisms in $\mathsf{P}(\mathsf{C})_{\mathsf{fin}}$ (1.4.196), the morphisms $Y \to X$ and $Y \to Z$ are of the form $p \to q$ and $p \to r$ for finite diagrams $q : L \to \mathsf{C}$ and $r : M \to \mathsf{C}$ with $K \subseteq L$ and $K \subseteq M$ with $q|_K = p = r|_K$. We may thus consider the pushout diagram $q \sqcup_p r : L \sqcup_K M \to \mathsf{C}$, which represents an object of $\mathsf{P}(\mathsf{C})_{\mathsf{fin}}$. There is now a tautological square diagram containing p, q, r, and $q \sqcup_p r$, and this diagram is a pushout in $\mathsf{P}(\mathsf{C})$ by Mayer–Vietoris (**??**). □

* *

⋆ **1.4.198 Definition** (Local presheaf)**.** Let C be an ∞-category, and let Λ be a *set* of morphisms in $\mathsf{P}(\mathsf{C})$. We denote by $\mathsf{P}_\Lambda(\mathsf{C}) \subseteq \mathsf{P}(\mathsf{C})$ the full subcategory spanned by right Λ-local objects (1.1.96).

1.4.199 Lemma (Locality as an assertion about limits)**.** *A presheaf $F \in \mathsf{P}(\mathsf{C})$ is right local with respect to a morphism $\operatorname{colim}_K^{\mathsf{P}(\mathsf{C})} p \to \operatorname{colim}_L^{\mathsf{P}(\mathsf{C})} q$ (for $K \subseteq L$ a subcomplex, $q : L \to \mathsf{C}$ a diagram, and $q|_K = p$) iff the map $\lim_L F(q) \to \lim_K F(p)$ is an isomorphism.*

Proof. The functor $\mathrm{Hom}_{\mathsf{P}(\mathsf{C})}(-,F)$ sends colimits in $\mathsf{P}(\mathsf{C})$ to limits in Spc (**??**), and the composition $\mathsf{C} \xrightarrow{\text{よ}_\mathsf{C}} \mathsf{P}(\mathsf{C}) \xrightarrow{\mathrm{Hom}(-,F)} \mathsf{Spc}^{\mathsf{op}}$ is F^{op} according to the Yoneda Lemma (1.4.191) (**??**). □

⋆ **1.4.200 Proposition.** *The full subcategory* $\mathsf{P}_\Lambda(\mathsf{C}) \subseteq \mathsf{P}(\mathsf{C})$ *is reflective.*

Proof. Represent Λ as a set of diagrams $\{A_\alpha \hookrightarrow X_\alpha \to \mathsf{C}\}_\alpha$ for simplicial set pairs (X_α, A_α). A right fibration $Q \to \mathsf{C}$ is Λ-local iff it satisfies the right lifting property with respect to all pairs $(X_\alpha, A_\alpha) \wedge (\Delta^k, \partial\Delta^k)$ (mapping to C via the given maps $A_\alpha \hookrightarrow X_\alpha \to \mathsf{C}$). For any right fibration $Q \to \mathsf{C}$ satisfying this lifting property, if a map $K' \to \mathsf{C}$ is obtained from $K \to \mathsf{C}$ (not necessarily a right fibration) by forming the pushout of such a right lifting problem against a right horn or a pair $(X_\alpha, A_\alpha) \wedge (\Delta^k, \partial\Delta^k)$, then the restriction map $\mathsf{Fun}_{/\mathsf{C}}(K', Q) \to \mathsf{Fun}_{/\mathsf{C}}(K, Q)$ is a trivial Kan fibration.

We can now argue that every object of $\mathsf{P}(\mathsf{C})$ has a reflection in $\mathsf{P}_\Lambda(\mathsf{C})$. Represent an arbitrary object of $\mathsf{P}(\mathsf{C})$ by a diagram $K \to \mathsf{C}$. The small object argument (**??**) produces a factorization $K \to \overline{K} \to \mathsf{C}$ where $K \to \overline{K}$ is filtered by pushouts of right horns and pairs $(X_\alpha, A_\alpha) \wedge (\Delta^k, \partial\Delta^k)$ over C and $\overline{K} \to \mathsf{C}$ has the right lifting property with respect to right horns and pairs $(X_\alpha, A_\alpha) \wedge (\Delta^k, \partial\Delta^k)$ over C. Thus $\overline{K}$ lies in $\mathsf{P}_\Lambda(\mathsf{C})$ and the restriction map $\mathrm{Hom}_{\mathsf{P}(\mathsf{C})}(\overline{K}, Q) = \mathsf{Fun}_{/\mathsf{C}}(\overline{K}, Q) \to \mathsf{Fun}_{/\mathsf{C}}(K, Q) = \mathrm{Hom}_{\mathsf{P}(\mathsf{C})}(K, Q)$ is a trivial Kan fibration for all left fibrations $Q \to \mathsf{C}$ in $\mathsf{P}_\Lambda(\mathsf{C})$. □

1.4.201 Warning. It is tempting to claim the converse to (1.4.200) (every reflective subcategory of $\mathsf{P}(\mathsf{C})$ is of the form $\mathsf{P}_\Lambda(\mathsf{C})$ for some set of morphisms Λ, namely the collection of all reflections) by applying (1.1.97), however this argument is flawed since the collection of all reflections in the large ∞-category $\mathsf{P}(\mathsf{C})$ need not be a set.

1.4.202 Lemma. *The reflector* $r_\Lambda : \mathsf{P}(\mathsf{C}) \to \mathsf{P}_\Lambda(\mathsf{C})$ *sends morphisms in* Λ *to isomorphisms.*

Proof. Let $\ell \in \Lambda$. The morphism $r_\Lambda \ell$ is an isomorphism iff $\mathrm{Hom}(r_\Lambda \ell, X)$ is an isomorphism for every $X \in \mathsf{P}_\Lambda(\mathsf{C})$. We have $\mathrm{Hom}(r_\Lambda \ell, X) = \mathrm{Hom}(\ell, X)$ for $X \in \mathsf{P}_\Lambda(\mathsf{C})$, and $\mathrm{Hom}(\ell, X)$ is an isomorphism for all $X \in \mathsf{P}_\Lambda(\mathsf{C})$ by definition of $\mathsf{P}_\Lambda(\mathsf{C})$. □

1.4.203 Lemma. *A functor* $F : \mathsf{P}(\mathsf{C}) \to \mathsf{E}$ *sending reflections* $X \to r_\Lambda X$ *to isomorphisms sends all morphisms in* Λ *to isomorphsms.*

Proof. Suppose F sends reflections to isomorphisms. Given a morphism $\ell : X \to Y$ in Λ, consider the diagram

$$\begin{tikzcd} F(X) \arrow[r, "F(\ell)"] \arrow[d, "F(X \to rX)"'] & F(Y) \arrow[d, "F(Y \to rY)"] \\ F(rX) \arrow[r, "F(r\ell)"] & F(rY) \end{tikzcd} \tag{1.4.203.1}$$

obtained by applying F to square $\ell \to r\ell$. The two vertical arrows are isomorphisms by hypothesis on F. The bottom horizontal arrow $F(r\ell)$ is an isomorphism since $r\ell$ is an isomorphism (1.4.202). Thus the top horizontal map $F(\ell)$ is also an isomorphism, as desired. □

1.4.204 Lemma. *A cocontinuous functor $F : \mathsf{P}(\mathsf{C}) \to \mathsf{E}$ sends reflections $X \to r_\Lambda X$ to isomorphisms iff it sends all morphisms in Λ to isomorphisms.*

Proof. One direction is given by (1.4.203), so we just need to prove the other.

Suppose F is cocontinuous and sends morphisms in Λ to isomorphisms. The construction of the reflector $r : \mathsf{P}(\mathsf{C}) \to \mathsf{P}_\Lambda(\mathsf{C})$ by the small object argument (1.4.200) exhibits the reflection $X \to rX$ as the colimit $X \to \underrightarrow{\operatorname{colim}}_i X_i$ of a diagram over a well ordered set whose transition maps $\underrightarrow{\operatorname{colim}}_{i<i_0} X_i \to X_{i_0}$ are pushouts of pairs $(Y, A) \wedge (\Delta^k, \partial\Delta^k)$ mapping to C via maps $A \hookrightarrow Y \to \mathsf{C}$ in Λ. Now F is cocontinuous, so to show that F sends such a reflection to an isomorphism, it suffices to show that it sends (the presheaf on C associated to) any such pair $(Y, A) \wedge (\Delta^k, \partial\Delta^k)$ to an isomorphism. Now this pair is simply the kth iterated codiagonal of the morphism $A \to Y$ in $\mathsf{P}(\mathsf{C})$, and F preserves codiagonals since it is cocontinuous, so we are done since F sends each map $A \to Y$ to an isomorphism by hypothesis. ☐

1.4.205 Exercise (Locality as an assertion about colimits)**.** Let $f : \mathsf{C} \to \mathsf{E}$ be a functor to a cocomplete ∞-category E. Show that its unique cocontinuous extension $よ_! f : \mathsf{P}(\mathsf{C}) \to \mathsf{E}$ sends a morphism $\operatorname{colim}_K^{\mathsf{P}(\mathsf{C})} p \to \operatorname{colim}_L^{\mathsf{P}(\mathsf{C})} q$ (for $K \subseteq L$ a subcomplex, $q : L \to \mathsf{C}$ a diagram, and $q|_K = p$) in $\mathsf{P}(\mathsf{C})$ to an isomorphism iff the induced map $\operatorname{colim}_K f(p) \to \operatorname{colim}_L f(q)$ is an isomorphism (use cocontinuity of $よ_! f$ and the fact that $よ^* よ_! f = f$).

⋆ **1.4.206 Proposition** (Universal property of local presheaves)**.** *For any cocomplete category* E*, the adjoint functors*

$$\mathsf{Fun}(\mathsf{C}, \mathsf{E}) \underset{よ^*}{\overset{よ_!}{\rightleftarrows}} \mathsf{Fun}(\mathsf{P}(\mathsf{C}), \mathsf{E}) \underset{r^*}{\overset{r_!}{\rightleftarrows}} \mathsf{Fun}(\mathsf{P}_\Lambda(\mathsf{C}), \mathsf{E}) \tag{1.4.206.1}$$

restrict to equivalences between the following ∞-categories of functors:

(1.4.206.2) *Functors* $\mathsf{P}_\Lambda(\mathsf{C}) \to \mathsf{E}$ *which are cocontinuous.*

(1.4.206.3) *Functors* $\mathsf{P}(\mathsf{C}) \to \mathsf{E}$ *which are cocontinuous and Λ-local.*

(1.4.206.4) *Functors* $\mathsf{C} \to \mathsf{E}$ *whose unique cocontinuous extensions to* $\mathsf{P}(\mathsf{C})$ *satisfy the above two equivalent conditions (see also* (1.4.205)*).*

Proof. This is a special case of the universal property of a reflective subcategory of presheaves (1.1.120). ☐

1.4.207 Exercise. Let $f : \mathsf{C} \to \mathsf{D}$ be a functor, and let Λ and Γ be sets of morphisms in $\mathsf{P}(\mathsf{C})$ and $\mathsf{P}(\mathsf{D})$, respectively. Show that if $f_!(\Lambda) \subseteq \Gamma$, then $f^*(\mathsf{P}_\Gamma(\mathsf{D})) \subseteq \mathsf{P}_\Lambda(\mathsf{C})$, and hence from (1.1.101) there is an adjunction $(r_\Gamma f_!, f^*)$.

$$r_\Gamma f_! : \mathsf{P}_\Lambda(\mathsf{C}) \rightleftarrows \mathsf{P}_\Gamma(\mathsf{D}) : f^* \tag{1.4.207.1}$$

Show that $f^*(\mathsf{P}_\Gamma(\mathsf{D})) \subseteq \mathsf{P}_\Lambda(\mathsf{C})$ and $r_\Lambda f^*$ is Γ-local iff $f^* : \mathsf{P}(\mathsf{D}) \to \mathsf{P}(\mathsf{C})$ sends Γ-reflections to Λ-reflections. Conclude from (1.1.102) that if f^* sends Γ-reflections to Λ-reflections, then $f : \mathsf{C} \to \mathsf{D}$ (equivalently $f_! : \mathsf{P}(\mathsf{C}) \to \mathsf{P}(\mathsf{D})$ (1.1.113)) fully faithful implies $r_\Gamma f_! : \mathsf{P}_\Lambda(\mathsf{C}) \to \mathsf{P}_\Gamma(\mathsf{D})$ is fully faithful.

⋆ **1.4.208 Definition** (Finite local presheaves)**.** We denote by $\mathsf{P}_\Lambda(\mathsf{C})_{\mathsf{fin}} \subseteq \mathsf{P}_\Lambda(\mathsf{C})$ the full subcategory spanned by finite colimits of objects of C. By reflectivity of $\mathsf{P}_\Lambda(\mathsf{C}) \subseteq \mathsf{P}(\mathsf{C})$, such colimits exist and are precisely the image of finite presheaves $\mathsf{P}(\mathsf{C})_{\mathsf{fin}} \subseteq \mathsf{P}(\mathsf{C})$ (1.4.195) under the reflector $\mathsf{P}(\mathsf{C}) \to \mathsf{P}_\Lambda(\mathsf{C})$.

1.4.209 Lemma. *Suppose Λ is a set of morphisms in $\mathsf{P}(\mathsf{C})_{\mathsf{fin}}$ (not $\mathsf{P}(\mathsf{C})$). For any finite diagram $p : K \to \mathsf{C}$, every morphism from the associated object $p \in \mathsf{P}_\Lambda(\mathsf{C})_{\mathsf{fin}}$ to another object of $\mathsf{P}_\Lambda(\mathsf{C})_{\mathsf{fin}}$ is induced from a finite diagram $q : L \to \mathsf{C}$, an inclusion $K \hookrightarrow L$, and an isomorphism $q|_K = p$.*

Proof. The special case $\Lambda = \varnothing$ was treated earlier (1.4.196), and the same argument applies here. The key point is that the colimit $\operatorname{colim}_K^{\mathsf{P}_\Lambda(\mathsf{C})} p$ is obtained from the diagram $p : K \to \mathsf{C}$ by iteratively attaching *finite* simplicial set pairs, which follows from the construction of the reflector $\mathsf{P}(\mathsf{C}) \to \mathsf{P}_\Lambda(\mathsf{C})$ (1.4.200) (noting that every morphism in $\mathsf{P}(\mathsf{C})_{\mathsf{fin}}$ may be realized as a finite simplicial set pair mapping to C (1.4.196)). □

1.4.210 Corollary. *For a set of morphisms Λ in $\mathsf{P}(\mathsf{C})_{\mathsf{fin}}$ (not $\mathsf{P}(\mathsf{C})$), the full subcategory $\mathsf{P}_\Lambda(\mathsf{C})_{\mathsf{fin}} \subseteq \mathsf{P}_\Lambda(\mathsf{C})$ is closed under finite colimits.*

Proof. The argument used to prove the result for $\mathsf{P}(\mathsf{C})_{\mathsf{fin}} \subseteq \mathsf{P}(\mathsf{C})$ (1.4.197) applies given the morphism classification in (1.4.209). □

* *

⋆ **1.4.211 Definition** (Filtered)**.** An ∞-category C is called *filtered* when it has the extension property for pairs $(K^\triangleright, K)$ for all finite simplicial sets K.

The extension property for maps $(K^\triangleright, K) \to \mathsf{C}$ is the assertion that every slice category $\mathsf{C}_{K/}$ is non-empty. Since formation of the slice category is invariant under equivalence (**??**), so too is the notion of being filtered.

1.4.212 Exercise. Show that if C is filtered then so is $\mathsf{C}_{L/}$ for any finite diagram $L \to \mathsf{C}$.

* *

We now study ∞-sifted colimits following [97] and [74, 5.5.8]. Sifted colimits were introduced (for ordinary categories) by Gabriel–Ulmer [33].

⋆ **1.4.213 Definition** (∞-sifted)**.** A simplicial set K is called ∞-*sifted* when the diagonal map $K \to K^n$ is ∞-final (1.4.122) for all $n \geq 0$.

1.4.214 Exercise. Use (1.4.126) to show that a sifted simplicial set is contractible.

1.4.215 Lemma. *A simplicial set K is ∞-sifted iff it is non-empty and the diagonal map $K \to K \times K$ is ∞-final.*

Proof. Suppose K is non-empty and $K \to K^2$ is ∞-final, and let us show that $K \to K^n$ is ∞-final for all $n \geq 0$ (the other direction is trivial). The case $n = 1$ is trivial, and the cases $n \geq 2$ follow by induction upon expressing the diagonal map $\Delta_n : K \to K^n$ as the composition of Δ_{n-1} and $\mathbf{1}_{K^{n-2}} \times \Delta_2$ and recalling that ∞-final maps are closed under composition and products (1.4.127).

For the case $n = 0$, it suffices (and, in fact, is necessary (1.4.126)) to show that K is contractible. Since $K \to K \times K$ is ∞-final, it is a homotopy equivalence, hence acts bijectively on homotopy groups/sets (1.3.31). On the other hand, the homotopy group/set functors preserve products, so the diagonal maps of the homotopy groups/sets of K are bijections. This implies they are trivial, so K is contractible by Whitehead's Theorem (1.3.34). □

1.4.216 Exercise. Let $K \to L$ be ∞-final. Show that if K is ∞-sifted then L is ∞-sifted. Show by example that if L is ∞-sifted then K need not be ∞-sifted.

1.4.217 Exercise. Show that if $K \to L$ is a categorical equivalence, then K is ∞-sifted iff L is ∞-sifted (use the fact that ∞-finality is a property of morphisms in hCat_∞ (1.4.133) and the fact that categorical equivalences are closed under products (1.4.72)).

1.4.218 Example. An ∞-category C is C is ∞-sifted iff the slice category $\mathsf{C}_{S/}$ is Kan contractible for every finite set $S \to \mathsf{C}$ (1.4.134). It follows that an ∞-category with finite coproducts is sifted (since in this case $\mathsf{C}_{S/}$ has an initial object, hence is Kan contractible (1.4.97)).

1.4.219 Exercise. Show that a filtered ∞-category is ∞-sifted (use (1.4.212) and (**??**)).

⋆ **1.4.220 Lemma.** *The simplex category* $\mathbf{\Delta}$ *is* ∞*-cosifted.*

Proof. We should show that $\mathbf{\Delta}_{/S}$ is Kan contractible for every finite set $S \to \mathbf{\Delta}$ (1.4.134). This slice category $\mathbf{\Delta}_{/S}$ is the slice category $\mathbf{\Delta}_{/\prod S}$ over the product of simplices $\prod S \in \mathsf{sSet}$, which is certainly Kan contractible. Now we note that for any simplicial set X, the slice category $\mathbf{\Delta}_{/X}$ is a model for the colimit in Spc of the functor $X : \mathbf{\Delta}^{\mathsf{op}} \to \mathsf{Set}$ (**??**), which is in turn given by the simplicial set X itself (**??**). □

1.4.221 Lemma. *Cocartesian fibrations with* ∞*-sifted fibers are closed under composition.*

Proof. It suffices to show that if D is an ∞-sifted ∞-category and $\mathsf{C} \to \mathsf{D}$ is a cocartesian fibration with ∞-sifted fibers, then C is ∞-sifted. To show that $\mathsf{C} \to \mathsf{C}^n$ is ∞-final, we should show that $\mathsf{C}_{S/}$ is Kan contractible for all finite sets $S \to \mathsf{C}$ (1.4.134). The map $\mathsf{C}_{S/} \to \mathsf{D}_{S/}$ is a cocartesian fibration (**??**), and its target $\mathsf{D}_{S/}$ is Kan contractible since D is ∞-sifted. The fiber of $\mathsf{C}_{S/} \to \mathsf{D}_{S/}$ over a map $S^\triangleright \to \mathsf{D}$ with cone point $d \in \mathsf{D}$ is equivalent to the slice category $(\mathsf{C} \times_{\mathsf{D}} \{d\})_{S/}$ of the fiber of $\mathsf{C} \to \mathsf{D}$ over d for the map $S \to \mathsf{C} \times_{\mathsf{D}} \{d\}$ obtained from $S \to \mathsf{D}$ via cocartesian lifting over the map $S^\triangleright \to \mathsf{D}$ (**??**). Such slice categories are Kan contractible since the fibers of $\mathsf{C} \to \mathsf{D}$ are ∞-sifted. Now use the fact that the total space of a cocartesian fibration with contractible fibers over a contractible base is contractible (**??**). □

⋆ **1.4.222 Lemma.** ∞*-sifted colimits commute with finite products in any* ∞*-category in which finite products distribute over colimits (for example* Spc *(**??**)).*

Proof. Given a finite set of diagrams $\{p_i : K \to \mathsf{E}\}_{i\in S}$, the comparison map in question admits the following factorization.

$$\operatorname*{colim}_K \prod_{i\in S} p_i \to \operatorname*{colim}_{K^S} \prod_{i\in S} p_i \to \prod_i \operatorname*{colim}_K p_i \tag{1.4.222.1}$$

The second map is an isomorphism since finite products distribute over colimits in E. The first map is an isomorphism since $K \to K^S$ is ∞-final since K is ∞-sifted and S is finite. □

⋆ **1.4.223 Corollary.** *Let* C *have finite products, and denote by* $\mathsf{Fun}_\times(\mathsf{C},\mathsf{E}) \subseteq \mathsf{Fun}(\mathsf{C},\mathsf{E})$ *the functors preserving finite products. The functor* $\mathsf{Fun}_\times(\mathsf{C},\mathsf{E}) \to \prod_{c\in\mathsf{C}} \mathsf{E}$ *reflects and lifts* ∞*-sifted colimits, for any* ∞*-category* E *in which finite products distribute over colimits (for example* Spc (**??**)*).*

Proof. Given that $\mathsf{Fun}(\mathsf{C},\mathsf{E}) \to \prod_{c\in\mathsf{C}} \mathsf{E}$ reflects and lifts all colimits (**??**), it suffices to show that $\mathsf{Fun}_\times(\mathsf{C},\mathsf{E}) \subseteq \mathsf{Fun}(\mathsf{C},\mathsf{E})$ is closed under ∞-sifted colimits which are preserved by the functor $\mathsf{Fun}(\mathsf{C},\mathsf{E}) \to \prod_{c\in\mathsf{C}} \mathsf{E}$. Consider such an ∞-sifted colimit $\operatorname{colim}_\alpha F_\alpha$ of finite product preserving functors $F_\alpha : \mathsf{C} \to \mathsf{E}$. To see that $\operatorname{colim}_\alpha F_\alpha$ preserves finite products, note that its comparison map for a finite collection of objects $x_1, \ldots, x_n \in \mathsf{C}$ is the following composition.

$$\operatorname*{colim}_\alpha F_\alpha\Big(\prod_i x_i\Big) \to \operatorname*{colim}_\alpha \prod_i F_\alpha(x_i) \to \prod_i \operatorname*{colim}_\alpha F_\alpha(x_i) \tag{1.4.223.1}$$

Each F_α preserves finite products, so the first map is a colimit of isomorphisms, hence is an isomorphism. The second map is an isomorphism since ∞-sifted colimits commute with finite products in E (1.4.222). □

⋆ **1.4.224 Definition** (Formal ∞-sifted colimits)**.** We define $\mathsf{Sif}(\mathsf{C}) \subseteq \mathsf{P}(\mathsf{C})$ to consist of those presheaves which are ∞-sifted colimits of representable presheaves.

1.4.225 Lemma. *A presheaf lies in* $\mathsf{Sif}(\mathsf{C})$ *iff the total space of its corresponding right fibration over* C *is* ∞*-sifted (recall that* ∞*-siftedness is invariant under equivalences of* ∞*-categories* (1.4.217)*).*

Proof. For any right fibration $\pi : \mathsf{E} \to \mathsf{C}$, the corresponding object of $\mathsf{P}(\mathsf{C})$ is the colimit $\operatorname{colim}_{\mathsf{E}}^{\mathsf{P}(\mathsf{C})} \pi$. Thus if E is ∞-sifted then the corresponding object of $\mathsf{P}(\mathsf{C})$ lies in $\mathsf{Sif}(\mathsf{C}) \subseteq \mathsf{P}(\mathsf{C})$.

Conversely, suppose K is ∞-sifted and $p : K \to \mathsf{C}$ is a diagram. Apply the small object argument (**??**) to factor p as the composition $K \to \hat{K} \to \mathsf{C}$ of a right fibration $\hat{p} : \hat{K} \to \mathsf{C}$ and a map $K \to \hat{K}$ filtered by pushouts of right horns. The map $K \to \hat{K}$ is ∞-final (1.4.123), so $\operatorname{colim}_K^{\mathsf{P}(\mathsf{C})} p = \operatorname{colim}_{\hat{K}}^{\mathsf{P}(\mathsf{C})} \hat{p}$ is the object corresponding to the right fibration $\hat{p} : \hat{K} \to \mathsf{C}$. The total space $\hat{K}$ is ∞-sifted since K is ∞-sifted and $K \to \hat{K}$ is ∞-final (1.4.216). □

1.4.226 Lemma. *The full subcategory* $\mathsf{Sif}(\mathsf{C}) \subseteq \mathsf{P}(\mathsf{C})$ *is closed under* ∞*-sifted colimits.*

Proof. Let K be ∞-sifted, fix a diagram $K \to \mathsf{Sif}(\mathsf{C})$, and let us show that its colimit in $\mathsf{P}(\mathsf{C})$ lies in $\mathsf{Sif}(\mathsf{C})$. Colimits and ∞-siftedness are unchanged by attaching inner horns to K, so we may assume K is an ∞-category. The functor $K \to \mathsf{Sif}(\mathsf{C}) \subseteq \mathsf{P}(\mathsf{C})$ is encoded by a left fibration $E \to K \times \mathsf{C}^{\mathsf{op}}$, and its colimit in $\mathsf{P}(\mathsf{C})$ is obtained by taking the composition $E \to K \times \mathsf{C}^{\mathsf{op}} \to \mathsf{C}^{\mathsf{op}}$ and applying the small object argument to attach left horns to make it a left fibration (**??**). Attaching left horns preserves ∞-siftedness (1.4.216), so it suffices to show that the total space E is ∞-sifted. Now the fibers of $E \to K$ are ∞-sifted since $K \to \mathsf{P}(\mathsf{C})$ has essential image contained in $\mathsf{Sif}(\mathsf{C})$, and K is ∞-sifted, so the total space E is ∞-sifted (1.4.221) (here is where we use the fact that K is an ∞-category). ☐

⋆ **1.4.227 Proposition** (Universal property of formal ∞-sifted colimits)**.** *For any category* C *and any cocomplete category* E*, the pair of adjoint functors* $\mathsf{Fun}(\mathsf{C},\mathsf{E}) \rightleftarrows \mathsf{Fun}(\mathsf{Sif}(\mathsf{C}),\mathsf{E})$ *(restriction and left Kan extension) define an equivalence between:*

(1.4.227.1) *Functors* $\mathsf{C} \to \mathsf{E}$.

(1.4.227.2) *Functors* $\mathsf{Sif}(\mathsf{C}) \to \mathsf{E}$ *which preserve sifted colimits.*

(1.4.227.3) *Functors* $\mathsf{Sif}(\mathsf{C}) \to \mathsf{E}$ *which preserve sifted colimits of objects of* C.

More generally, the same holds for E *not assumed cocomplete, once we restrict to those functors* $\mathsf{C} \to \mathsf{E}$ *which send every sifted diagram in* C *to a diagram in* E *whose colimit exists.*

Proof. Given (1.4.226), this is a special case of the universal property of a full subcategory of presheaves (1.1.119). ☐

1.4.228 Lemma. *If* C *has finite coproducts, then a presheaf* $F \in \mathsf{P}(\mathsf{C})$ *lies in* $\mathsf{Sif}(\mathsf{C}) \subseteq \mathsf{P}(\mathsf{C})$ *precisely when it sends finite coproducts in* C *to products in* Spc.

Proof. Let us write $\mathsf{P}_{\sqcup\mapsto\sqcap}(\mathsf{C}) \subseteq \mathsf{P}(\mathsf{C})$ for those presheaves sending finite coproducts in C to products in Spc.

Representable presheaves lie in $\mathsf{P}_{\sqcup\mapsto\sqcap}(\mathsf{C})$ since they send *all* colimits in C to limits in Spc (**??**), and the full subcategory $\mathsf{P}_{\sqcup\mapsto\sqcap}(\mathsf{C}) \subseteq \mathsf{P}(\mathsf{C})$ is closed under ∞-sifted colimits (1.4.223)(**??**), so we have $\mathsf{Sif}(\mathsf{C}) \subseteq \mathsf{P}_{\sqcup\mapsto\sqcap}(\mathsf{C})$.

Now let us show that if $F \in \mathsf{P}_{\sqcup\mapsto\sqcap}(\mathsf{C})$ then $F \in \mathsf{Sif}(\mathsf{C})$. Represent F as a right fibration $\mathsf{E} \to \mathsf{C}$, so it suffices to show E is ∞-sifted (1.4.225). It suffices to show that E (which is an ∞-category) has finite coproducts (1.4.218). Since the base C has finite coproducts and $\mathsf{E} \to \mathsf{C}$ is a cartesian fibration, to show that E has finite coproducts it is enough to show that $\mathsf{E} \to \mathsf{C}$ has finite relative coproducts (1.4.155). That is, for a finite set S and a diagram of solid arrows

$$\begin{tikzcd} S \arrow[r] \arrow[d] & \mathsf{E} \arrow[d] \\ S^{\triangleright} \arrow[r] \arrow[ur, dashed] & \mathsf{C} \end{tikzcd} \tag{1.4.228.1}$$

we should ask that the ∞-category of dotted lifts has an initial object. This is an ∞-groupoid since $\mathsf{E} \to \mathsf{C}$ is a right fibration, and its contractibility amounts to the assertion that the pullback $\mathsf{E} \times_{\mathsf{C}} S^{\triangleright} \to S^{\triangleright}$ encodes a limit diagram $S^{\triangleright} \to \mathsf{Spc}$ (**??**). ☐

⋆ **1.4.229 Corollary.** *If* C *has finite coproducts, then* $\mathsf{Sif}(\mathsf{C}) \subseteq \mathsf{P}(\mathsf{C})$ *is the full subcategory of local objects* (1.4.198) *with respect to the set of morphisms*

$$\coprod_{i\in S}^{\mathsf{P}(\mathsf{C})} x_i \to \coprod_{i\in S}^{\mathsf{C}} x_i \tag{1.4.229.1}$$

for all finite sets $(x_i)_{i\in S}$ *of objects of* C *(equivalently, the morphisms* $\varnothing^{\mathsf{P}(\mathsf{C})} \to \varnothing^{\mathsf{C}}$ *and* $X \sqcup^{\mathsf{P}(\mathsf{C})} Y \to X \sqcup^{\mathsf{C}} Y$ *for pairs* $X, Y \in \mathsf{C}$*).*

Proof. Combine (1.4.228) and (1.4.199). □

1.4.230 Definition (Siftedization)**.** If C has finite coproducts, then the reflection $\mathsf{P}(\mathsf{C}) \to \mathsf{Sif}(\mathsf{C})$ (1.4.229)(1.4.200) is called the *siftedization* of formal colimits in C.

⋆ **1.4.231 Exercise.** Let C have finite coproducts. Show that for any formal colimit $p \in \mathsf{P}(\mathsf{C})$, the colimit of p exists in C iff the colimit of its siftedization $p_{\mathrm{sif}} \in \mathsf{Sif}(\mathsf{C})$ exists in C, and that in this case there is a canonical isomorphism $\operatorname{colim} p \xrightarrow{\sim} \operatorname{colim} p_{\mathrm{sif}}$. Conclude that C has all colimits iff it has all sifted colimits.

1.4.232 Corollary. *If* C *has finite coproducts, then the functor* $\mathsf{C} \to \mathsf{Sif}(\mathsf{C})$ *preserves finite coproducts.*

Proof. To say that $\mathsf{C} \to \mathsf{Sif}(\mathsf{C})$ preserves finite coproducts is to say that siftedization $\mathsf{P}(\mathsf{C}) \to \mathsf{Sif}(\mathsf{C})$ sends the comparison maps (1.4.229.1) to isomorphisms, and this follows from the fact that $\mathsf{Sif}(\mathsf{C})$ is the subcategory of local objects (1.4.229) with respect to these morphisms (1.4.202). □

1.4.233 Lemma. *If* $f : \mathsf{C} \to \mathsf{D}$ *preserves finite coproducts, then* $f_! : \mathsf{P}(\mathsf{C}) \to \mathsf{P}(\mathsf{D})$ *commutes with siftedization.*

Proof. Since $f_!$ is cocontinuous, we have $f_!(\mathsf{Sif}(\mathsf{C})) \subseteq \mathsf{Sif}(\mathsf{D})$. We would like to show that $f_!$ sends siftedizations to siftedizations, equivalently that $\mathrm{sif} \circ f_!$ sends siftedizations to isomorphisms. According to (1.4.229)(1.4.204), this holds iff $\mathrm{sif} \circ f_!$ sends the finite coproduct comparison morphisms (1.4.229.1) to isomorphisms. Now $f_!$ sends finite coproduct comparison morphisms for C to finite coproduct comparison morphisms for D since f preserves finite coproducts, and the latter are sent to isomorphisms by siftedization in D by (1.4.229)(1.4.202). □

⋆ **1.4.234 Exercise.** Let C have finite coproducts, and let $f : \mathsf{C} \to \mathsf{D}$ preserve finite coproducts. Use (1.4.233) to show that a formal colimit $p \in \mathsf{P}(\mathsf{C})$ is preserved by f iff its siftedization p_{sif} is preserved by f. Conclude that f preserves all colimits iff it preserves all sifted colimits.

The description of $\mathsf{Sif}(\mathsf{C}) \subseteq \mathsf{P}(\mathsf{C})$ as a full subcategory of local objects (1.4.229) gives it another universal property distinct from (1.4.227):

⋆ **1.4.235 Proposition** (Universal property of formal ∞-sifted colimits)**.** *Suppose* C *admits finite coproducts. For any cocomplete* ∞*-category* E*, the adjoint functors*

$$\mathsf{Fun}(\mathsf{C},\mathsf{E}) \underset{y^*}{\overset{y_!}{\rightleftarrows}} \mathsf{Fun}(\mathsf{P}(\mathsf{C}),\mathsf{E}) \underset{r^*}{\overset{r_!}{\rightleftarrows}} \mathsf{Fun}(\mathsf{Sif}(\mathsf{C}),\mathsf{E}) \tag{1.4.235.1}$$

restrict to equivalences between the following ∞*-categories of functors:*

(1.4.235.2) *Functors* $\mathsf{Sif}(\mathsf{C}) \to \mathsf{E}$ *which are cocontinuous.*

(1.4.235.3) *Functors* $\mathsf{P}(\mathsf{C}) \to \mathsf{E}$ *which are cocontinuous and send morphisms* $\coprod_{i\in S}^{\mathsf{P}(\mathsf{C})} x_i \to \coprod_{i\in S}^{\mathsf{C}} x_i$ *to isomorphisms for all finite sets* $(x_i)_{i\in S}$ *of objects of* C*.*

(1.4.235.4) *Functors* $\mathsf{C} \to \mathsf{E}$ *which preserve finite coproducts.*

Proof. Since $\mathsf{Sif}(\mathsf{C}) \subseteq \mathsf{P}(\mathsf{C})$ is a full subcategory of local objects (1.4.229), the universal property of such full subcategories (1.4.206) implies that the functors (1.4.235.1) induce equivalences between (1.4.235.2), (1.4.235.3), and functors $\mathsf{C} \to \mathsf{E}$ whose unique cocontinuous extension to $\mathsf{P}(\mathsf{C})$ satisfies (1.4.235.3). The latter is equivalent to (1.4.235.4) by (1.4.205). □

1.4.236 Definition (Finite sifted colimit)**.** Let C have finite coproducts. The full subcategory of $\mathsf{Sif}(\mathsf{C})$ spanned by colimits in $\mathsf{Sif}(\mathsf{C})$ of finite diagrams in $\mathsf{C} \subseteq \mathsf{Sif}(\mathsf{C})$ (which exist by (1.4.229)(1.4.200)) is denoted $\mathsf{Sif}(\mathsf{C})_{\mathsf{fin}} \subseteq \mathsf{Sif}(\mathsf{C})$ and called *finite sifted colimits.* Recall (1.4.210) which guarantees that $\mathsf{Sif}(\mathsf{C})_{\mathsf{fin}} \subseteq \mathsf{Sif}(\mathsf{C})$ is closed under finite colimits.

1.4.237 Lemma. $\mathsf{Sif}(\mathsf{C})_{\mathsf{fin}}$ *has finite sifted colimits, and they remain colimits in* $\mathsf{P}(\mathsf{C})$*.*

Proof. A finite sifted formal colimit in $\mathsf{Sif}(\mathsf{C})_{\mathsf{fin}}$ is, by definition, the siftedization p_{sif} of a finite diagram $p : K \to \mathsf{Sif}(\mathsf{C})_{\mathsf{fin}}$. Note that siftedization means siftedization of formal colimits in $\mathsf{Sif}(\mathsf{C})_{\mathsf{fin}}$, namely the reflection $\mathsf{P}(\mathsf{Sif}(\mathsf{C})_{\mathsf{fin}}) \to \mathsf{Sif}(\mathsf{Sif}(\mathsf{C})_{\mathsf{fin}})$. We now claim that

$$\operatorname*{colim}^{\mathsf{P}(\mathsf{C})} p_{\mathrm{sif}} = \operatorname*{colim}^{\mathsf{Sif}(\mathsf{C})} p_{\mathrm{sif}} = \operatorname*{colim}^{\mathsf{Sif}(\mathsf{C})} p \in \mathsf{Sif}(\mathsf{C})_{\mathsf{fin}}. \tag{1.4.237.1}$$

The identification $\operatorname{colim}^{\mathsf{P}(\mathsf{C})} p_{\mathrm{sif}} = \operatorname{colim}^{\mathsf{Sif}(\mathsf{C})} p_{\mathrm{sif}}$ holds since $\mathsf{Sif}(\mathsf{C}) \subseteq \mathsf{P}(\mathsf{C})$ is closed under sifted colimits (1.4.226). The identification $\operatorname{colim}^{\mathsf{Sif}(\mathsf{C})} p_{\mathrm{sif}} = \operatorname{colim}^{\mathsf{Sif}(\mathsf{C})} p$ holds since p_{sif} is the siftedization of p in $\mathsf{Sif}(\mathsf{C})$ (it is, by definition, the siftedization of p in $\mathsf{Sif}(\mathsf{C})_{\mathsf{fin}}$, which is the same as the siftedization in $\mathsf{Sif}(\mathsf{C})$ since $\mathsf{Sif}(\mathsf{C})_{\mathsf{fin}} \subseteq \mathsf{Sif}(\mathsf{C})$ is closed under finite coproducts, in fact under all finite colimits (1.4.210)(1.4.233)). Finally, we have $\operatorname{colim}^{\mathsf{Sif}(\mathsf{C})} p \in \mathsf{Sif}(\mathsf{C})_{\mathsf{fin}}$ since $\mathsf{Sif}(\mathsf{C})_{\mathsf{fin}} \subseteq \mathsf{Sif}(\mathsf{C})$ is closed under finite colimits (1.4.210). □

We now recall a well known explicit construction of the siftedization functor (1.4.230) and study its basic properties.

1.4.238 Definition (Bousfield–Kan formula [14, Chapter XI, §5])**.** Given a simplicial set K and a complete ∞-category E, we may consider push/pull of diagrams valued in E via the following correspondence (involving the 'last vertex map' (1.4.188)):

$$\begin{array}{ccc} \mathbf{\Delta}_{/K} & \xrightarrow{\ \ell\ } & K \\ {\scriptstyle\pi}\downarrow & & \\ \mathbf{\Delta} & & \end{array} \qquad \mathsf{Fun}(K,\mathsf{E}) \xrightarrow{\ell^*} \mathsf{Fun}(\mathbf{\Delta}_{/K},\mathsf{E}) \xrightarrow{\pi_*} \mathsf{Fun}(\mathbf{\Delta},\mathsf{E}) \tag{1.4.238.1}$$

The last vertex map ℓ is ∞-initial (**??**), so the natural transformation $1 \to \ell_*\ell^*$ induces an isomorphism on limits over K (**??**). For a diagram $p : K \to \mathsf{E}$, we call $p_\triangle = \pi_*\ell^*p$ the *Bousfield–Kan transform* of p, and the resulting isomorphism

$$\lim_K p \xrightarrow{\sim} \lim_{\mathbf{\Delta}_{/K}} \ell^*p = \lim_{\mathbf{\Delta}} \pi_*\ell^*p = \lim_{\mathbf{\Delta}} p_\triangle \tag{1.4.238.2}$$

is called the *Bousfield–Kan formula*. The functor π is a right fibration, so the right Kan extension π_* is given by the fiberwise limit (**??**), namely $(\pi_*D)^n = \prod_{f:\Delta^n\to K} D(f)$, and hence

$$p_\triangle^n = \prod_{f:\Delta^n\to K} p(f(n)). \tag{1.4.238.3}$$

In particular, the existence of the right Kan extension π_* requires only that E have products (or even just finite products if K is degreewise finite).

1.4.239 Lemma. *Suppose C admits finite products. For any diagram $p : K \to \mathsf{C}$ where K is degreewise finite, there is a canonical isomorphism of formal limits $p_\triangle \xrightarrow{\sim} p_{\mathrm{cosif}}$ from the Bousfield–Kan transform $p_\triangle$ (1.4.238) to the cosiftedization p_{cosif} (1.4.230).*

Proof.

$$p_{\mathrm{cosif}} = \overset{\mathsf{Cosif(C)}}{\lim_K}\, p \xleftarrow{\sim} \overset{\mathsf{Cosif(C)}}{\lim_{\mathbf{\Delta}}}\, p_{\triangle\mathsf{Cosif(C)}} \xleftarrow{\sim} \overset{\mathsf{Cosif(C)}}{\lim_{\mathbf{\Delta}}}\, p_{\triangle\mathsf{C}} = (p_{\triangle\mathsf{C}})_{\mathrm{cosif}} = p_{\triangle\mathsf{C}} = p_\triangle \tag{1.4.239.1}$$

This is a formal consequence of the Bousfield–Kan formula (1.4.238.2) (the first isomorphism above) and the fact that the Bousfield–Kan transform $\triangle$ is defined via finite (since K is degreewise finite) products (1.4.238.3) hence commutes with the inclusion $\mathsf{C} \hookrightarrow \mathsf{Cosif(C)}$ (1.4.232) (the second isomorphism above), and the fact that $\mathbf{\Delta}$ is ∞-cosifted (1.4.220) (the identification $(p_{\triangle\mathsf{C}})_{\mathrm{cosif}} = p_{\triangle\mathsf{C}}$ above). ☐

1.4.240 Definition. A diagram $D : \mathbf{\Delta}_{/K} \to \mathsf{E}$ shall be called *flat* when it sends every surjection $\Delta^n \twoheadrightarrow \Delta^m \to K$ in $\mathbf{\Delta}_{/K}$ to an isomorphism (equivalently, when it sends morphisms in $\mathbf{\Delta}_{/K}$ over the same non-degenerate simplex of K to isomorphisms). For example, the last vertex map $\mathbf{\Delta}_{/K} \to K$ is flat, hence so is any diagram pulled back from it.

1.4.241 Lemma. *Let $D : \mathbf{\Delta}_{/K} \to \mathsf{E}$ be flat* (1.4.240)*, and suppose its right Kan extension $\pi_*D : \mathbf{\Delta} \to \mathsf{E}$ along $\pi : \mathbf{\Delta}_{/K} \to \mathbf{\Delta}$ exists. If K has dimension $\leq d$, then π_*D is d-truncated.*

Proof. Recall that the matching maps of a cosimplicial object detect whether or not it is truncated (1.2.16). The matching map $(\pi_*D)^n \to M^n(\pi_*D)^\bullet$ (1.2.14)(1.2.15) is given by

$$\prod_{f:\Delta^n\to K} D(f) \to \lim_{\substack{[n]\twoheadrightarrow[k]\\ k<n}} \prod_{g:\Delta^k\to K} D(g). \tag{1.4.241.1}$$

This map is the product over all $f : \Delta^n \to K$ of the maps

$$D(f) \to \lim_{\substack{([n]\twoheadrightarrow[k]\xrightarrow{g}K)=f\\ k<n}} D(g). \tag{1.4.241.2}$$

Since D is flat, each map $D(f) \to D(g)$ is an isomorphism, so this map is an isomorphism iff the indexing category $([n] \downarrow [k] \downarrow K)_{f,k<n}$ is Kan contractible. Now the category $([n] \downarrow [k] \downarrow K)_f$ of factorizations of f has a final object, namely the non-degenerate simplex underlying f (1.2.10), hence is Kan contractible. If f is degenerate, then its full subcategory $([n] \downarrow [k] \downarrow K)_{f,k<n}$ omitting the isomorphism $k = n$ retains this final object, hence remains Kan contractible. We conclude that if all n-simplices of K are degenerate, then the nth matching map of $\pi_* D$ is an isomorphism. In particular, if K is d-dimensional, then the matching maps of $\pi_* D$ of all degrees $> d$ are isomorphisms, hence $\pi_* D$ is d-truncated (1.2.16). $\square$

1.4.242 Lemma. *Suppose C has finite coproducts. An object of $\mathsf{P}(\mathsf{C})$ lies in $\mathsf{Sif}_{\mathsf{fin}}(\mathsf{C})$ iff it is (the formal colimit of) a simplicial object of the form $(\mathbf{\Delta}_{/K} \to \mathbf{\Delta})_! D : \mathbf{\Delta}^{\mathsf{op}} \to \mathsf{C}$ for some finite simplicial set K and some flat* (1.4.240) *diagram $\mathbf{\Delta}_{/K} \to \mathsf{C}$.*

Proof. An object of $\mathsf{Sif}_{\mathsf{fin}}(\mathsf{C})$ is by definition the siftedization of a finite diagram $p : K \to \mathsf{C}$. The siftedization of a finite diagram is given by the Bousfield–Kan transform $p_\triangle$ (1.4.239), which is of the desired form by definition.

For the converse, we should show that $(\mathbf{\Delta}_{/K} \to \mathbf{\Delta})_! D = \pi_! D$ is always a formal finite sifted colimit, that is $\operatorname{colim}^{\mathsf{P}(\mathsf{C})} \pi_! D \in \mathsf{Sif}_{\mathsf{fin}}(\mathsf{C})$. Since $\mathbf{\Delta}$ is cosifted (1.4.220), we have $\operatorname{colim}^{\mathsf{P}(\mathsf{C})} \pi_! D \in \mathsf{Sif}(\mathsf{C})$, so it suffices to show that $\operatorname{colim}^{\mathsf{Sif}(\mathsf{C})} \pi_! D \in \mathsf{Sif}_{\mathsf{fin}}(\mathsf{C})$. Now the comparison morphism

$$\pi_! D = \pi_!^{\mathsf{C}} D \leftarrow \pi_!^{\mathsf{Sif}(\mathsf{C})} D \tag{1.4.242.1}$$

is an isomorphism since the left Kan extension $\pi_!$ takes finite coproducts (1.4.238), which are preserved by $\mathsf{C} \hookrightarrow \mathsf{Sif}(\mathsf{C})$ (1.4.232). It therefore suffices to show that $\operatorname{colim}^{\mathsf{Sif}(\mathsf{C})} \pi_!^{\mathsf{Sif}(\mathsf{C})} D \in \mathsf{Sif}_{\mathsf{fin}}(\mathsf{C})$. The simplicial object $\pi_!^{\mathsf{Sif}(\mathsf{C})} D : \mathbf{\Delta}^{\mathsf{op}} \to \mathsf{Sif}(\mathsf{C})$ is truncated since K is finite-dimensional (1.4.241), hence its colimit in $\mathsf{Sif}(\mathsf{C})$ is a finite iterated colimit of its values (**??**), which lie in C since $\pi_!^{\mathsf{Sif}(\mathsf{C})} D = \pi_!^{\mathsf{C}} D$. Finally, recall that $\mathsf{Sif}_{\mathsf{fin}}(\mathsf{C}) \subseteq \mathsf{Sif}(\mathsf{C})$ is closed under finite colimits (1.4.236). $\square$

1.4.243 Lemma. *Let $f : \mathsf{C} \to \mathsf{D}$ be a functor, and let $F \to G$ be a morphism in $\mathsf{P}(\mathsf{C})$. The left Kan extension functor $f_! : \mathsf{P}(\mathsf{C}) \to \mathsf{P}(\mathsf{D})$ preserves all pullbacks of $F \to G$ iff it preserves the pullback diagrams*

$$\begin{tikzcd} F \times_G c' \arrow[r] \arrow[d] & F \times_G c \arrow[d] \\ c' \arrow[r] & c \end{tikzcd} \tag{1.4.243.1}$$

for all morphisms $c' \to c \to G$ from $c', c \in \mathsf{C}$.

Proof. The diagram (1.4.243.1) is the pullback of $F \to G$ along $c' \to c \to G$, which is more fully illustrated as follows.

$$\begin{tikzcd} F \times_G c' \arrow[r] \arrow[d] & F \times_G c \arrow[r] \arrow[d] & F \arrow[d] \\ c' \arrow[r] & c \arrow[r] & G \end{tikzcd} \tag{1.4.243.2}$$

Now if $f_!$ preserves every pullback of $F \to G$, then it preserves the right fiber square and composite fiber square above, hence preserves the left fiber square (1.4.243.1) by cancellation (1.1.57).

It remains to show that if $f_!$ preserves all fiber squares (1.4.243.1), then it preserves all pullbacks of $F \to G$. Consider the pullback of $F \to G$ under a morphism $Z \to G$ from arbitrary $Z \in \mathsf{P}(\mathsf{C})$. Writing Z as a colimit of representables and appealing to the fact that presheaf fiber product is cocontinuous (**??**) and $f_!$ is cocontinuous, we may reduce to the case that Z is representable. That is, we are to show that $f_!$ preserves the pullback square

$$\begin{tikzcd} F \times_G c_0 \arrow[r] \arrow[d] & F \arrow[d] \\ c_0 \arrow[r] & G \end{tikzcd} \tag{1.4.243.3}$$

for any map $c_0 \to G$ from $c_0 \in \mathsf{C}$. Present G via the tautological colimit diagram $G = \operatorname{colim}_{(\mathsf{C}\downarrow G)} c$ (1.4.193) and note $F = \operatorname{colim}_{(\mathsf{C}\downarrow G)} F \times_G c$ (**??**). Thus we would like to show that the following pullback square

$$\begin{tikzcd} F \times_G c_0 \arrow[r] \arrow[d] & \operatorname*{colim}_{(\mathsf{C}\downarrow G)} F \times_G c \arrow[d] \\ c_0 \arrow[r] & \operatorname*{colim}_{(\mathsf{C}\downarrow G)} c \end{tikzcd} \tag{1.4.243.4}$$

in $\mathsf{P}(\mathsf{C})$ is preserved by $f_!$. Note that we can see the square (1.4.243.4) to be a pullback using (1.4.189) since the diagram $(\mathsf{C} \downarrow G) \to \mathsf{Fun}(\Delta^1, \mathsf{P}(\mathsf{C}))$ given by $F \times_G c \to c$ sends edges in $(\mathsf{C} \downarrow G)$ to pullback squares (1.4.243.1) in $\mathsf{P}(\mathsf{C})$. This property is preserved by $f_!$ by assumption (1.4.243.1), so since $f_!$ is cocontinuous, we conclude it sends (1.4.243.4) to a pullback square. $\square$

⋆ **1.4.244 Corollary** (Pullbacks and properties preserved by presheaf left Kan extension)**.** *Let $f : \mathsf{C} \to \mathsf{D}$ be a functor. If f preserves pullbacks, then $f_! : \mathsf{P}(\mathsf{C}) \to \mathsf{P}(\mathsf{D})$ preserves pullbacks of representable morphisms (in particular, sends representable morphisms to representable morphisms). More generally, for properties of morphisms $\mathcal{P}$ (in C) and $\mathcal{Q}$ (in D) preserved under pullback, if f sends pullbacks of $\mathcal{P}$-morphisms to pullbacks of $\mathcal{Q}$-morphisms (in particular, sends $\mathcal{P}$-morphisms to $\mathcal{Q}$-morphisms), then so does $f_!$.*

Proof. We apply (1.4.243). If $F \to G$ is a $\mathcal{P}$-morphism in $\mathsf{P}(\mathsf{C})$, then the pullbacks (1.4.243.1) are pullbacks of $\mathcal{P}$-morphisms in C, hence are preserved by f by assumption. Thus (1.4.243) guarantees that $f_!$ preserves all pullbacks of $F \to G$. To see that $f_!(F \to G)$ is a $\mathcal{Q}$-morphism, note that every morphism $d \to f_!G$ from $d \in \mathsf{D}$ factors through $f_!(c \to G)$ for some $c \in \mathsf{C}$, and the pullback $f_!(F \times_G c \to c)$ of $f_!(F \to G)$ is a $\mathcal{Q}$-morphism in D by hypothesis on f. $\square$

Stable ∞-categories

Chapter 2

Topology

2.1 Topological spaces

Basic notions

We assume familiarity with basic point-set topology. Nevertheless, we include a brief review to set notation and terminology.

⋆ **2.1.1 Definition** (Topological space)**.** A *topology* on a set X is a collection of subsets $T \subseteq 2^X$ (called 'open subsets') satisfying the following axioms:

(2.1.1.1) $\varnothing$ and X are open.

(2.1.1.2) If U and V are open, then is $U \cap V$ is open.

(2.1.1.3) If U_α are open, then $\bigcup_\alpha U_\alpha$ is open.

A subset is called *closed* when its complement is open. A *topological space* is a set equipped with a topology. A map between topological spaces is called *continuous* when the inverse image of every open subset is open. The category of topological spaces and continuous maps is denoted Top.

2.1.2 Exercise. Let X be a set, and let $U_i \subseteq X$ be subsets. Show that there is a coarsest topology on X in which all U_i are open, namely the topology consisting of arbitrary unions of finite intersections of the U_i. This is called the topology on X *generated* by (declaring all) the $U_i \subseteq X$ (to be open).

2.1.3 Definition (Neighborhood)**.** Let X be a topological space, and let $x \in X$ be a point. A *neighborhood* of x is a subset $N \subseteq X$ containing an open subset $U \subseteq X$ which contains x. A *neighborhood base* at x is a collection $\mathcal{N}$ of neighborhoods of x with the property that every open subset $U \subseteq X$ containing x contains some $N \in \mathcal{N}$. To say that x has 'arbitrarily small' neighborhoods with some property means that the collection of all neighborhoods of x with this property is a neighborhood base of x.

2.1.4 Definition (Locally compact)**.** A topological space is called *locally compact* iff every point has arbitrarily small compact neighborhoods. See [86, §29] for basic properties of locally compact Hausdorff topological spaces (or the reader may take them as exercises).

2.1.5 Exercise (Semi-continuity of cohomology)**.** Show that the dimension of the middle cohomology of $A \xrightarrow{f} B \xrightarrow{g} C$ equals $\dim B - \operatorname{rank} f - \operatorname{rank} g$. Conclude that for any complex of vector bundles $V^\bullet$ on a topological space X, the locus $\{x \in X \mid \dim H^i V^\bullet_x \leq r\} \subseteq X$ is open for all i and r (use the fact that the set of matrices of rank $\geq a$ is open).

2.1.6 Lemma (Minimizing the support of a complex)**.** *Let $V^\bullet$ be a complex of vector bundles on a topological space X, and let $x \in X$ be a point. If $H^i V^\bullet_x = 0$ for $i \notin I \subseteq \mathbb{Z}$, then $V^\bullet$ is quasi-isomorphic (in a neighborhood of x) to a complex of vector bundles supported in degrees I.*

Proof. It suffices to show that if $H^i V^\bullet_x = 0$, then there exists a surjective quasi-isomorphism from $V^\bullet$ onto a complex which vanishes in degree i.

Suppose $H^i V_x^\bullet = 0$. Choose a basis for the image of $V_x^{-1} \to V_x^0$, and lift it to an injection $\underline{\mathbb{R}}^n \to V^{-1}$ in a neighborhood of x. Now there is an induced injection from the acyclic complex $\underline{\mathbb{R}}^n \to V^{-1} \to V^{-1}/\mathbb{R}^n$ to $V^\bullet$. The quotient is the desired surjective quasi-isomorphism. $\square$

Properties of morphisms

We recall here some important properties of morphisms (1.1.41) of topological spaces.

⋆ **2.1.7 Exercise.** Show that the following properties of morphisms of topological spaces $f : X \to Y$ are closed under composition (1.1.44).

(2.1.7.1) f is open (i.e. the image of any open set is open).
(2.1.7.2) f is closed (i.e. the image of any closed set is closed).
(2.1.7.3) f is an embedding (i.e. a homeomorphism onto its image).
(2.1.7.4) f has local sections (i.e. there is an open cover $Y = \bigcup_i U_i$ such that each inclusion $U_i \to Y$ factors through $X \to Y$).

$$\begin{tikzcd} & X \arrow[d] \\ U_i \arrow[ur, dashed] \arrow[r] & Y \end{tikzcd}$$

Show that the following properties of morphisms of topological spaces $f : X \to Y$ are preserved under pullback (1.1.60).

(2.1.7.5) f is open.
(2.1.7.6) f is an embedding.
(2.1.7.7) f is a closed embedding.
(2.1.7.8) f is has local sections.

Show that being closed is not preserved under pullback, but that it is preserved under pullback along open embeddings.

2.1.8 Exercise (Locally closed embedding)**.** A map of topological spaces $X \to Y$ is called a *locally closed embedding* iff it can be factored as a closed embedding $X \to U$ followed by an open embedding $U \to Y$. Show that locally closed embeddings are preserved under pullback and closed under composition.

2.1.9 Exercise (Locally trivial)**.** A map of topological spaces $X \to Y$ is called *locally trivial* iff there exists an open cover $Y = \bigcup_i U_i$ such that each restriction $X \times_Y U_i \to U_i$ is isomorphic to the projection $U_i \times F_i \to U_i$ for some topological space F_i. Show that being locally trivial is preserved under pullback.

⋆ **2.1.10 Exercise** (Local isomorphism)**.** A map of topological spaces $X \to Y$ is called a *local isomorphism* iff there exists a collection of open embeddings $\{V_i \to X\}$ which is jointly surjective (an 'open covering') such that each composition $V_i \to X \to Y$ is an open embedding. Show that local isomorphisms are preserved under pullback and closed under composition.

⋆ **2.1.11 Definition** (Target-local property)**.** Let $\mathcal{P}$ be a property of morphisms of topological spaces. We say $\mathcal{P}$ is *local on the target* when for every open cover $Y = \bigcup_i U_i$, a morphism $X \to Y$ has $\mathcal{P}$ iff every pullback $X \times_Y U_i \to U_i$ has $\mathcal{P}$. In particular, $\mathcal{P}$ is preserved under pullback by open embeddings.

2.1.12 Exercise. Show that the properties of morphisms of topological spaces (2.1.7.1)–(2.1.7.4) and (2.1.8)–(2.1.10) are local on the target.

2.1.13 Exercise. Let $\mathcal{P}$ be a property of morphisms of topological spaces which is preserved under pullback. Show that $\mathcal{P}$ is local on the target iff it satisfies the following two properties.

(2.1.13.1) For every map $X \to Y$ and every map $Z \twoheadrightarrow Y$ admitting local sections, if $X \times_Y Z \to Z$ has $\mathcal{P}$ then so does $X \to Y$.

(2.1.13.2) If a collection of maps $f_i : X_i \to Y_i$ all have $\mathcal{P}$, then so does their disjoint union $\bigsqcup_i f_i : \bigsqcup_i X_i \to \bigsqcup_i Y_i$.

2.1.14 Exercise. Let $\mathcal{P}$ be a property of morphisms of topological spaces which is local on the target (hence, in particular, preserved under pullback by open embeddings). Show that $\mathcal{P}$ is preserved under pullback by local isomorphisms.

⋆ **2.1.15 Definition** (Source-local property)**.** Let $\mathcal{P}$ be a property of morphisms of topological spaces. We say $\mathcal{P}$ is *local on the source* when for every open cover $X = \bigcup_i V_i$ and every collection of open sets $U_i \subseteq Y$ on the same index set, a map $f : X \to Y$ with $f(V_i) \subseteq U_i$ satisfies $\mathcal{P}$ iff all its restrictions $V_i \to U_i$ satisfy $\mathcal{P}$.

2.1.16 Exercise. Show that being open (2.1.7.1) is local on the source.

2.1.17 Exercise. Show that being a local isomorphism (2.1.10) is local on the source. Conversely, show that if $\mathcal{P}$ is local on the source and contains all isomorphisms, then it contains all local isomorphisms. This justifies the term 'local isomorphism'.

2.1.18 Exercise. Show that a property which is local on the source is also local on the target.

2.1.19 Exercise. Show that if $\mathcal{P}$ is local on the source, then $\varnothing \to Y$ has $\mathcal{P}$ for every topological space Y.

Recall that closed maps of topological spaces are not preserved under pullback (2.1.7). 'Universally closed' is the weakest property which is preserved under pullback and implies closed (2.1.20). It turns out that this notion is a relative form of compactness. We will see that, like compactness, it has equivalent characterizations in terms of coverings and subnet convergence (2.1.29).

2.1.20 Definition (Universally closed)**.** A map of topological spaces $X \to Y$ is called *universally closed* when for every map $Z \to Y$, the pullback $X \times_Y Z \to Z$ is closed.

2.1.21 Exercise. Show that being universally closed is preserved under pullback, closed under composition, and local on the target.

2.1.22 Exercise. Show that an embedding of topological spaces is closed iff it is universally closed.

2.1.23 Exercise. Show that if the composition $X \to Y \to Z$ is universally closed and $X \to Y$ is surjective, then $Y \to Z$ is universally closed.

2.1.24 Definition (Limit pointed topological space)**.** A *limited pointed topological space* $(X, 0)$ is a topological space X together with a point $0 \in X$ whose complement is dense ($\overline{X \setminus 0} = X$); we set $X^* = X \setminus 0$. A map of limit pointed topological spaces $f : (X, 0_X) \to (Y, 0_Y)$ is a map satisfying $f^{-1}(0_Y) = 0_X$ (equivalently, $f(0_X) = 0_Y$ and $f(X^*) \subseteq Y^*$).

A limit pointed topological space X is called *discrete* when X^* has the discrete topology. Given any limit pointed topological space X, we can consider the topology on it obtained by adjoining all subsets of X^* as open sets; this is called the discretization X^δ. There is an evident map of limit pointed topological spaces $X^\delta \to X$, composition with which induces, for discrete limit pointed topological spaces Y, a bijection between maps of limit pointed topological spaces $Y \to X^\delta$ and $Y \to X$.

2.1.25 Exercise. Show that there is a unique topology on $\mathbb{Z}_{\geq 0} \cup \{\infty\}$ with the property that a map $f : \mathbb{Z}_{\geq 0} \cup \{\infty\} \to X$ is continuous iff the sequence $f(0), f(1), f(2), \ldots$ converges to $f(\infty)$. Show that $\mathbb{Z}_{\geq 0} \cup \{\infty\}$ with this topology is a discrete limit pointed topological space.

2.1.26 Definition (Swarm)**.** A *swarm* in a topological space X is a limited pointed topological space S and a map $S^* \to X$. A *completed swarm* is a map $S \to X$. A *limit* of a swarm $S^* \to X$ is a point $x \in X$ for which sending $0 \mapsto x$ defines a completed swarm, and a swarm is called *convergent* when it has at least one limit. A *subswarm* of a swarm $S^* \to X$ is its pre-composition with a map of limit pointed topological spaces $T \to S$.

A *relative swarm* on a map $X \to Y$ is a commuting diagram of solid arrows

$$\begin{tikzcd} S^* \arrow[r] \arrow[d] & X \arrow[d] \\ S \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd} \tag{2.1.26.1}$$

and a *completed relative swarm* is a relative swarm along with a dotted arrow making the diagram commute. The definition of limits, convergence, and subswarms carry over analogously to the relative context.

A (relative) swarm is called discrete when its underlying limit pointed topological space is discrete. Pre-composition with discretization is a subswarm.

2.1.27 Exercise. Show that the closure of a subset A of a topological space X is the set of limits of swarms $S^* \to X$ landing inside A.

2.1.28 Definition (Compact)**.** A topological space X is called *compact* when for every open cover $X = \bigcup_i U_i$ there exists a finite subcollection which cover X.

$\star$ **2.1.29 Proposition.** *For a map of topological spaces $f : X \to Y$, the following are equivalent:*

(2.1.29.1) (Universally closed) *For every map $Z \to Y$, the pullback $X \times_Y Z \to Z$ is closed.*

(2.1.29.2) (Finite subcover property) *For collection of open subsets $\{U_i \subseteq X\}_i$ covering $f^{-1}(y)$, there exists a finite subcollection which cover $f^{-1}(V)$ for some open neighborhood $y \in V \subseteq Y$.*

(2.1.29.3) (Subswarm lifting property) *Every relative swarm on $X \to Y$ has a convergent relative subswarm.*

(2.1.29.4) *The map $X \to Y$ is closed and has compact fibers.*

These conditions are a relative version of compactness: a topological space X is compact iff the map $X \to *$ is universally closed.

Proof. Let us show the subswarm lifting property (2.1.29.3) implies universal closedness (2.1.29.1). Since the subswarm lifting property is evidently preserved under pullback, it suffices to show that it implies that $X \to Y$ is closed. Let $A \subseteq X$ be closed, and let us show that $f(A)$ is closed. Suppose $S^* \to Y$ is a swarm contained in $f(A)$ converging to some $y \in Y$, and let us show that $y \in f(A)$. By passing to a subswarm, we can assume that S^* has the discrete topology, and hence we can lift $S^* \to Y$ to X so that it lands inside A. This is now a relative swarm, which (after passing to a further subswarm) has a limit by the subswarm lifting property, which lies in A since A is closed. We have thus shown $y \in f(A)$ as desired.

We show that universal closedness (2.1.29.1) implies the subswarm lifting property (2.1.29.3). Let $X \to Y$ be universally closed, and fix a relative swarm $(S, S^*) \to (Y, X)$. Consider the image in S of the closure of the image of $S^* \to X \times_Y S$. It is closed since $X \times_Y S \to S$ is closed, and it evidently contains S^*, so it must also contain 0_S. In other words, the closure of the image of $S^* \to X \times_Y S$ contains a point lying over 0_S. By the characterization of closure in terms of subswarms (2.1.27), there exists a completed swarm $T \to X \times_Y S$ sending T^* inside the image of $S^* \to X \times_Y S$ and sending 0_T to a point over 0_S (so $T \to S$ is a map of limit pointed topological spaces).

$$\begin{tikzcd} T^* \arrow[r, dashed] \arrow[d] & S^* \arrow[r] \arrow[d] & X \arrow[d] \\ T \arrow[r] \arrow[urr] & S \arrow[r] & Y \end{tikzcd} \tag{2.1.29.5}$$

To make this a convergent relative subswarm, we must lift the composition $T^* \to T \to X \times_Y S$ to $S^* \to X \times_Y S$. By construction, the image of T^* lies inside the image of S^*, so lifts exist pointwise. By replacing T with its discretization $T^\delta \to T$, we may assume wlog that T^* has the discrete topology, so every lift is continuous.

□

Now for some properties of morphisms of topological spaces which are defined using the relative diagonal (1.1.63). Recall that for any property of morphisms $\mathcal{P}$, a morphism is said to have $\mathcal{P}_\Delta$ when its diagonal has $\mathcal{P}$ (1.1.65). Recall that if $\mathcal{P}$ is preserved under pullback then so is $\mathcal{P}_\Delta$ (1.1.66).

2.1.30 Lemma. *Let $\mathcal{P}$ be a property of morphisms of topological spaces. If $\mathcal{P}$ is local on the target* (2.1.11)*, then so is $\mathcal{P}_\Delta$.*

Proof. Let $X \to Y$ be a morphism, and let $Y = \bigcup_i U_i$ be an open cover with the property that every pullback $X \times_Y U_i \to U_i$ has $\mathcal{P}_\Delta$. The diagonal of the pullback $X \times_Y U_i \to U_i$ is the pullback of the diagonal $X \to X \times_Y X$ to the inverse image of $U_i \subseteq Y$ inside $X \times_Y X$ (1.1.66). It follows that $X \to X \times_Y X$ has $\mathcal{P}$ since $\mathcal{P}$ is local on the target. □

2.1.31 Exercise. Show that the diagonal of any map of topological spaces is an embedding. Show that the diagonal of any injective map of topological spaces is an isomorphism.

2.1.32 Exercise. Show that the diagonal of a local isomorphism of topological spaces is an open embedding.

⋆ **2.1.33 Exercise** (Separated)**.** Show that for a morphism of topological spaces $f : X \to Y$, the following are equivalent:

(2.1.33.1) Every pair of distinct points $x_1, x_2 \in X$ in the same fiber $f(x_1) = f(x_2)$ have disjoint open neighborhoods $U_1 \cap U_2 = \varnothing$, $x_i \in U_i \subseteq X$.

(2.1.33.2) The relative diagonal $X \to X \times_Y X$ is a closed embedding.

(2.1.33.3) Every relative swarm on $X \to Y$ has at most one limit.

A morphism satisfying these conditions is called *separated*; this is a relative version of the Hausdorff property (X is Hausdorff iff $X \to *$ is separated). Show that being separated is preserved under pullback, closed under composition, and local on the target.

⋆ **2.1.34 Exercise** (Proper)**.** A map of topological spaces is called *proper* iff all its iterated diagonals are universally closed. Show that a map has proper diagonal iff it is separated. Conclude that a map is proper iff it is separated and universally closed (in particular, $X \to *$ is proper iff X is compact Hausdorff).

2.1.35 Exercise. Show that a map of topological spaces is a proper local isomorphism iff it is locally trivial (2.1.9) with finite fibers.

Now that we have seen the notions of separatedness and properness, let us have a more abstract discussion of properties of morphisms of topological spaces defined in terms of their diagonal (1.1.65).

2.1.36 Exercise. Let $\mathcal{P}$ be a property of morphisms of topological spaces which is local on the target. Show that $\mathcal{P}_\Delta$ is also local on the target.

One reason to consider properties of the diagonal is to apply cancellation (1.1.68).

2.1.37 Exercise. Prove both directly and using cancellation that if $X \to Y \to Z$ are maps of topological spaces whose composition $X \to Z$ is separated, then the first map $X \to Y$ is separated.

2.1.38 Exercise. Prove both directly and using cancellation that if $X \to Y \to Z$ are maps of topological spaces whose composition $X \to Z$ is an embedding, then the first map $X \to Y$ is an embedding.

2.1.39 Exercise. Prove both directly and using cancellation that if $X \to Y \to Z$ are maps of topological spaces with $X \to Z$ an open embedding and $Y \to Z$ is a local isomorphism, then $X \to Y$ is an open embedding. Conclude that any section of a local isomorphism is an open embedding.

2.1.40 Exercise. Prove both directly and using cancellation that if $X \to Y \to Z$ are maps of topological spaces with $X \to Z$ universally closed and $Y \to Z$ separated, then $X \to Y$ is universally closed. Deduce that a compact subspace of a Hausdorff space is closed.

Paracompactness and partitions of unity

⋆ **2.1.41 Definition** (Bump function)**.** Let X be a topological space. Given a point $x \in X$, we say that X *has bump functions at* x when x has arbitrarily small (2.1.3) closed neighborhoods $x \in N \subseteq X$ for which there exists a continuous function $\varphi : X \to \mathbb{R}$ (called a 'bump function') satisfying $\varphi(x) > 0$ and $\varphi|_{X \setminus N} \equiv 0$. We say that X has *local* bump functions at x when x has an open neighborhood U which has bump functions at x (having bump functions at x implies having local bump functions at x, and the converse holds provided x has arbitrarily small closed neighborhoods, e.g. if X is locally compact Hausdorff). We say that X has (local) bump functions when it has (local) bump functions at every point. The term 'bump function' often tacitly implies the additional property that $\varphi \geq 0$; note this can always be achieved by relacing φ with φ^2.

2.1.42 Definition (Paracompact [20])**.** Let X be a topological space. A *refinement* of an open cover $X = \bigcup_i U_i$ is another open cover $X = \bigcup_j V_j$ where each V_j is contained in some U_i. An open cover $X = \bigcup_i U_i$ is called *locally finite* when every point of X has an open neighborhood which intersects at most finitely many U_i. The topological space X is called *paracompact* when every open cover has a locally finite refinement.

2.1.43 Exercise. Let X be a locally compact Hausdorff topological space. Show that if X is σ-compact (is a countable union of interiors of compact subspaces), then X is paracompact.

⋆ **2.1.44 Definition** (Partition of unity)**.** Let X be a topological space. A *partition of unity* on X is a collection of functions $\varphi_i : X \to \mathbb{R}_{\geq 0}$ which is locally finite (every point of X has a neighborhood over which all but finitely many φ_i are identically zero) and satisfies $\sum_i \varphi_i \equiv 1$. A partition of unity *subordinate* to an open cover $X = \bigcup_i U_i$ is a partition of unity $\sum_i \varphi_i \equiv 1$ (with the same index set) on X satisfying $\operatorname{supp} \varphi_i \subseteq U_i$. A topological space is said to *admit partitions of unity* when it has a partition of unity subordinate to every open cover.

2.1.45 Remark. For the purpose of proving the existence of a partition of unity, the condition that $\sum_i \varphi_i \equiv 1$ may be weakened to $\sum_i \varphi_i > 0$. Indeed, in the latter case, the functions $\varphi_i / \sum_j \varphi_j$ form a partition of unity in the former sense.

2.1.46 Proposition (Dieudonné [20][86, Theorem 41.7])**.** *A paracompact Hausdorff topological space admits partitions of unity.* □

2.1.47 Remark. The *numerable topology* of Dold [22] is a 'Grothendieck topology' in which a collection of open subsets $U_i \subseteq X$ counts as a covering iff it has a subordinate partition of unity. Every (ordinary) open cover of a paracompact Hausdorff space is a numerable open cover by (2.1.46). Most (all?) results about paracompact Hausdorff spaces are based on (2.1.46), hence can be viewed more generally as results about the numerable topology on arbitrary topological spaces.

2.1.48 Exercise. Let X be a paracompact Hausdorff topological space, and let V/X be a vector bundle. Show that there exists a positive definite inner product on V. Conclude that every short exact sequence of vector bundles $0 \to V' \to V \to V'' \to 0$ on X splits.

2.1.49 Lemma. *Let $V \to M$ be a vector bundle over a paracompact Hausdorff topological space. Every open neighborhood of the zero section contains the image of an open embedding $V \hookrightarrow V$ which is the identity in a neighborhood of the zero section.*

Proof. Fix a metric (positive definite inner product) on V (sum up local metrics via a partition of unity), and find a continuous function $\varepsilon : V \to \mathbb{R}_{>0}$ (also using partition of unity) so that the fiberwise ε-balls of V are contained within the given open neighborhood of the zero section $U \subseteq V$. Now consider the map $V \to V$ given by $v \mapsto \alpha(\varepsilon^{-1}|v|) \cdot v$ for some function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{>0}$ which equals 1 in a neighborhood of the origin and for which $x \mapsto \alpha(x)x$ is a diffeomorphism $[0, \infty) \to [0, 1)$. □

⋆ **2.1.50 Nagata–Smirnov Metrization Theorem** ([87][100][86, Theorem 42.1])**.** *A topological space is metrizable iff it is paracompact Hausdorff and locally metrizable.* □

It follows that every open subset of a locally metrizable paracompact Hausdorff topological space is paracompact.

⋆ **2.1.51 Lemma** (From bump functions to partitions of unity)**.** *Let M be a paracompact Hausdorff topological space. If M is locally compact, then for any class of 'regular' bump functions on M, there exist partitions of unity on M comprised of finite sums of regular bump functions divided by locally finite sums of regular bump functions.*

Proof. Let $M = \bigcup_i U_i$ be an open cover. By passing to a refinement, we may assume that each U_i has compact closure in M. By passing to a further refinement, we may assume that the open cover is also locally finite.

Fix a continuous partition of unity $\varphi_i : X \to \mathbb{R}_{\geq 0}$ (2.1.46) subordinate to the open cover. Now $\operatorname{supp} \varphi_i$ is closed and contained U_i, whose closure is assumed compact, so $\operatorname{supp} \varphi_i$ is also compact. Since $\operatorname{supp} \varphi_i$ is compact, there is a finite sum of regular bump functions $\psi_i : M \to \mathbb{R}_{\geq 0}$ supported inside U_i which is positive everywhere on $\operatorname{supp} \varphi_i$. Since the open cover $M = \bigcup_i U_i$ is locally finite, so is the collection of functions ψ_i. The sum $\sum_i \psi_i$ is everywhere positive since the $\operatorname{supp} \varphi_i$ cover M. □

Topological groups

2.1.52 Definition (Topological group)**.** A topological group is a group object (1.1.128) in Top.

2.1.53 Lemma. *A topological group is Hausdorff iff the identity is a closed point.*

Proof. For a group object G in any category, the diagonal map $G \to G \times G$ is a pullback of the inclusion of the identity $* \to G$ (1.1.129). $\square$

2.1.54 Lemma. *A locally compact Hausdorff topological group is paracompact.*

Proof. Since G is locally compact Hausdorff, there exists a compact neighborhood of the identity $K \subseteq G$. Consider the infinite ascending union $K_\infty = \bigcup_i K^i \subseteq G$, which is evidently a subgroup of G. Since $K \cdot K_\infty \subseteq K_\infty$ and K contains a neighborhood of the identity, it follows that $K_\infty \subseteq G$ is open, thus also locally compact Hausdorff. Being a countable union of compact subspaces (the images of $K^i \to G$), the subgroup K_∞ is paracompact (2.1.43). Since K_∞ is open, the quotient G/K_∞ is discrete. Choosing a section of the projection $G \to G/K_\infty$ defines a homeomorphism $G = (G/K_\infty) \times K_\infty$. It is immediate that an open disjoint union of paracompact spaces is paracompact. $\square$

2.2 Sheaves

A presheaf F on a topological space X assigns to each open subset $U \subseteq X$ a set $F(U)$ and to each inclusion $V \subseteq U$ a 'restriction' map $F(U) \to F(V)$, compatible with composition for triples $W \subseteq V \subseteq U$. A presheaf F is called a *sheaf* when, roughly speaking, an element of $F(U)$ amounts to *local data* on U, where 'locality' is understood via the restriction maps. Sheaves originated in work of Leray [68, 82], though the modern definition of a sheaf was formulated a bit later, notably by Cartan. It makes sense to consider presheaves and sheaves valued in any category, not just the category of sets. Sheaves valued in 2-categories were first considered by Giraud [34], who introduced sheaves valued in the 2-category of groupoids. More generally, one can consider sheaves valued in any ∞-category.

Here we review the basic theory of sheaves, sheafification, pushforward, pullback, etc. We will also explain the meaning and the utility of ∞-categories in the context of sheaves.

Basic notions

2.2.1 Definition (Category of open subsets)**.** Let X be a topological space. We denote by $\mathsf{Open}(X)$ the poset of open subsets of X, regarded as a category as in (1.1.3); that is, an object of $\mathsf{Open}(X)$ is an open subset $U \subseteq X$, and there is a single morphism from U to V when $U \subseteq V$.

⋆ **2.2.2 Definition** (Presheaf on a topological space)**.** A *presheaf* on a topological space X is a presheaf (1.1.78) on the category of open sets $\mathsf{Open}(X)$ of X, and we denote by $\mathsf{P}(X) = \mathsf{P}(\mathsf{Open}(X)) = \mathsf{Fun}(\mathsf{Open}(X)^{\mathsf{op}}, \mathsf{Set})$ the category of presheaves on X. More generally, $\mathsf{P}(X; \mathsf{E}) = \mathsf{P}(\mathsf{Open}(X); \mathsf{E}) = \mathsf{Fun}(\mathsf{Open}(X)^{\mathsf{op}}, \mathsf{E})$ denotes the category of presheaves on X valued in a category E. Dually, a *precosheaf* on X is a presheaf on $\mathsf{Open}(X)^{\mathsf{op}}$.

2.2.3 Example. Here are some examples of presheaves.

(2.2.3.1) For any topological space X, we can assign to $U \subseteq X$ the set $C(U)$ of continuous functions $U \to \mathbb{R}$, and to an inclusion $U \subseteq V$ the restriction map $C(V) \to C(U)$. Thus $U \mapsto C(U)$ is a presheaf on X.

(2.2.3.2) $U \mapsto C(U \times U)$ is a presheaf on any topological space.

(2.2.3.3) Associating to $U \subseteq X$ the set of embeddings of U into $\mathbb{R}^n$ (some fixed n) is a presheaf (the restriction of an embedding is an embedding).

(2.2.3.4) The *constant presheaf* assigns to every $U \subseteq X$ a fixed set S and to every inclusion the identity map $\mathbf{1}_S$.

(2.2.3.5) Associating to U the set of isomorphism classes of vector bundles on U is a presheaf.

(2.2.3.6) On a smooth manifold, the assignment $U \mapsto C^\infty(U)$ is a presheaf.

(2.2.3.7) On a smooth manifold, assigning to U the set of smooth embeddings of U into a fixed $\mathbb{R}^n$ is a presheaf.

(2.2.3.8) On a smooth manifold, assigning to U the set of smooth immersions of U into a fixed $\mathbb{R}^n$ is a presheaf.

We have omitted an explicit description of the restriction maps for most of these examples since they are quite obvious. The same holds for most presheaves we will encounter.

⋆ **2.2.4 Definition** (Čech descent)**.** Let F be a presheaf on a topological space X. Given an open covering $U = \bigcup_i U_i$ of an open subset $U \subseteq X$, there is a natural map

$$F(U) \to \lim\Big(\prod_i F(U_i) \rightrightarrows \prod_{i,j} F(U_i \cap U_j)\Big), \tag{2.2.4.1}$$

and we say that F satisfies *descent* for the open cover $U = \bigcup_i U_i$ when this map is an isomorphism.

⋆ **2.2.5 Definition** (Sheaf on a topological space)**.** A presheaf F on a topological space X is called a *sheaf* when it satisfies descent (2.2.4) for all open covers of all open subsets of X. The category of sheaves $\mathsf{Shv}(X;\mathsf{C})$ is the full subcategory of $\mathsf{P}(X;\mathsf{C})$ spanned by sheaves. As with presheaves, the default is $\mathsf{C} = \mathsf{Set}$ unless specified otherwise. Dually, a *cosheaf* is a precosheaf satisfying descent (for its opposite presheaf).

2.2.6 Example. The presheaf of continuous real valued functions (2.2.3.1) is a sheaf on any topological space X. This amounts to two separate assertions: (i) to specify a function, it is equivalent to specify it on an open cover subject to the requirement of agreement on overlaps, and (ii) a function is continuous iff it is locally continuous. There is nothing special about the target being $\mathbb{R}$: the same holds for continuous functions $C(-,Z)$ valued in any fixed topological space Z.

2.2.7 Exercise. Of the remaining presheaves (2.2.3.2)–(2.2.3.8), which are sheaves? For those which are not, what exactly fails?

2.2.8 Exercise. Show that the identity functor $\mathsf{Top} \to \mathsf{Top}$ is a cosheaf.

It turns out that the sheaf property, namely Čech descent (2.2.4), admits an equivalent formulation in terms of so-called 'covering sieves'. This alternative formulation is often useful.

2.2.9 Definition (Sieve)**.** A *sieve* S on a topological space X is a subset of the set of open subsets of X with the property that $V \subseteq U \in S$ implies $V \in S$. A *covering sieve* S on X is a sieve for which $\bigcup_{U \in S} U = X$. The set of covering sieves on a topological space X is denoted $J(X)$.

2.2.10 Definition (Sieve descent)**.** Let F be a presheaf on a topological space X. Associated to any covering sieve $S \in J(X)$ is a map

$$F(X) \to \lim_{U \in S} F(U). \tag{2.2.10.1}$$

When this map is an isomorphism, we say that F satisfies *descent* for the covering sieve S.

2.2.11 Exercise (Sieve descent equals Čech descent)**.** Show that a presheaf satisfies descent for all covering sieves on X iff it satisfies descent for all open coverings of X.

* *

2.2.12 Exercise (Étale space)**.** Fix a topological space X. Let $\mathsf{Top}_{/X}^{\mathsf{lociso}} \subseteq \mathsf{Top}_{/X}$ denote the full subcategory spanned by local isomorphisms to X. Consider the functor

$$\mathsf{Top}_{/X}^{\mathsf{lociso}} \to \mathsf{Shv}(X) \tag{2.2.12.1}$$

sending a local isomorphism $A \to X$ to its sheaf of sections. Show that this functor is an equivalence of categories, and that it identifies pullback of sheaves with pullback of local isomorphisms. The local isomorphism over X corresponding to a sheaf F on X is called the *étale space* of F.

2.2.13 Definition (Plus construction)**.** Here is an operation on a presheaf F which produces another presheaf F^+ which looks like it should be 'closer' to being a sheaf (this construction may originate in SGA4 [1, Exposé II Section 3] and was adapted to the higher categorical setting by Lurie [74, 6.2.2]). Informally, it is given by the following:

$$F^+(U) = \varinjlim_{S \in J(U)} \lim_{V \in S} F(V) \tag{2.2.13.1}$$

Recall that $J(U)$ denotes the set of covering sieves of U. There is an evident natural map $F(U) \to F^+(U)$ which is a isomorphism if F is a sheaf.

To make the definition of $F \mapsto F^+$ precise, consider the following diagram:

$$\mathsf{Open}(X) \xleftarrow{v} \tilde{J} \rtimes \mathsf{Open}(X) \xrightarrow{s} J \rtimes \mathsf{Open}(X) \xrightarrow{j} \mathsf{Open}(X) \tag{2.2.13.2}$$

Here $J \rtimes \mathsf{Open}(X) \to \mathsf{Open}(X)$ is the cartesian fibration associated to the functor $J : \mathsf{Open}(X)^{\mathsf{op}} \to \mathsf{Po}$ sending an open set $U \subseteq X$ to its poset of covering sieves (concretely, an object of $J \rtimes \mathsf{Open}(X)$ is an open set $U \subseteq X$ together with a covering sieve $S \in J(U)$, and there is at most one morphism $(U, S) \to (U', S')$, which exists iff $U \subseteq U'$ and $S \subseteq S'|_U$). The category $\tilde{J} \rtimes \mathsf{Open}(X)$ consists of triples (U, S, V) with $U \subseteq X$ open and $V \in S \in J(U)$ (a morphism $(U, S, V) \to (U', S', V')$ means $U \subseteq U'$, $S \subseteq S'|_U$, and $V \subseteq V'$), and the leftmost functor v above sends $(U, S, V) \mapsto V$. The forgetful functor $s : \tilde{J} \rtimes \mathsf{Open}(X) \to J \rtimes \mathsf{Open}(X)$ is evidently a cocartesian fibration, classifying the functor $J \rtimes \mathsf{Open}(X) \to \mathsf{Cat}$ given by $(U, S) \mapsto S$.

Now the plus construction functor $F \mapsto F^+$ is the composition $j_! s_* v^*$ (pull back under v, right Kan extend along s, and left Kan extend along j). To see that this coincides with the informal prescription (2.2.13.1), recall that left (resp. right) Kan extension along a cocartesian (resp. cartesian) fibration is computed by taking fiberwise colimits (resp. limits) (**??**).

Now the natural transformation $F \to F^+$ is defined as follows. The forgetful functor j has a section f sending each open set U to the 'identity' sieve consisting of all open subsets of U. Since $jf = 1$, there is a canonical natural transformation $f^* \to f^* j^* j_! = j_!$ (concretely, this is just the map $G(U, f(U)) \to \varinjlim_{S \in J(U)} G(U, S)$ for a functor G on $J \rtimes \mathsf{Open}(X)$). The section f lifts to a section $\tilde{f} : \mathsf{Open}(X) \to \tilde{J} \rtimes \mathsf{Open}(X)$ sending each open set U to the object U inside the identity sieve $f(U) \in J(U)$. The induced natural transformation $f^* s_* = \tilde{f}^* s^* s_* \to \tilde{f}^*$ is an isomorphism since $\tilde{f}(U)$ is initial in the fiber over $f(U)$. We now have a composition

$$1 = \tilde{f}^* v^* \xleftarrow{\sim} \tilde{f}^* s^* s_* v^* = f^* s_* v^* \to f^* j^* j_! s_* v^* = j_! s_* v^* \tag{2.2.13.3}$$

which is the desired natural transformation $1 \to j_! s_* v^*$.

2.2.14 Lemma. *The natural transformation $F \to F^+$ induces an isomorphism* $\operatorname{Hom}(F^+, G) \to \operatorname{Hom}(F, G)$ *for every sheaf G (considering here presheaves valued in any ∞-category E which has limits and filtered colimits).*

Proof. According to the definition of the map $F \to F^+$ (2.2.13.3), the map $\operatorname{Hom}(F^+, G) \to \operatorname{Hom}(F, G)$ is the bottom horizontal map in the following commuting diagram:

$$\begin{tikzcd}
\operatorname{Hom}(j_! s_* v^* F, G) \arrow[r, "j^*"] \arrow[dr, equal, "jf=1"] & \operatorname{Hom}(j^* j_! s_* v^* F, j^* G) \arrow[r, "1\to j^* j_!"] \arrow[d, "f^*"] & \operatorname{Hom}(s_* v^* F, j^* G) \arrow[d, "f^*"] \\
& \operatorname{Hom}(f^* j^* j_! s_* v^* F, f^* j^* G) \arrow[r, "1\to j^* j_!"] \arrow[d, equal, "jf=1"] & \operatorname{Hom}(f^* s_* v^* F, f^* j^* G) \arrow[d, equal, "jf=1"] \\
& \operatorname{Hom}(f^* j^* j_! s^* v^* F, G) \arrow[r, "1\to j^* j_!"] & \operatorname{Hom}(f^* s_* v^* F, G)
\end{tikzcd} \tag{2.2.14.1}$$

The composition of the top two horizontal arrows is an isomorphism by the adjunction $(j_!, j^*)$. It therefore suffices to show that the right upper vertical arrow f^* is an isomorphism. By the adjunction (f^*, f_*), this is implied by the unit map $1 \to f_* f^*$ sending $j^* G$ to an isomorphism. Applying $1 \to f_* f^*$ to $j^* G$ and evaluating at $(U, S) \in J \rtimes \mathsf{Open}(X)$, we obtain the descent map $G(U) \to \lim_S G$ (to compute f_*, note that the slice category $\mathsf{Open}(X)_{f(\cdot)/(U,S)}$ is precisely S), which is an isomorphism since G is a sheaf. $\square$

⋆ **2.2.15 Proposition** (Sheafification)**.** *The inclusion* $\mathsf{Shv}(X) \subseteq \mathsf{P}(X)$ *is a reflective subcategory whose left adjoint, termed* sheafification *and denoted $F \mapsto F^\#$, is given by applying the plus construction* (2.2.13) *twice.*

2.2.16 Corollary (Sheaf fiber product is cocontinuous)**.** *For any map of sheaves $B \to A$, the functor* $\mathsf{Shv}(X)_{/A} \xrightarrow{\times_A B} \mathsf{Shv}(X)_{/B}$ *preserves colimits.*

Proof. Use the corresponding fact about presheaves (**??**), the fact that $\mathsf{Shv}(X) \subseteq \mathsf{P}(X)$ is reflective (2.2.15), and the fact that sheafification preserves finite limits (**??**). $\square$

2.2.17 Exercise (Support)**.** Let $F \in \mathsf{Shv}(X, \mathsf{Set}_*)$ be a sheaf of pointed sets on X, and let $s \in F(X)$ be a section. The *support* of s is the set of points $p \in X$ for which the stalk $s_x \in F_x$ is not the basepoint. Show that for any collection of open sets U_α with $s|_{U_\alpha} = *$, we have $s|_{\bigcup_\alpha U_\alpha} = *$. Conclude that there exists a largest open subset $U \subseteq X$ with the property that $s|_U = *$. Show that this open set is $X \setminus \operatorname{supp} s$.

⋆ **2.2.18 Definition** (Čech nerve)**.** Let $X = \bigcup_{i\in I} U_i$ be an open cover. Denote by 2^I_{fin} the category of finite subsets of I, and consider the functor $(2^I_{\mathsf{fin}})^{\mathsf{op}} \to \mathsf{Top}$ given by $A \mapsto \bigcap_{i\in A} U_i$ (in particular $\varnothing \mapsto X$). We may regard 2^I_{fin} as the cone $(2^I_{\mathsf{fin}} \setminus \{\varnothing\})^{\triangleleft}$ and thus obtain a comparison map

$$N(X,\{U_i\}_{i\in I}) = \operatorname*{colim}^{\mathsf{P}(\mathsf{Open}(X))}_{\substack{\varnothing\neq A\subseteq I\\ |I|<\infty}} \bigcap_{i\in A} U_i \to X. \tag{2.2.18.1}$$

The *Čech nerve* $N(X,\{U_i\}_{i\in I})$ of the open cover $X = \bigcup_{i\in I} U_i$ is the above formal colimit (i.e. colimit in $\mathsf{P}(\mathsf{Open}(X))$) over $(2^I_{\mathsf{fin}} \setminus \{\varnothing\})^{\mathsf{op}}$.

⋆ **2.2.19 Exercise.** Conclude from (1.4.199) that a presheaf $F \in \mathsf{P}(X)$ satisfies descent for an open cover $U = \bigcup_i U_i$ iff it is right local with respect to the morphism $N(U,\{U_i\}_i) \to U$ in $\mathsf{P}(X)$. Conclude that $\mathsf{Shv}(X) \subseteq \mathsf{P}(X)$ is the full subcategory of local objects (1.4.198) with respect to the morphisms $N(U,\{U_i\}_i) \to U$ in $\mathsf{P}(X)$ associated to open covers $U = \bigcup_i U_i$. In particular, conclude that $\mathsf{Shv}(X) \subseteq \mathsf{P}(X)$ is a reflective subcategory (1.4.200) (reflection denoted $\# : \mathsf{P}(X) \to \mathsf{Shv}(X)$ and termed sheafification) and that (1.4.206) for any cocomplete ∞-category E, pullback and left Kan extension

$$\mathsf{Fun}(\mathsf{Open}(X);\mathsf{E}) \underset{(\#よ)^*}{\overset{(\#よ)_!}{\rightleftarrows}} \mathsf{Fun}(\mathsf{Shv}(X);\mathsf{E}) \tag{2.2.19.1}$$

restrict to equivalences between coheaves $\mathsf{Shv}(X;\mathsf{E}^{\mathsf{op}})^{\mathsf{op}} \subseteq \mathsf{Fun}(\mathsf{Open}(X);\mathsf{E})$ and cocontinuous functors $\mathsf{Shv}(X) \to \mathsf{E}$.

2.2.20 Lemma (Čech descent vs sieve descent)**.** *For any open cover $U = \bigcup_i U_i$ and $S \subseteq \mathsf{Open}(X)$ the sieve it generates, the natural map on formal colimits*

$$\operatorname*{colim}^{\mathsf{P}(\mathsf{Open}(X))}_{\substack{\varnothing\neq A\subseteq I\\ |A|<\infty}} \bigcap_{i\in A} U_i \to \operatorname*{colim}^{\mathsf{P}(\mathsf{Open}(X))}_{V\in S} V \tag{2.2.20.1}$$

is an isomorphism. It follows (2.2.19) *that a presheaf (valued in any ∞-category E) satisfies descent for an open cover* (2.2.4) *iff it satisfies descent for the covering sieve it generates* (2.2.10).

Proof. It suffices to show that the functor $(2^I_{\mathsf{fin}} \setminus \{\varnothing\})^{\mathsf{op}} \to S$ is ∞-final (**??**). To show this functor is final, it suffices to check that for every $V \in S$, the slice category $(V \downarrow (2^I_{\mathsf{fin}} \setminus \{\varnothing\})^{\mathsf{op}})$ is Kan contractible (1.4.134). This slice category is $(2^{I_V}_{\mathsf{fin}} \setminus \{\varnothing\})^{\mathsf{op}}$, where $I_V \subseteq I$ denotes the set of indices $i \in I$ for which $U_i \supseteq V$ (note that I_V is non-empty for $V \in S$, by definition of S). This is evidently filtered, hence Kan contractible (**??**). □

2.2.21 Exercise (Support)**.** Let $F \in \mathsf{Shv}(X,\mathsf{Spc}_*)$ be a sheaf of pointed spaces on X, and let $s \in F(X)$ be a section. The *support* of s is the set of points $p \in X$ for which the stalk $s_x \in F_x$ is not (in the component of) the basepoint. Show that $\operatorname{supp} s \subseteq X$ is closed. Show by example that $s|_{X\setminus\operatorname{supp} s}$ need not be the basepoint (where does the proof that $s|_{X\setminus\operatorname{supp} s} = *$ for sheaves of pointed sets (2.2.17) fail for sheaves of spaces?).

2.2.22 Definition (Homotopy coherent pushforward and pullback)**.** Consider the category $\mathsf{Open} \rtimes \mathsf{Top}$ of pairs (X, U) where X is a topological space and $U \subseteq X$ is an open subset, in which a morphism $(X, U) \to (Y, V)$ is a map $f : X \to Y$ with $f(U) \subseteq V$ (equivalently $U \subseteq f^{-1}(V)$). The functor

$$\mathsf{Open} \rtimes \mathsf{Top} \to \mathsf{Top} \tag{2.2.22.1}$$
$$(X, U) \mapsto X \tag{2.2.22.2}$$

is (by inspection) cartesian, the cartesian edges being the morphisms $(X, f^{-1}(V)) \to (Y, V)$ for $f : X \to Y$. This cartesian functor encodes the categories $\mathsf{Open}(X)$ for topological spaces X and the functors $f^{-1} = \mathsf{Open}(f) : \mathsf{Open}(Y) \to \mathsf{Open}(X)$ for maps $f : X \to Y$.

Sheaves on compact sets

For locally compact Hausdorff topological spaces, it is helpful to reformulate the notion of a sheaf in terms of compact sets instead of open sets (for example, as in Lurie [74, 7.3.4]). This reformulation is not merely for the sake of aesthetics. Certain operations and results (for example, sheafification (**??**) and proper base change (2.2.34)) become clearer in this context.

In this discussion, we will indicate open vs compact sets with the prefixes 'o-' and 'k-' (so presheaves and sheaves in the usual sense (2.2.2)(2.2.5) will be called o-presheaves $\mathsf{P}(X_{\text{open}})$ and o-sheaves $\mathsf{Shv}(X_{\text{open}})$, for clarity).

2.2.23 Definition (k-presheaf)**.** Let X be a locally compact Hausdorff topological space. A *k-presheaf* on X is a presheaf on the category $\mathsf{Cpt}(X)$ of compact subsets of X. The ∞-category of k-presheaves is denoted $\mathsf{P}(X_{\text{cpt}})$.

⋆ **2.2.24 Definition** (k-sheaf)**.** A k-presheaf F (valued in any ∞-category E) is called a *k-sheaf* when it satisfies the following two properties:

(2.2.24.1) (Continuity) For compact K, the map

$$\varinjlim_{K \subseteq (K')^\circ} F(K') \to F(K)$$

is an isomorphism (where the directed colimit is over the collection of compact sets K' whose interior contains K).

(2.2.24.2) (Descent) For every finite collection of compact sets $K_1, \ldots, K_n$ ($n \geq 0$), the descent morphism

$$F(K_1 \cup \cdots \cup K_n) \to \lim_{\varnothing \neq S \subseteq \{1,\ldots,n\}} F\Big(\bigcap_{i \in S} K_i\Big)$$

is an isomorphism.

The full subcategory spanned by k-sheaves is denoted $\mathsf{Shv}(X_{\text{cpt}}) \subseteq \mathsf{P}(X_{\text{cpt}})$.

2.2.25 Exercise. Show that the continuity property (2.2.24.1) is equivalent to the assertion that the map

$$\varinjlim_{K\in\mathcal{K}} F(K) \to F\Big(\bigcap_{K\in\mathcal{K}} K\Big) \tag{2.2.25.1}$$

is an isomorphism for any subset $\mathcal{K} \subseteq \mathsf{Cpt}(X)$ which is cofiltered (meaning, concretely, that $\mathcal{K}$ is non-empty and that for all $K, L \in \mathcal{K}$ there exists $M \in \mathcal{K}$ contained in $L \cap K$).

2.2.26 Definition (Regular k-presheaf)**.** A k-presheaf will be called *regular* when it satisfies the continuity axiom (2.2.24.1) and the descent axiom (2.2.24.2) for $n = 0$ (in other words, sends the empty set to the terminal object). We denote the full subcategory of regular k-presheaves by $\mathsf{P}(X_{\mathrm{cpt}})_{\mathrm{reg}} \subseteq \mathsf{P}(X_{\mathrm{cpt}})$.

2.2.27 Definition (Plus construction)**.** Here is a variant of the plus construction (2.2.13) for regular k-presheaves (2.2.26) (not all k-presheaves). Informally, it is given by the following:

$$F^+(K) = \varinjlim_{K=K_1\cup\cdots\cup K_n} \lim_{\varnothing\neq S\subseteq\{1,\ldots,n\}} F\Big(\bigcap_{i\in S} K_i\Big) \tag{2.2.27.1}$$

There is an evident natural map $F(K) \to F^+(K)$ which is an isomorphism if F is a k-sheaf (that is, satisfies descent (2.2.24.2)).

To make the definition of $F \mapsto F^+$ precise, the approach given in the case of o-presheaves (2.2.13) based on Kan extension may be adapted easily. We consider the following diagram:

$$\mathsf{Cpt}(X) \xleftarrow{v} \tilde{J} \rtimes \mathsf{Cpt}(X) \xrightarrow{s} J \rtimes \mathsf{Cpt}(X) \xrightarrow{j} \mathsf{Cpt}(X) \tag{2.2.27.2}$$

Here $J(K)$ consists of those sieves $S \subseteq \mathsf{Cpt}(K)$ which 'finitely cover' K in the sense that there exist $K_1, \ldots, K_n \in S$ with $K \subseteq \bigcup_{i=1}^n K_i$. As before, the category $\tilde{J} \rtimes \mathsf{Cpt}(X)$ consists of triples (K, S, V) with $V \in S \in J(K)$ and $K \in \mathsf{Cpt}(X)$. The plus construction functor $F \mapsto F^+$ is now given by the composition $j_! s_* v^*$. Now there is a natural transformation $1 \to +$ defined in the same was as before (2.2.13.3).

2.2.28 Exercise. Show that the natural map

$$\varinjlim_{K\subseteq K_1^\circ\cup\cdots\cup K_n^\circ} \lim_{\varnothing\neq S\subseteq\{1,\ldots,n\}} F\Big(\bigcap_{i\in S} K_i\Big) \to \varinjlim_{K=K_1\cup\cdots\cup K_n} \lim_{\varnothing\neq S\subseteq\{1,\ldots,n\}} F\Big(\bigcap_{i\in S} K_i\Big) \tag{2.2.28.1}$$

is an isomorphism for any k-presheaf F satisfying continuity (2.2.24.1) valued in an ∞-category in which filtered colimits commute with finite limits.

2.2.29 Lemma. *The natural transformation $F \to F^+$ induces an isomorphism* $\operatorname{Hom}(F^+, G) \to \operatorname{Hom}(F, G)$ *for every k-sheaf G.*

Proof. The proof of the corresponding result for o-presheaves (2.2.14) applies without change. $\square$

2.2.30 Definition (Comparing o-presheaves and k-presheaves)**.** We define an adjoint pair $(\kappa_!, \kappa^*)$ of functors

$$\kappa_! : \mathsf{P}(X_{\mathrm{open}}; \mathsf{E}) \rightleftarrows \mathsf{P}(X_{\mathrm{cpt}}; \mathsf{E}) : \kappa^* \tag{2.2.30.1}$$

whenever E has filtered colimits and cofiltered limits. Informally speaking, we take

$$(\kappa_! F)(K) = \varinjlim_{K \subseteq U} F(U), \tag{2.2.30.2}$$

$$(\kappa^* F)(U) = \varprojlim_{K \subseteq U} F(K). \tag{2.2.30.3}$$

To make these formulae precise (and functorial), we use Kan extensions. Consider the category $\mathsf{Cpt}(X) \cup \mathsf{Open}(X)$ of subsets of X which are compact or open. Then we have inclusion functors from $\mathsf{Cpt}(X)$ and $\mathsf{Open}(X)$ into $\mathsf{Cpt}(X) \cup \mathsf{Open}(X)$, and we take $\alpha_!$ and α^* to be the compositions of the following Kan extensions.

$$\begin{tikzcd}
\mathsf{P}(X_{\mathrm{open}}) \arrow[r, shift left, "(\mathsf{Open}\to\mathsf{Open}\cup\mathsf{Cpt})_!"] \arrow[rr, bend left, "\kappa_!"] & \mathsf{P}(X_{\mathrm{opencpt}}) \arrow[l, shift left, "(\mathsf{Open}\to\mathsf{Open}\cup\mathsf{Cpt})^*"] \arrow[r, shift left, "(\mathsf{Cpt}\to\mathsf{Open}\cup\mathsf{Cpt})^*"] & \mathsf{P}(X_{\mathrm{cpt}}) \arrow[l, shift left, "(\mathsf{Cpt}\to\mathsf{Open}\cup\mathsf{Cpt})_*"] \arrow[ll, bend left, "\kappa^*"]
\end{tikzcd} \tag{2.2.30.4}$$

The left Kan extension $(\mathsf{Open} \to \mathsf{Open}{\cup}\mathsf{Cpt})_!$ and the right Kan extension $(\mathsf{Cpt} \to \mathsf{Open}{\cup}\mathsf{Cpt})_*$ are, by definition (1.4.173), given by exactly the directed colimit (2.2.30.2) and inverse limit (2.2.30.3), respectively (in particular, they exist provided E has filtered colimits and cofiltered limits).

⋆ **2.2.31 Proposition** (Comparing o-sheaves and k-sheaves)**.** *The functors* $(\kappa_!, \kappa_*)$ (2.2.30) *restrict to an adjoint (hence inverse) pair of equivalences*

$$\kappa_! : \mathsf{Shv}(X_{\mathrm{open}}; \mathsf{E}) \rightleftarrows \mathsf{Shv}(X_{\mathrm{cpt}}; \mathsf{E}) : \kappa^* \tag{2.2.31.1}$$

whenever E *has limits and filtered colimits and they commute.*

Proof. Let $F \in \mathsf{P}(X_{\mathrm{open}}; \mathsf{E})$, and let us show that $\kappa_! F \in \mathsf{P}(X_{\mathrm{cpt}}; \mathsf{E})$ satisfies the continuity axiom (2.2.24.1). Inserting the definition of $\kappa_! F$ (2.2.30.2) into the continuity axiom yields

$$\varinjlim_{K \subseteq (K')^\circ} \varinjlim_{K' \subseteq U'} F(U') \to \varinjlim_{K \subseteq U} F(U), \tag{2.2.31.2}$$

which is an isomorphism since the map $(K', U') \mapsto U'$ is final.

Let $F \in \mathsf{Shv}(X_{\mathrm{open}}; \mathsf{E})$, and let us show that $\kappa_! F \in \mathsf{Shv}(X_{\mathrm{cpt}}; \mathsf{E})$. We saw already that $\kappa_! F$ satisfies the continuity axiom (2.2.24.1), so it remains to check descent (2.2.24.2). Inserting the definition of $\kappa_! F$ (2.2.30.2) into the descent axiom yields

$$\varinjlim_{K_1 \cup \cdots \cup K_n \subseteq U} F(U) \to \lim_{\varnothing \neq S \subseteq \{1,\ldots,n\}} \varinjlim_{\bigcap_{i \in S} K_i \subseteq U} F(U). \tag{2.2.31.3}$$

Now consider instead the directed system of tuples of open sets $K_i \subseteq U_i$. The 'union' map to open sets containing $K_1 \cup \cdots \cup K_n$ is final, as is the 'intersection' map to open sets containing $\bigcap_{i\in S} K_i$ for $\varnothing \neq S \subseteq \{1,\ldots,n\}$ (exercise: every open set containing $K \cap K'$ contains the intersection of open sets $K \subseteq U$ and $K' \subseteq U'$, since X is locally compact Hausdorff), hence we may rewrite the map in question as

$$\operatorname*{colim}_{\overrightarrow{\{K_i\subseteq U_i\}_i}} F(U_1 \cup \cdots \cup U_n) \to \lim_{\varnothing\neq S\subseteq\{1,\ldots,n\}} \operatorname*{colim}_{\overrightarrow{\{K_i\subseteq U_i\}_i}} F\Big(\bigcap_{i\in S} U_i\Big). \tag{2.2.31.4}$$

Since limits commute with filtered colimits in E, this map is a filtered colimit of descent maps for F, hence is an isomorphism since F is a sheaf.

Let $F \in \mathsf{Shv}(X_{\mathrm{cpt}};\mathsf{E})$, and let us show that $\kappa^*F \in \mathsf{Shv}(X_{\mathrm{open}};\mathsf{E})$. Consider the descent morphism for κ^*F associated to an open cover $U = \bigcup_{i\in I} U_i$ and insert the definition of κ^*F (2.2.30.3) to obtain

$$\varprojlim_{K\subseteq U} F(K) \to \lim_{S\in 2^I_{\mathsf{fin}}\setminus\{\varnothing\}} \varprojlim_{K\subseteq\bigcap_{i\in S} U_i} F(K). \tag{2.2.31.5}$$

Now consider instead the inverse system $\{K_i \subseteq U_i\}_i$ of tuples of compact sets $K_i \subseteq U_i$ all but finitely many of which are empty. The 'intersection' map to compact subsets of $\bigcap_{i\in S} U_i$ is final for all finite non-empty $S \subseteq I$, as is the 'union' map to compact subsets of $U = \bigcup_i U_i$ (exercise: every compact subset of $U \cup U'$ is the union of a compact subsets $K \subseteq U$ and $K' \subseteq U'$, since X is locally compact Hausdorff), hence we may rewrite the map in question as

$$\varprojlim_{\{K_i\subseteq U_i\}_i} F\Big(\bigcup_i K_i\Big) \to \lim_{S\in 2^I_{\mathsf{fin}}\setminus\{\varnothing\}} \varprojlim_{\{K_i\subseteq U_i\}_i} F\Big(\bigcap_{i\in S} K_i\Big). \tag{2.2.31.6}$$

Since limits commute with limits, this is the inverse limit over $\{K_i \subseteq U_i\}_i$ of the descent morphisms

$$F\Big(\bigcup_i K_i\Big) \to \lim_{S\in 2^I_{\mathsf{fin}}\setminus\{\varnothing\}} F\Big(\bigcap_{i\in S} K_i\Big), \tag{2.2.31.7}$$

which we now argue are all isomorphisms. Let $I' \subseteq I$ denote the (finite!) set of indices $i \in I$ for which $K_i \neq \varnothing$. The descent morphism for I' is an isomorphism since F is a k-sheaf, so it suffices to show that the map

$$\lim_{S\in 2^I_{\mathsf{fin}}\setminus\{\varnothing\}} F\Big(\bigcap_{i\in S} K_i\Big) \to \lim_{S\in 2^{I'}\setminus\{\varnothing\}} F\Big(\bigcap_{i\in S} K_i\Big) \tag{2.2.31.8}$$

is an isomorphism. According to (1.4.110), it suffices to observe that for every $S \subseteq I$ not contained in I', we have $F(\bigcap_{i\in S} K_i) = F(\varnothing) = *$ (the terminal object of E) since F is a k-sheaf.

It remains to show that the unit and counit maps $F \to \kappa^*\kappa_!F$ and $\kappa_!\kappa^*F \to F$ are isomorphisms for F an o-sheaf and a k-sheaf (respectively).

The counit map $\kappa_!\kappa^*F \to F$ for a k-presheaf F has the form

$$\operatorname*{colim}_{\overrightarrow{K\subseteq U}} \varprojlim_{L\subseteq U} F(L) \to F(K). \tag{2.2.31.9}$$

This is the directed colimit over open U containing K, but we may just as well allow U to be any subset of X whose interior contains K (open U are final inside this). Now compact U whose interior contains K are also final inside this, and for such U the inner inverse limit is achieved at the initial object $L = U$. The map above now becomes the map (2.2.24.1), hence is an isomorphism whenever F satisfies the continuity axiom (in particular, when F is a k-sheaf).

The unit map $F \to \kappa^*\kappa_! F$ for an o-presheaf F has the form

$$F(U) \to \varprojlim_{K \subseteq U} \varinjlim_{K \subseteq V} F(V). \tag{2.2.31.10}$$

Arguing as in the case of the unit map, we may replace the inverse limit over compact $K \subseteq U$ with the inverse limit over open $K \subseteq U$ contained in a compact subset of U, whereby we obtain the map $F(U) \to \varprojlim_{K \subseteq U} F(K^\circ)$, which is an isomorphism whenever F is a sheaf (it is the descent morphism for the covering sieve of U given by open subsets contained in a compact subspace of U). □

2.2.32 Definition (Pushforward and pullback)**.** Consider a proper map of locally compact Hausdorff spaces $f : X \to Y$. There is an associated pushforward operation $f_*^{\mathrm{pre}} : \mathsf{P}(X_{\mathrm{cpt}}) \to \mathsf{P}(Y_{\mathrm{cpt}})$ given by pulling back under $f^{-1} : \mathsf{Cpt}(Y) \to \mathsf{Cpt}(X)$ (note that f is assumed proper so f^{-1} sends compact sets to compact sets). The continuity and descent axioms for F imply the same for f_*F (inspection), so we have a k-sheaf pushforward functor $f_* : \mathsf{Shv}(X_{\mathrm{cpt}}) \to \mathsf{Shv}(Y_{\mathrm{cpt}})$.

Now consider any map of locally compact Hausdorff spaces $f : X \to Y$. We can define pullback $f^*_{\mathrm{pre}} : \mathsf{P}(Y_{\mathrm{cpt}}) \to \mathsf{P}(X_{\mathrm{cpt}})$ by pulling back under $f : \mathsf{Cpt}(X) \to \mathsf{Cpt}(Y)$ (the image of a compact set is always compact). Now the continuity axiom for F implies the same for f^*F (2.2.25) (note that the trivial inclusion $f(\bigcap_{K \in \mathcal{K}} K) \subseteq \bigcap_{K \in \mathcal{K}} f(K)$ is an equality for $\mathcal{K} \subseteq \mathsf{Cpt}(X)$ since the fibers of f are compact). Moreover, regularity of F implies regularity of f^*F (2.2.26). Thus we may compose f^* with k-sheafification of regular k-presheaves (**??**) to define the k-sheaf pullback map $f^* : \mathsf{Shv}(Y_{\mathrm{cpt}}) \to \mathsf{Shv}(X_{\mathrm{cpt}})$.

For a proper map of locally compact Hausdorff spaces $f : X \to Y$, there is an adjunction $(f^*_{\mathrm{pre}}, f_*^{\mathrm{pre}})$ of functors on k-presheaves $f_*^{\mathrm{pre}} : \mathsf{P}(X_{\mathrm{cpt}}) \rightleftarrows \mathsf{P}(Y_{\mathrm{cpt}}) : f^*_{\mathrm{pre}}$ since $f : \mathsf{Cpt}(X) \rightleftarrows \mathsf{Cpt}(Y) : f^{-1}$ are adjoint (f, f^{-1}) (1.1.104). Both f_*^{pre} and f^*_{pre} preserve regularity as just noted. The adjunction $(f^*_{\mathrm{pre}}, f_*^{\mathrm{pre}})$ of functors on regular k-presheaves now descends (1.1.101) to an adjunction (f^*, f_*) of functors on k-sheaves (since they are reflective inside regular k-presheaves (**??**)).

Proper base change

⋆ **2.2.33 Definition** (Base change morphism)**.** Consider a diagram of topological spaces.

$$\begin{array}{ccc} X' & \xrightarrow{\alpha} & X \\ \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi} \\ Y' & \xrightarrow{\beta} & Y \end{array} \tag{2.2.33.1}$$

Since $\pi\alpha = \beta\pi'$, we have $\pi_*\alpha_* = \beta_*\pi'_*$, and hence there is a resulting natural transformation $\beta^*\pi_* \to \pi'_*\alpha^*$ (1.1.86) called the *base change morphism*.

$$\begin{tikzcd} \mathsf{Shv}(X') \arrow[r, "\alpha_*"] \arrow[d, "\pi'_*"] & \mathsf{Shv}(X) \arrow[d, "\pi_*"] \\ \mathsf{Shv}(Y') \arrow[r, "\beta_*"] & \mathsf{Shv}(Y) \end{tikzcd} \rightsquigarrow \begin{tikzcd} \mathsf{Shv}(X') \arrow[d, "\pi'_*"] & \mathsf{Shv}(X) \arrow[l, "\alpha^*"'] \arrow[d, "\pi_*"] \arrow[dl, phantom, "\Leftarrow"] \\ \mathsf{Shv}(Y') & \mathsf{Shv}(Y) \arrow[l, "\beta^*"] \end{tikzcd} \tag{2.2.33.2}$$

The base change morphism is a sort of generalized pullback operation (consider the special case $Y' = Y = *$, for example).

The base change morphism is compatible with composition of squares (1.1.86):

$$\begin{tikzcd} X'' \arrow[r, "\gamma"] \arrow[d, "\pi''"] & X' \arrow[r, "\alpha"] \arrow[d, "\pi'"] & X \arrow[d, "\pi"] \\ Y'' \arrow[r, "\delta"] & Y' \arrow[r, "\beta"] & Y \end{tikzcd} \qquad \begin{tikzcd} X' \arrow[r, "\alpha"] \arrow[d, "\pi'"] & X \arrow[d, "\pi"] \\ Y' \arrow[r, "\beta"] \arrow[d, "\rho'"] & Y \arrow[d, "\rho"] \\ Z' \arrow[r, "\gamma"] & Z \end{tikzcd} \tag{2.2.33.3}$$

The base change morphism of the left composition of squares $\delta^*\beta^*\pi_* \to \pi''_*\gamma^*\alpha^*$ is the composition of base change morphisms $\delta^*(\beta^*\pi_* \to \pi'_*\alpha^*)$ and $(\delta^*\pi'_* \to \pi''_*\gamma^*)\alpha^*$. The base change morphism of the right composition of squares $\gamma^*\rho_*\pi_* \to \rho'_*\pi'_*\alpha^*$ is the composition of base change morphisms $(\gamma^*\rho_* \to \rho'_*\beta^*)\pi_*$ and $\rho'_*(\beta^*\pi_* \to \pi'_*\alpha^*)$.

⋆ **2.2.34 Proper Base Change Theorem.** *Given a pullback diagram of locally compact locally Hausdorff topological spaces*

$$\begin{tikzcd} X' \arrow[r, "\alpha"] \arrow[d, "\pi'"] & X \arrow[d, "\pi"] \\ Y' \arrow[r, "\beta"] & Y \end{tikzcd} \tag{2.2.34.1}$$

the associated base change morphism $\beta^*\pi_* \to \pi'_*\alpha^*$ (2.2.33.2) *is an isomorphism provided* π *is proper.*

Proof. Since the assertion is local on Y', we may assume that both Y and Y' (hence also X and X') are locally compact Hausdorff. We may thus employ the formalism of k-sheaves (2.2.24).

We first consider the case that the bottom map $Y' \to Y$ is a closed embedding. Now pullback of k-sheaves under a closed embedding is simply restriction of functors. Thus for a k-sheaf F on X, the base change morphism is simply the tautological identification $(\beta^*\pi_*F)(K) = F(\pi^{-1}(\beta(K))) = F(\alpha((\pi')^{-1}(K))) = (\pi'_*\alpha^*F)(K)$. This proves the result in the case $Y' \to Y$ is a closed embedding.

We now reduce to the case that Y and Y' are compact. The inclusion of a compact set $K \to Y'$ is a closed embedding, so the base change morphism of the pullback square of

$X' \to Y'$ along $K \to Y'$ is an isomorphism. Now recall that the base change morphism of a composition of squares is the composition of base change morphisms (2.2.33). It follows that the base change morphism associated to the pullback of $X \to Y$ along $K \to Y$ is the restriction to K of the base change morphism associated to the pullback of $X \to Y$ along $Y' \to Y$. It thus suffices to treat the case that Y' is compact. Now if Y' is compact, then there is a factorization $Y' \to K \to Y$ for $K \to Y$ the inclusion of a compact subset (for example, K could be the image of $Y' \to Y$). The base change morphism for pulling back along $K \to Y$ is an isomorphism, so using again the compatibility of base change with composition of squares, we reduce to the case that Y is also compact.

Now factor the map $Y' \to Y$ into the composition of its graph $Y' \to Y' \times Y$ and the projection $Y' \times Y \to Y$. The graph is a closed embedding, so it suffices to prove base change for the projection $Y' \times Y \to Y$. We are thus reduced to the case of squares of the following form with Y and L compact Hausdorff:

$$\begin{tikzcd} L \times X \arrow[r] \arrow[d] & X \arrow[d] \\ L \times Y \arrow[r] & Y \end{tikzcd} \tag{2.2.34.2}$$

Now to check that the base change morphism is an isomorphism, it suffices to check that it is an isomorphism on sections over products $L_0 \times Y_0$ for compact $L_0 \subseteq L$ and $Y_0 \subseteq Y$. Indeed, it follows from descent (2.2.24.2) that the base change morphism is then an isomorphism over all finite unions of such products $L_0 \times Y_0$, and then from continuity (2.2.24.1) that it is an isomorphism over all compact sets (since every compact set has arbitrarily small neighborhoods which are finite unions of products $L_0 \times Y_0$). Using base change for pullbacks along closed embeddings, we may replace (L, Y) with (L_0, Y_0) and thereby reduce to checking that the base change morphism is an isomorphism on global sections for squares of the above form.

Now consider the stacking:

$$\begin{tikzcd} L \times X \arrow[r] \arrow[d] & X \arrow[d] \\ L \times Y \arrow[r] \arrow[d] & Y \arrow[d] \\ L \arrow[r] & * \end{tikzcd} \tag{2.2.34.3}$$

The base change morphism of the composition of squares is (on global sections) the composition of the base change morphisms for the top and bottom squares. It therefore suffices to check that the base change morphism is an isomorphism on global sections for squares of the form:

$$\begin{tikzcd} L \times X \arrow[r] \arrow[d] & X \arrow[d] \\ L \arrow[r] & * \end{tikzcd} \tag{2.2.34.4}$$

Now the base change morphism on global sections for such a square may be described concretely as follows. Recall that pullback of k-sheaves is given by pullback of k-presheaves followed by the plus construction (2.2.32). The base change morphism thus takes the form

$$\operatorname*{colim}_{S\in J(L)}\lim_{V\in S} F\left(\begin{cases} X & V\neq\varnothing \\ \varnothing & V=\varnothing\end{cases}\right) \to \operatorname*{colim}_{S\in J(L\times X)}\lim_{V\in S} F(p_X(V)) \tag{2.2.34.5}$$

for the evident pullback map on indexing categories $J(L)\to J(L\times X)$. Intuitively, the reason this map from 'sheafifying over L' to 'sheafifying over $L\times X$' is an isomorphism is that F is already a sheaf on X. To formalize this, we would like to replace the colimit over $J(L\times X)$ with the colimit of the pullback diagram under the product map $\otimes : J(L)\times J(X)\to J(L\times X)$. To perform this replacement, we must first replace the colimits over J with colimits over the full subcategories $J^\circ\subseteq J$ consisting of those sieves of compact sets whose *interiors* cover; recall that directed colimit over J° maps isomorphically to that over J for continuous k-presheaves (2.2.28). Now the product map $\otimes : J^\circ(L)\times J^\circ(X)\to J^\circ(L\times X)$ is ∞-final (use compactness of L and X), and for any $(S,T)\in J^\circ(L)\times J^\circ(X)$, the map $S\times T\to S\otimes T$ is ∞-initial (the relevant slice categories (1.4.134) are filtered, hence contractible (**??**)). Our base change morphism thus takes the following form:

$$\operatorname*{colim}_{S\in J(L)}\lim_{V\in S} F\left(\begin{cases} X & V\neq\varnothing \\ \varnothing & V=\varnothing\end{cases}\right) \to \operatorname*{colim}_{S\in J(L)}\operatorname*{colim}_{T\in J(X)}\lim_{V\in S}\lim_{A\in T} F\left(\begin{cases} A & V\neq\varnothing \\ \varnothing & V=\varnothing\end{cases}\right) \tag{2.2.34.6}$$

To show this is an isomorphism, it suffices to check that for any fixed $S\in J(L)$, any fixed $T\in J(X)$ (note that $J(X)$ is filtered, hence contractible (**??**)), and any fixed $V\in S$, the following is an isomorphism:

$$F\left(\begin{cases} X & V\neq\varnothing \\ \varnothing & V=\varnothing\end{cases}\right) \to \lim_{A\in T} F\left(\begin{cases} A & V\neq\varnothing \\ \varnothing & V=\varnothing\end{cases}\right) \tag{2.2.34.7}$$

In fact, it suffices to check this for T in an ∞-final full subcategory of $J(X)$, such as those sieves generated by a finite compact cover. For such T (and $V\neq\varnothing$), this map is just the descent map for F with respect to the finite compact cover, hence is an isomorphism. In the case $V=\varnothing$, recall that $F(\varnothing)=*$ is the terminal object, and note that $\lim_K * = *$ for any diagram shape K. □

2.2.35 Corollary (Preservation of relative limit diagrams under proper pushforward)**.** *Let $p,q : K^\triangleleft\to\mathsf{Top}$ be two diagrams of locally compact locally Hausdorff spaces, and let $p\to q$ be a morphism of diagrams (that is, a diagram $\Delta^1\times K^\triangleleft\to\mathsf{Top}$) which is proper (meaning $p(k)\to q(k)$ is proper for all vertices $k\in K^\triangleleft$) and which sends edges $*\to k$ of $K^\triangleleft$ ($k\in K$) to pullback diagrams of spaces. For a relative limit diagram $F\in\mathsf{Shv}(p)^{\mathsf{op}}$ (where $\mathsf{Shv}(p)^{\mathsf{op}}$ denotes the ∞-category of sections of $\mathsf{Shv}^{\mathsf{op}}\rtimes\mathsf{Top}\to\mathsf{Top}$ over p), the following are equivalent:*

(2.2.35.1) *The pushforward $(p\to q)_*F\in\mathsf{Shv}(q)^{\mathsf{op}}$ is a relative limit diagram.*

(2.2.35.2) *The pushforward $(p(*)\to q(*))_*\overline{F}\in\mathsf{Shv}(q(*))^{\mathsf{op}}$ is a relative limit diagram, where $\overline{F} : K^\triangleleft\to\mathsf{Shv}(p(*))^{\mathsf{op}}$ denotes the cartesian transport* (1.4.154) *of F.*

Proof. The pushforward $(p \to q)_* F \in \mathsf{Shv}(q)^{\mathsf{op}}$ is a relative limit diagram iff its cartesian transport $\overline{(p \to q)_* F} \in \mathsf{Shv}(q(*))^{\mathsf{op}}$ is a limit diagram (1.4.154). The canonical comparison map $\overline{(p \to q)_* F} \to (p(*) \to q(*))_* \overline{F}$ (2.2.33) is an isomorphism by proper base change (2.2.34). □

Partitions of unity and acyclicity

There are a number of classical results of the following general flavor: a sheaf which is sufficiently 'flexible' is 'acyclic'. We now present one particular such result (2.2.41), in which 'flexibility' is characterized in terms of partitions of unity (2.2.36) and 'acyclicity' is measured by sheaves being closed under ∞-sifted colimits inside presheaves (2.2.39). The engine behind these results is the fact (2.2.38) that presheaves of modules over sheaves of rings with partitions of unity are in fact sheaves provided they have just a bit of extra structure, specifically certain 'extend by zero' operations.

2.2.36 Definition (Compact partitions of unity)**.** Let R be a sheaf of ring spaces (**??**) on a locally compact Hausdorff space X. A *partition of unity* in R over a compact subset $K \subseteq X$ subordinate to a finite open cover $K \subseteq U_1 \cup \cdots \cup U_n$ is a collection of sections $\varphi_1, \ldots, \varphi_n \in R(K)$ with $\varphi_i|_{K \setminus U_i} = 0$ (beware that $\varphi_i|_{K \setminus U_i} = 0$ is stronger than $\operatorname{supp} \varphi_i \subseteq K \cap U_i$ (2.2.17)(2.2.21)). We say that R *admits compact partitions of unity* when every compact $K \subseteq X$ and every finite open cover $K \subseteq U_1 \cup \cdots \cup U_n$ has a subordinate partition of unity.

2.2.37 Definition (Multiply and extend by zero)**.** Let R be a sheaf of ring spaces (**??**) on a locally compact Hausdorff space X. A *multiply and extend by zero operation* on a k-presheaf of R-modules M for an element $\varphi \in R(K) \times_{R(B)} 0$ (for some $B \subseteq K \subseteq X$ compact) is a dotted lift in the following diagram, functorial in compact $A \subseteq K$.

$$\begin{tikzcd} & M(A \cup B) \arrow[d] \\ M(A) \arrow[ur, dashed, "\varphi\odot"] \arrow[r, "\varphi\cdot"'] & M(A) \end{tikzcd} \tag{2.2.37.1}$$

Functoriality in A means, precisely, that the above is a diagram of presheaves on $\mathsf{Cpt}(K)$. We say an R-module k-presheaf M *has multiply and extend by zero operations* when such operations exist for every fixed $\varphi \in R(K) \times_{R(B)} 0$ and $B \subseteq K \subseteq X$.

Let us note that a multiply and extend by zero operation (2.2.37.1) canonically extends to a diagram

$$\begin{tikzcd} M(A \cup B) \arrow[r, "\varphi\cdot"] \arrow[d] & M(A \cup B) \arrow[d] \\ M(A) \arrow[ur, dashed, "\varphi\odot"] \arrow[r, "\varphi\cdot"'] & M(A) \end{tikzcd} \tag{2.2.37.2}$$

functorial in A. Indeed, this follows from considering functoriality of the diagram (2.2.37.1) under $A = (C \to C \cup B)$.

Any k-sheaf of R-modules M has canonical (and functorial) multiply and extend by zero operations (2.2.37). Indeed, writing $M(A\cup B)=M(A)\times_{M(A\cap B)}M(B)$, we see that lifting $(R(A\cup B)\times_{R(B)}0)\times M(A)\to M(A)$ to $M(A\cup B)\to M(A)$ is equivalent to lifting the composition $(R(A\cup B)\times_{R(B)}0)\times M(A)\to M(A)\to M(A\cap B)$ to $M(B)\to M(A\cap B)$. This latter composition is (canonically identified with) zero, a canonical lift of which is given by zero.

$$\begin{tikzcd} & M(A\cup B) \arrow[r] \arrow[d] & M(B) \arrow[d] \\ (R(A\cup B)\times_{R(B)}0)\times M(A) \arrow[r, "\cdot"] \arrow[ur, dashed, "\odot"] \arrow[urr, dashed] & M(A) \arrow[r] & M(A\cap B) \end{tikzcd} \tag{2.2.37.3}$$

We now explore a converse of sorts to this statement, namely we argue that (under certain conditions) a presheaf with multiply and extend by zero operations is a sheaf (2.2.38).

2.2.38 Lemma (Extension by zero implies the sheaf property)**.** *Let X be a locally compact Hausdorff space. Let R be a sheaf of ring spaces* (**??**) *on X, and let M be a k-presheaf of R-modules* (**??**)*. Suppose that:*

(2.2.38.1) *R admits compact partitions of unity* (2.2.36)*.*

(2.2.38.2) *M satisfies the continuity property* (2.2.24.1)*.*

(2.2.38.3) *M has multiply and extend by zero operations over R* (2.2.37)*.*

In this case, M satisfies descent (2.2.24.2)*, hence is a k-sheaf.*

Proof. We would like to show that for every finite collection of compact sets $K_1,\ldots,K_n\subseteq X$ with union $K=\bigcup_{i=1}^n K_i$, the map

$$M(K)\to\lim_{\varnothing\ne S\subseteq\{1,\ldots,n\}}M\Big(\bigcap_{i\in S}K_i\Big) \tag{2.2.38.4}$$

is an isomorphism.

The 'multiply and extend by zero' operations on M are not far from giving an inverse to the descent map (2.2.38.4) (intuitively, this is because they allow one to 'localize' the descent question to a given set K_i, where it becomes trivial). Indeed, suppose $\varphi\in R(K)$ has $\varphi|_{K\setminus K_j^\circ}=0$, and consider the resulting diagrams (2.2.37.2) for $A=\bigcap_{i\in S}K_i$ and $B=K\setminus K_j^\circ$.

$$\begin{tikzcd} M\big(\bigcap_{i\in S}K_i\cup(K\setminus K_j^\circ)\big) \arrow[r, "\varphi\cdot"] \arrow[d] & M\big(\bigcap_{i\in S}K_i\cup(K\setminus K_j^\circ)\big) \arrow[d] \\ M\big(\bigcap_{i\in S}K_i\big) \arrow[r, "\varphi\cdot"] \arrow[ur, dashed, "\varphi\odot"] & M\big(\bigcap_{i\in S}K_i\big) \end{tikzcd} \tag{2.2.38.5}$$

Now take the limit of these diagrams over $\varnothing\ne S\subseteq\{1,\ldots,n\}$. We claim that the natural retraction

$$M(K)\to\lim_{\varnothing\ne S\subseteq\{1,\ldots,n\}}M\Big(\bigcap_{i\in S}K_i\cup(K\setminus K_j^\circ)\Big)\to M(K) \tag{2.2.38.6}$$

(presheaf restriction and evaluation at $S = \{j\}$, respectively) is an isomorphism. To see that restriction to $\{j\} \subseteq \{\varnothing \neq S \subseteq \{1,\ldots,n\}\}$ is an isomorphism, filter the inclusion of indexing categories in two steps $\{j\} \subseteq \{j \in S \subseteq \{1,\ldots,n\}\} \subseteq \{\varnothing \neq S \subseteq \{1,\ldots,n\}\}$; the first step is the inclusion of an initial object, and the second step induces an isomorphism on limits by (1.4.109) (it is a pushout of $(\Delta^1,1)^\# \times \{\varnothing \neq S \subseteq \{1,\ldots,n\} \setminus \{j\}\}$ since our diagram sends morphisms $S \to S \cup \{j\}$ in the indexing category to isomorphisms). In view of this isomorphism (2.2.38.6), the limit of (2.2.38.5) over $\varnothing \neq S \subseteq \{1,\ldots,n\}$ has the following form.

$$\begin{tikzcd} M(K) \arrow[r, "\varphi\cdot"] \arrow[d] & M(K) \arrow[d] \\ \lim\limits_{\varnothing \neq S \subseteq \{1,\ldots,n\}} M\big(\bigcap\limits_{i\in S} K_i\big) \arrow[r, "\varphi\cdot"] \arrow[ur, dashed, "\varphi\odot"] & \lim\limits_{\varnothing \neq S \subseteq \{1,\ldots,n\}} M\big(\bigcap\limits_{i\in S} K_i\big) \end{tikzcd} \tag{2.2.38.7}$$

In other words, the 'multiply and extend by zero' operation $\varphi\,\odot$ defines for us an inverse to the descent map (2.2.38.4), up to multiplication by any particular $\varphi \in R(K)$ with $\varphi|_{K\setminus K_j^\circ} = 0$ for some $j \in \{1,\ldots,n\}$.

If there exists a decomposition $1 = \varphi_1 + \cdots + \varphi_n$ in $R(K)$ with $\varphi_i|_{K\setminus K_i^\circ} = 0$ for all i, then the sum of the resulting lifts (2.2.38.7) defines an inverse to the descent map (2.2.38.4), as desired. Our hypothesis that R admits compact partitions of unity does not quite give such a decomposition, since $\bigcup_i K_i^\circ$ may not cover all of K. To bridge this gap, we will create a bit of extra 'room' by enlarging the sets K_i using the fact that M satisfies the continuity property.

Consider the directed system of all tuples of compact sets $K_1^+,\ldots,K_n^+ \subseteq X$ with $K_i \subseteq (K_i^+)^\circ$ (let $K^+ = \bigcup_{i=1}^n K_i^+$). Since M satisfies the continuity property, we may express $M(K) = \varinjlim M(K^+)$ and $M(\bigcap_{i\in S} K_i) = \varinjlim M(\bigcap_{i\in S} K_i^+)$ as directed colimits over this directed system. Now filtered colimits commute with finite limits in Spc (**??**), so we conclude that the descent map (2.2.38.4) may be expressed as

$$\varinjlim_{\{K_i \subseteq K_i^+\}_i} \left(M(K^+) \to \lim_{\varnothing \neq S \subseteq \{1,\ldots,n\}} M\Big(\bigcap_{i\in S} K_i^+\Big)\right). \tag{2.2.38.8}$$

To show that this map on filtered colimits is an isomorphism, it suffices (according to (1.4.190)) to construct a dotted lift

$$\begin{tikzcd} M(K^+) \arrow[r] \arrow[d] & M(K) \arrow[d] \\ \lim\limits_{\varnothing \neq S \subseteq \{1,\ldots,n\}} M\big(\bigcap\limits_{i\in S} K_i^+\big) \arrow[r] \arrow[ur, dashed] & \lim\limits_{\varnothing \neq S \subseteq \{1,\ldots,n\}} M\big(\bigcap\limits_{i\in S} K_i\big) \end{tikzcd} \tag{2.2.38.9}$$

for every fixed tuple $K_1^+,\ldots,K_n^+$. Our earlier construction (2.2.38.7) applied to $K_1^+,\ldots,K_n^+$ provides such a lift, but with the horizontal maps multiplied by some $\varphi \in R(K^+)$ with $\varphi|_{K^+\setminus(K_i^+)^\circ} = 0$ for some i. We conclude that the desired lift (2.2.38.9) exists if there exist $\varphi_1,\ldots,\varphi_n \in R(K^+)$ with $(\varphi_1 + \cdots + \varphi_n)|_K = 1$ and $\varphi_i|_{K^+\setminus(K_i^+)^\circ} = 0$. The existence of compact partitions of unity in R provides $\varphi_1, \ldots, \varphi_n \in R(K)$ with $\varphi_1 + \cdots + \varphi_n = 1$ and $\varphi_i|_{K \setminus (K_i^+)^\circ} = 0$. Using the continuity axiom for R, we may lift this to a decomposition $1 = \varphi_1 + \cdots + \varphi_n$ in $R(K')$ with $\varphi_i|_{K' \setminus (K_i^+)^\circ} = 0$ for some compact K' with $K \subseteq (K')^\circ$. This gives a lift (2.2.38.9) not for the given tuple $K_1^+, \ldots, K_n^+$, but for its intersection with K'; in particular, the set of tuples $\{K_i \subseteq K_i^+\}_i$ for which a lift (2.2.38.9) exists is final. This is enough, since we can just replace the directed colimit (2.2.38.8) over all tuples $\{K_i \subseteq K_i^+\}_i$ with the directed colimit over just those tuples $\{K_i \subseteq K_i^+\}_i$ for which the lift (2.2.38.9) exists. □

⋆ **2.2.39 Definition** (k-acyclic)**.** Let X be locally compact Hausdorff. A colimit of sheaves on X which is preserved by the inclusion of sheaves into k-presheaves is called *k-acyclic*.

2.2.40 Exercise. Show that k-acyclic colimits are preserved by proper pushforward.

We will presently be interested in ∞-sifted colimits of ring-module sheaves and k-presheaves (**??**). Such colimits coincide with colimits of underlying (k-pre)sheaves (1.4.223)(**??**) (formally, the forgetful functors $\mathsf{RngShv}(X) \to \mathsf{Shv}(X)$ and $\mathsf{Mod} \rtimes \mathsf{RngShv}(X) \to \mathsf{Shv}(X) \times \mathsf{Shv}(X)$ as well as $\mathsf{RngP}(X_{\mathrm{cpt}}) \to \mathsf{P}(X_{\mathrm{cpt}})$ and $\mathsf{Mod} \rtimes \mathsf{RngP}(X_{\mathrm{cpt}}) \to \mathsf{P}(X_{\mathrm{cpt}}) \times \mathsf{P}(X_{\mathrm{cpt}})$ reflect and lift ∞-sifted colimits). In particular, the notion of k-acyclicity (2.2.39) has an unambiguous meaning for ∞-sifted colimits of ring-module sheaves.

⋆ **2.2.41 Corollary** (Partitions of unity imply k-acyclicity)**.** *Let X be locally compact Hausdorff. An ∞-sifted colimit $(R, M) : K^\triangleleft \to \mathsf{Shv}(X, \mathsf{Mod} \rtimes \mathsf{RngSpc})$ is k-acyclic provided R_i has compact partitions of unity for at least one $i \in K$.*

Proof. We are to show that the k-presheaf colimits $\operatorname{colim}_K^{\mathsf{P}(X_{\mathrm{cpt}})} R$ and $\operatorname{colim}_K^{\mathsf{P}(X_{\mathrm{cpt}})} M$ are k-sheaves. We may restrict attention to the latter, as the former is a special case of it (consider the free module of rank one $(R, R) : K^\triangleleft \to \mathsf{Shv}(X, \mathsf{Mod} \rtimes \mathsf{RngSpc})$).

The k-presheaf colimit $\operatorname{colim}_K^{\mathsf{P}(X_{\mathrm{cpt}})} M$ satisfies continuity (2.2.24.1) since colimits commute with colimits. To show that it satisfies descent (2.2.24.2), we appeal to (2.2.38) and its R_i-module structure. It thus suffices to equip $\operatorname{colim}_K^{\mathsf{P}(X_{\mathrm{cpt}})} M$ with multiply and extend by zero operations over R_i (2.2.37).

Recall that every ring-module sheaf (R, M) has *functorial* operations (2.2.37.3):

$$\begin{tikzcd} & M(A \cup B) \arrow[d] \\ (R(A \cup B) \times_{R(B)} 0) \times M(A) \arrow[ur, dashed, "\odot"] \arrow[r, "\cdot"] & M(A) \end{tikzcd} \tag{2.2.41.1}$$

Taking the colimit over K and using the fact that ∞-sifted colimits commute with finite products (1.4.222), we obtain operations:

$$\begin{tikzcd} & \operatorname*{colim}_K M(A \cup B) \arrow[d] \\ \operatorname*{colim}_K (R(A \cup B) \times_{R(B)} 0) \times \operatorname*{colim}_K M(A) \arrow[ur, dashed, "\odot"] \arrow[r, "\cdot"] & \operatorname*{colim}_K M(A) \end{tikzcd} \tag{2.2.41.2}$$

Pre-composing with the map $R_i(A \cup B) \times_{R_i(B)} 0 \to \operatorname{colim}_K (R(A \cup B) \times_{R(B)} 0)$, we obtain the desired multiply and extend by zero operations for the R_i-module structure on $\operatorname{colim}_K^{\mathsf{P}(X_{\mathrm{cpt}})} M$. $\square$

2.2.42 Exercise. Conclude from (2.2.41) that ∞-sifted colimits of R-modules are k-acyclic for any sheaf of ring spaces R (on a locally compact Hausdorff space) which admits compact partitions of unity.

2.3 Topological stacks

In (2.2), we studied sheaves on a fixed topological space. We now turn to sheaves on the category of all topological spaces, where the discussion takes a markedly different, more geometric, flavor. We will call a sheaf on the category of topological spaces a *topological stack* (we find this terminology the most descriptive, though it is not standard). The Yoneda functor gives a fully faithful embedding from the category of topological spaces into the category of topological stacks

$$\mathsf{Top} \subseteq \mathsf{Shv}(\mathsf{Top}), \tag{2.3.0.1}$$

and it is helpful to regard topological stacks as 'generalized topological spaces'. We will see how to generalize many natural notions and constructions from topological spaces to topological stacks. Arbitrary topological stacks are a bit like arbitrary topological spaces: they can be very pathological and are not of so much interest. There is a particularly nice class of topological stacks, namely those which admit a *representable atlas*; they are equivalent, in a certain sense, to 'topological groupoids' as introduced by Ehresmann [27] and developed by Haefliger [37, 38, 39] and others. Examples of such topological stacks include orbifolds [98, 104] and graphs/complexes of groups [39].

References for the theory we are about to discuss include Noohi [89] and Heinloth [40]. It is a topological analogue of the theory of algebraic stacks originating from Grothendieck, Deligne–Mumford [19], and Artin [9], for which a comprehensive reference is Laumon–Moret-Bailly [65]. This topological analogue is an easier, more elementary, version of the algebraic theory; it was documented only much later in Noohi [89]. An intuitive geometric introduction may be found in Behrend [12].

We will work in the generality of sheaves of ∞-groupoids $\mathsf{Shv}(-) = \mathsf{Shv}(-;\mathsf{Spc})$. We emphasize, however, that the reader may restrict to the technically and conceptually simpler setting of sheaves of groupoids $\mathsf{Shv}(-;\mathsf{Grpd}) \subseteq \mathsf{Shv}(-;\mathsf{Spc})$ and retain the essence of the discussion (in fact, this is the setting addressed by all of the aforementioned references).

⋆ **2.3.1 Definition** (Topological stack)**.** A *topological stack* is a sheaf on Top valued in the ∞-category Spc (that is, a functor $\mathsf{Top}^{\mathsf{op}} \to \mathsf{Spc}$ satisfying descent for open covers (2.2.4)(2.2.10)(2.2.20)). Topological stacks form the ∞-category $\mathsf{Shv}(\mathsf{Top}) \subseteq \mathsf{P}(\mathsf{Top})$ (a full subcategory of presheaves).

2.3.2 Example ($\mathsf{Shv}(\mathsf{Top})$ is not locally small)**.** Just like $\mathsf{P}(\mathsf{Set})$ (1.1.79), the category $\mathsf{Shv}(\mathsf{Top})$ is not locally small. Given a topological space X and a point $x \in X$, we may consider the set $Q(X,x)$ of functions $\alpha : 2^X \to \mathrm{Card}$ with the property that $\alpha(S) \leq |S|$, modulo the equivalence relation $\alpha \sim \alpha'$ when they agree on 2^U for some open set $U \subseteq X$ containing x. Given a continuous map $f : X \to Y$, there is a map $f^* : Q(Y, f(x)) \to Q(X,x)$ given by taking $(f^*\alpha)(S) = \alpha(f(S))$. Now $Q(X) = \prod_{x\in X} Q(X,x)$ defines a sheaf $Q : \mathsf{Top}^{\mathsf{op}} \to \mathsf{Set}$. As in (1.1.79), every endomorphism $\gamma : \mathrm{Card} \to \mathrm{Card}$ with $\gamma(\kappa) \leq \kappa$ gives a distinct endomorphism of Q, proving that $\mathrm{Hom}_{\mathsf{Shv}(\mathsf{Top})}(Q,Q)$ is not small.

2.3.3 Proposition (Universal property of topological stacks)**.** *For any cocomplete ∞-category* E*, pullback along the functors* $\mathsf{Top} \xrightarrow{y_{\mathsf{Top}}} \mathsf{P}(\mathsf{Top}) \xrightarrow{\#} \mathsf{Shv}(\mathsf{Top})$ *defines equivalences between the following ∞-categories of functors:*

(2.3.3.1) *Cocontinuous functors* $\mathsf{Shv}(\mathsf{Top}) \to \mathsf{E}$*.*

(2.3.3.2) *Cocontinuous functors* $\mathsf{P}(\mathsf{Top}) \to \mathsf{E}$ *which send sheafifications to isomorphisms.*

(2.3.3.3) *Cocontinuous functors* $\mathsf{P}(\mathsf{Top}) \to \mathsf{E}$ *which send Čech nerves $N(X, \{U_i\}_i) \to X$ to isomorphisms.*

(2.3.3.4) *Cosheaves* $\mathsf{Top} \to \mathsf{E}$*.*

Proof. This is a special case of the universal property of local presheaves (1.4.206), given the fact that a presheaf is a sheaf iff it is right local (1.1.96) with respect to Čech nerves $N(X, \{U_i\}_i) \to X$ (2.2.19). □

⋆ **2.3.4 Definition** (Point)**.** A *point* x of a topological stack X is a map $x : * \to X$, i.e. it is an object $x \in X(*)$ (also simply written $x \in X$).

⋆ **2.3.5 Exercise** (Coarse space)**.** The *coarse space* $|X|$ of a topological stack X is its image under the functor

$$|\cdot| : \mathsf{Shv}(\mathsf{Top}) \to \mathsf{Top} \quad (2.3.5.1)$$
$$X \mapsto |X| \quad (2.3.5.2)$$

left adjoint to the Yoneda embedding $\mathsf{Top} \hookrightarrow \mathsf{Shv}(\mathsf{Top})$. Use formal reasoning to show that this left adjoint exists on small sheaves $\mathsf{Shv}(\mathsf{Top})_{\mathsf{small}} \subseteq \mathsf{Shv}(\mathsf{Top})$ since Top is cocomplete. Next, show that $|X|$ is defined for all $X \in \mathsf{Shv}(\mathsf{Top})$ by giving an explicit construction (the underlying set of $|X|$ is $\pi_0 \mathrm{Hom}(*, X)$, and it is equipped with the topology in which a subset is open iff its inverse image under any map $Z \to X$ from a topological space Z is open).

Properties of morphisms

We now discuss properties of morphisms of topological stacks. Recall the notion of a *representable morphism* of presheaves (1.1.122), which applies in particular to morphisms of topological stacks by restriction $\mathsf{Shv}(\mathsf{Top}) \subseteq \mathsf{P}(\mathsf{Top})$. Also recall that for every property of morphisms of topological spaces preserved under pullback, there is an induced property of representable morphisms in $\mathsf{P}(\mathsf{Top})$ (hence, by restriction, to representable morphisms of topological stacks) (1.1.124).

2.3.6 Example (Open cover)**.** A morphism of topological stacks $U \to X$ is called an open embedding (1.1.124) when for every morphism $Z \to X$ from a topological space Z, the pullback $U \times_X Z \to Z$ is an open embedding of topological spaces. A collection of open embeddings $\{U_i \to X\}_i$ is called an open cover when for every morphism $Z \to X$ from a topological space Z, the collection of pullbacks $\{U_i \times_X Z \to Z\}_i$ is an open cover.

Sometimes properties of morphisms of topological spaces have a natural generalization to topological stacks which coincides with the induced property for representable morphisms but is more general (compare (1.1.125)). Here is a first example:

⋆ **2.3.7 Definition** (Admits local sections)**.** A map of topological stacks $X \to Y$ is said to *admit local sections* (indicated by the arrow $\twoheadrightarrow$) iff for every map $U \to Y$ from a topological space U, there exists an open cover $U = \bigcup_i U_i$ so that each restriction $U_i \to Y$ lifts to X.

$$\begin{tikzcd} & & X \arrow[d] \\ U_i \arrow[r] \arrow[urr, dashed] & U \arrow[r] & Y \end{tikzcd} \tag{2.3.7.1}$$

2.3.8 Exercise. Show that a representable map of topological stacks admits local sections in the sense of (2.3.7) iff it does so in the sense of (1.1.124).

2.3.9 Exercise. Show that admitting local sections is preserved under pullback and closed under composition.

⋆ **2.3.10 Lemma.** *For any map of topological stacks $U \twoheadrightarrow X$ admitting local sections, the natural map*

$$\operatorname*{colim}^{\mathsf{Shv(Top)}}\Big(\cdots \substack{\to\\\to\\\to\\\to} U \times_X U \times_X U \substack{\to\\\to\\\to} U \times_X U \rightrightarrows U\Big) \to X \tag{2.3.10.1}$$

is an isomorphism.

Proof. Recall (**??**) that for any map of spaces $U \to X$, the natural map

$$\operatorname*{colim}^{\mathsf{Spc}}\Big(\cdots \substack{\to\\\to\\\to\\\to} U \times_X U \times_X U \substack{\to\\\to\\\to} U \times_X U \rightrightarrows U\Big) \to X \tag{2.3.10.2}$$

is the inclusion of the components of X in the image of $\pi_0(U \to X)$. Since limits and colimits in presheaf categories are computed pointwise (**??**), it follows that the natural map

$$\operatorname*{colim}^{\mathsf{P(Top)}}\Big(\cdots \substack{\to\\\to\\\to\\\to} U \times_X U \times_X U \substack{\to\\\to\\\to} U \times_X U \rightrightarrows U\Big) \to X \tag{2.3.10.3}$$

is the inclusion of the full subpresheaf of X spanned by those maps $Z \to X$ which lift to U. Since every map $Z \to X$ lifts locally to U (by hypothesis), it follows that this map is a sheafification (**??**). □

⋆ **2.3.11 Definition** (Target-local property)**.** Let $\mathcal{P}$ be a property of morphisms of topological stacks preserved under pullback. We say that a morphism $X \to Y$ is *(target-)locally* $\mathcal{P}$ when there exists a collection of maps $U_i \to Y$ such that every pullback $X \times_Y U_i \to U_i$ has $\mathcal{P}$ and $\bigsqcup_i U_i \twoheadrightarrow Y$ admits local sections (2.3.7). It is evident that $\mathcal{P}$ implies locally $\mathcal{P}$. When locally $\mathcal{P}$ implies $\mathcal{P}$, we say that $\mathcal{P}$ is *local on the target.*

2.3.12 Lemma. *Isomorphism of topological stacks is local on the target* (2.3.11).

Proof. We refine the argument of (**??**). Suppose $X \to Y$ is a morphism of topological stacks and there exist morphisms of topological stacks $U_i \to Y$ such that $\bigsqcup_i U_i \twoheadrightarrow Y$ admits local sections and every pullback $X \times_Y U_i \to U_i$ is an isomorphism. By (2.2.16), the full subcategory of $\mathsf{Shv}(\mathsf{Top})_{/Y}$ spanned by those morphisms $Z \to Y$ for which the pullback $X \times_Y Z \to Z$ is an isomorphism is closed under colimits. Given a topological space Z and a map $Z \to Y$, the hypothesis implies that there exists an open cover $Z = \bigcup_i V_i$ for which every pullback $X \times_Y V_i \to V_i$ is an isomorphism. Since Z is the colimit in $\mathsf{Shv}(\mathsf{Top})$ of this open cover (**??**), it follows that $X \times_Y Z \to Z$ is an isomorphism. If Y is a colimit of topological spaces (that is, if $Y \in \mathsf{Shv}(\mathsf{Top})_{\mathsf{small}}$ (**??**)), then we conclude that $X \to Y$ is an isomorphism. For general Y, note that to show $X \to Y$ is an isomorphism, it suffices to show that its restriction to $\mathsf{Shv}(\mathsf{Top}_\alpha)$ is an isomorphism for every cardinal α, which reduces us to the small case. □

⋆ **2.3.13 Exercise.** Use locality of isomorphism (2.3.12) to show that a topological stack which admits an open cover (2.3.6) by topological spaces is itself a topological space (note that $X \in \mathsf{Shv}(\mathsf{Top})$ is representable iff $X \to |X|$ is an isomorphism, and use the relation between open embeddings into X and open subsets of $|X|$ (**??**)).

2.3.14 Corollary. *Representability of morphisms of topological stacks is local on the target.*

Proof. Let $X \to Y$ be a morphism of topological stacks which is locally (on the target) representable. We are to show that $X \times_Y Z$ is representable for any map $Z \to Y$ from a topological space Z. The pullback $X \times_Y Z \to Z$ is also locally representable (**??**), so by renaming $Y = Z$, it suffices to show that if $X \to Y$ is locally representable and Y is representable, then X is representable. Since Y is representable and $\bigsqcup_i U_i \twoheadrightarrow Y$ admits local sections, there exists an open cover $Y = \bigcup_j V_j$ with every $X \times_Y V_j \to V_j$ representable. Thus every $X \times_Y V_j$ is representable, which implies that X is representable (2.3.13). □

$*\quad*$

2.3.15 Exercise. Let X be a topological stack. A subset $E \subseteq |X(*)|$ ($|\cdot|$ denotes isomorphism classes) determines an assignment to every map $f : Z \to X$ of a subset $Z_{E,f} \subseteq Z$ which is compatible with pullback in the sense that $Z'_{E,f\circ g} = g^{-1}(Z_{E,f})$ for every map $g : Z' \to Z$. Show that this defines a bijection between subsets of $|X|$ and pullback compatible assignments of subsets of Z to maps $Z \to X$.

2.3.16 Exercise (Classification of embedded substacks)**.** Let X be a topological stack, let $E \subseteq |X(*)|$ be any subset, and let X_E denote the topological stack for which a map $Z \to X_E$ is a map $Z \to X$ whose specialization to every point of Z lies in E. Show that for $f : Z \to X$, the natural diagram

$$\begin{tikzcd} Z_{E,f} \arrow[r] \arrow[d] & X_E \arrow[d] \\ Z \arrow[r, "f"] & X \end{tikzcd} \tag{2.3.16.1}$$

is a pullback square. Conclude that $X_E \to X$ is an embedding (2.1.7.3)(1.1.124) and that $X_E \to X$ satisfies a property $\mathcal{P}$ preserved under pullback iff every $Z_{E,f} \subseteq Z$ satisfies $\mathcal{P}$. Moreover, show that every embedding $X' \to X$ is uniquely isomorphic to a unique $X_E \to X$.

$*\quad*$

⋆ **2.3.17 Definition** (Universally closed)**.** A map of topological stacks $X \to Y$ is said to be *universally closed* when it satisfies the subswarm lifting property (2.1.29.3), namely that for any commuting diagram of solid arrows

$$\begin{tikzcd} T^* \arrow[r, dashed] \arrow[d, dashed] & S^* \arrow[r] \arrow[d] & X \arrow[d] \\ T \arrow[r, dashed] \arrow[urr, dashed] & S \arrow[r] & Y \end{tikzcd} \tag{2.3.17.1}$$

in which (S, S^*) is a limit pointed topological space (2.1.24), there exists a map of limit pointed topological spaces $(T, T^*) \to (S, S^*)$ and a diagonal dotted arrow making the diagram commute.

2.3.18 Exercise. Show that universal closedness (2.3.17) is preserved under pullback, closed under composition, and local on the target (2.3.11).

2.3.19 Exercise. Show that a representable morphism of topological stacks is universally closed in the sense of (2.3.17) (satisfies the subswarm lifting property) iff it is universally closed in the sense of (1.1.124) (every pullback to a topological space is universally closed).

⋆ **2.3.20 Definition** (Proper)**.** A map of topological stacks is called *proper* when its kth diagonal (1.1.63) is universally closed for all $k \geq 0$.

⋆ **2.3.21 Definition** (Separated)**.** A map of topological stacks is called *separated* when its diagonal is proper (that is, when its kth diagonal is universally closed for all $k \geq 1$).

2.3.22 Lemma. *If $|X|$ is Hausdorff, then $X \to X \times X$ is proper iff it is so locally on $|X|$.*

Proof. Let us show that $X \to X \times X$ is proper provided it is so locally on $|X|$. Properness is local on the target (2.3.18)(**??**), so it suffices to verify that $X \to X \times X$ is proper over the elements of an open cover of $X \times X$. By hypothesis, there is an open cover $X = \bigcup_i U_i$ so that each U_i has proper diagonal, so $X \to X \times X$ is proper over each open substack $U_i \times U_i \subseteq X \times X$. Since $|X|$ is Hausdorff, the complement of $|X| \to |X| \times |X|$ is open, and the restriction of $X \to X \times X$ to this open substack is certainly proper (as its domain is empty). □

Atlases

⋆ **2.3.23 Definition** (Atlas)**.** Let X be a topological stack. An *atlas* for X is a topological space U and a map $U \twoheadrightarrow X$ admitting local sections. For any property of morphisms $\mathcal{P}$, an atlas is said to have $\mathcal{P}$ (or to be a $\mathcal{P}$-atlas) when the map $U \twoheadrightarrow X$ has $\mathcal{P}$.

2.3.24 Lemma. *If $U \twoheadrightarrow X$ is an atlas, then the induced map $U \to |X|$ is the quotient by the equivalence relation given by the image of $U \times_X U \to U \times U$.*

Proof. The coarse space functor $\mathsf{Shv}(\mathsf{Top}) \to \mathsf{Top}$ is a left adjoint, hence preserves colimits, and the colimit of the simplicial object $\cdots \substack{\to\\\to\\\to} U \times_X U \rightrightarrows U$ is the quotient of U by the equivalence relation given by the image of $U \times_X U \to U \times U$. □

2.3.25 Lemma. *If $U \twoheadrightarrow X$ is an open atlas, then $U \to |X|$ is open.*

Proof. Since $U \to |X|$ is a topological quotient (2.3.24), a subset of $|X|$ is open iff its inverse image in U is open. Let $V \subseteq U$ be open, and let us show that its image in $|X|$ is open. The inverse image in U of the image of $V \to U \to |X|$ is, by the description of the equivalence relation (2.3.24), the image of $V \times_X U \to U$, which is open since $V \times_X U \to U$ is a pullback of the composition of open maps $V \to U \to X$. □

2.3.26 Lemma. *If X has an open atlas and $X \to X \times X$ is universally closed, then $|X|$ is Hausdorff.*

Proof. Let $U \twoheadrightarrow X$ be an open atlas. Since $U \to |X|$ is an open topological quotient map (2.3.25)(2.3.24), the quotient $|X|$ is Hausdorff iff the equivalence relation is closed. This equivalence relation is the image of $U \times_X U \to U \times U$, which is closed since $X \to X \times X$ is universally closed. □

2.3.27 Lemma. *Let X and Y be topological stacks admitting open atlases. The natural map $|X \times Y| \to |X| \times |Y|$ is an isomorphism.*

Proof. The map $|X \times Y| \to |X| \times |Y|$ is a bijection since $|X| = \pi_0 \mathrm{Hom}(*, X)$ (2.3.5) and π_0 preserves products. To show that $|X \times Y| \to |X| \times |Y|$ is open, we need to understand open subsets of $|X| \times |Y|$. Choose open atlases $U \twoheadrightarrow X$ and $V \twoheadrightarrow Y$. The maps $U \to |X|$ and $V \to |Y|$ are open topological quotient maps (2.3.24)(2.3.25). It follows that their product $U \times V \to |X| \times |Y|$ is a topological quotient map (open topological quotient maps are closed under products). On the other hand, $U \times V \to |X \times Y|$ is also a topological quotient map (a product of atlases is an atlas), and $|X \times Y| \to |X| \times |Y|$ is a bijection, so we are done. □

⋆ **2.3.28 Exercise.** Let $U \twoheadrightarrow X$ be an atlas, and let $\mathcal{P}$ be a property of morphisms of topological stacks preserved under pullback and local on the target (2.3.11). Show that a map of topological stacks $Z \to X$ has $\mathcal{P}$ iff its pullback $Z \times_X U \to U$ has $\mathcal{P}$. Conclude that:

(2.3.28.1) $U \to X$ satisfies $\mathcal{P}$ iff $U \times_X U \to U$ satisfies $\mathcal{P}$.

(2.3.28.2) $X \to X \times X$ satisfies $\mathcal{P}$ iff $U \times_X U \to U \times U$ satisfies $\mathcal{P}$.

In particular, conclude that the diagonal of X is representable, and hence that $\underline{\mathrm{Aut}}(x)$ is a topological group for every point $x : * \to X$.

2.3.29 Example (Proper group action)**.** Let $G \curvearrowright X$ be an action of a topological group G on a topological space X. The stack quotient X/G (**??**) is separated (has proper diagonal) iff the map $G \times X \to X \times X$ given by $(g, x) \mapsto (x, gx)$ is proper (2.3.28.2) (such a group action is called *proper*). The quotient map $X \to X/G$ is proper if G is compact Hausdorff (and the converse holds when $X \neq \varnothing$).

2.3.30 Exercise. Show that if G is compact Hausdorff, then an action $G \curvearrowright X$ is proper iff X is separated.

2.3.31 Exercise. Let $G \curvearrowright X$ and $H \curvearrowright Y$ be group actions, and let $Y \to X$ be a map which is equivariant with respect to a homomorphism $H \to G$. Show that the induced map $Y/H \to X/G$ is an isomorphism iff the map $(G \times Y)/H \to X$ is an isomorphism (note that the latter is a pullback of the former and use descent (2.3.28.1)).

2.3.32 Proposition (Proper atlas from proper diagonal)**.** *Let X have a representable atlas and proper diagonal, and let $U \to X$ be a map from a locally compact* (2.1.4) *Hausdorff topological space U. Suppose $p \in U$ is such that $\underline{\mathrm{Aut}}(p) = p \times_X p \subseteq p \times_X U$ is open (note that $p \to U$ is a closed embedding, hence so is its pullback $p \times_X p \to p \times_X U$). For every sufficiently small open neighborhood $V \subseteq U$ of p, we have $p \times_X p = p \times_X V$ and the map $V \to X$ is proper over an open substack of X containing the image of p.*

Proof. By hypothesis, $p \times_X (U \setminus p) \subseteq p \times_X U$ is closed, and $p \times_X U \to U$ is proper since it is a pullback of the diagonal of X. Thus the image of $p \times_X (U \setminus p) \to U$ is a closed set disjoint from p. By replacing U with the complement of this closed set, we may assume wlog that $p \times_X p = p \times_X U$ (p is unique in its orbit).

Since U is locally compact, there exists a compact neighborhood $K \subseteq U$ of p. Since $K \to *$ is proper and $X \to *$ has proper diagonal, it follows that the map $K \to X$ is proper (1.1.68). Suppose $V \subseteq U$ is open and contained in K. Hence $V \subseteq K$ is open, so $K \setminus V \to K$ is a closed embedding, hence proper, so $K \setminus V \to X$ is also proper. It is representable, hence its image (embedded substack of X) is closed (consider its pullback under any map from a topological space Z to X). Let $Y \subseteq X$ denote the complement of the image of $K \setminus V \to X$ (thus Y is an open substack of X); note that Y contains the image of p since p is unique in its orbit. Thus $V \times_X Y = K \times_X Y \to Y$ is a pullback of $K \to X$, hence is proper. $\square$

Artin morphisms

⋆ **2.3.33 Definition** (n-Artin morphism)**.** A morphism of topological stacks $X \to Y$ is called *n-Artin* (for integers $n \geq 0$) when for every map $U \to Y$ from a topological space U, the pullback $X \times_Y U$ admits an $(n-1)$-Artin atlas (this is an inductive definition, the base case being that a morphism is (-1)-Artin iff it is an isomorphism). It is immediate that n-Artin morphisms are preserved under pullback.

2.3.34 Exercise. Show that n-Artin implies $(n+1)$-Artin.

2.3.35 Exercise. Let Y be representable. Show that a morphism of topological stacks $X \to Y$ is n-Artin iff X has an $(n-1)$-Artin atlas.

2.3.36 Lemma. *n-Artin morphisms are closed under composition.*

Proof. Fix n-Artin maps $X \to Y \to Z$, and consider a map $U \to Z$ from representable U. Since $Y \to Z$ is n-Artin, there exists an $(n-1)$-Artin atlas $V \twoheadrightarrow Y \times_Z U$. Since $X \to Y$ is n-Artin, there exists an $(n-1)$-Artin atlas $W \twoheadrightarrow X \times_Y V$.

$$\begin{tikzcd}
W \arrow[r, two heads] & X \times_Y V \arrow[r] \arrow[d] & X \times_Z U \arrow[r] \arrow[d] & X \arrow[d] \\
& V \arrow[r, two heads] & Y \times_Z U \arrow[r] \arrow[d] & Y \arrow[d] \\
& & U \arrow[r] & Z
\end{tikzcd} \tag{2.3.36.1}$$

The maps $W \twoheadrightarrow X \times_Y V \twoheadrightarrow X \times_Z U$ are both $(n-1)$-Artin and admit local sections, hence so does their composition (by induction on n). This is the desired $(n-1)$-Artin atlas for $X \times_Z U$. □

2.3.37 Lemma. *The diagonal of an n-Artin morphism is* $\max(n-1, 0)$*-Artin.*

Proof. Let $X \to Y$ is n-Artin.

By (**??**), every pullback of $X \to X \times_Y X$ to a topological space U is a pullback of the diagonal of the pullback $X \times_Y U \to U$. We may thus assume wlog that Y is representable.

Since Y is representable, there exists an $(n-1)$-Artin atlas $U \twoheadrightarrow X$. We consider the following fiber square.

$$\begin{tikzcd} U \times_X U \arrow[r] \arrow[d] & X \arrow[d] \\ U \times_Y U \arrow[r] & X \times_Y X \end{tikzcd} \tag{2.3.37.1}$$

The pullback $U \times_Y U$ is representable since U and Y are representable. The pullback $U \times_X U$ has a $\max(-1, n-2)$-Artin atlas since U is representable and $U \to X$ is $(n-1)$-Artin. Thus the morphism $U \times_X U \to U \times_Y U$ is $\max(0, n-1)$-Artin (2.3.35). The map $U \times_Y U \twoheadrightarrow X \times_Y X$ admits local sections since $U \twoheadrightarrow X$ does (1.1.62), hence $X \to X \times_Y X$ is $\max(0, n-1)$-Artin (**??**). □

2.3.38 Exercise. Conclude from (1.1.68)(2.3.37) the following cancellation property for Artin morphisms: if the composition $X \to Y \to Z$ is n-Artin and $Y \to Z$ is $(n+1)$-Artin, then $X \to Y$ is n-Artin (any $n \geq 0$).

Mapping stacks

Here we study topological stacks parameterizing maps between topological spaces.

⋆ **2.3.39 Definition** (Mapping stack $\underline{\mathrm{Hom}}(X, Y)$)**.** For topological spaces X and Y, the topological stack $\underline{\mathrm{Hom}}(X, Y)$ is defined by declaring a map $Z \to \underline{\mathrm{Hom}}(X, Y)$ to be a continuous map $Z \times X \to Y$.

2.3.40 Example. The set of maps $* \to \underline{\mathrm{Hom}}(X, Y)$ is the set $\mathrm{Hom}(X, Y)$ of continuous maps $X \to Y$.

2.3.41 Exercise. Show that $\underline{\mathrm{Hom}}(X, Y)$ is a sheaf on Top.

2.3.42 Exercise. Show that the natural map $\underline{\mathrm{Hom}}(X, Y \times Y') \to \underline{\mathrm{Hom}}(X, Y) \times \underline{\mathrm{Hom}}(X, Y')$ is an isomorphism.

2.3.43 Exercise. Show that there is a tautological 'evaluation' map $X \times \underline{\mathrm{Hom}}(X, Y) \to Y$.

⋆ **2.3.44 Definition** (Condition on morphisms)**.** A *condition* $\mathcal{C}$ on morphisms $X \to Y$ is a subset $\operatorname{Hom}_{\mathcal{C}}(X,Y) \subseteq \operatorname{Hom}(X,Y)$. Equivalently, a condition is the assignment to every map $f : Z \to \underline{\operatorname{Hom}}(X,Y)$ of a subset $Z_{\mathcal{C},f} \subseteq Z$ which is compatible with pullback in the sense that $Z'_{\mathcal{C},f\circ g} = g^{-1}(Z_{\mathcal{C},f})$ for any map $g : Z' \to Z$ (2.3.15). Given a condition $\mathcal{C}$, we can consider the embedded substack $\underline{\operatorname{Hom}}_{\mathcal{C}}(X,Y) \subseteq \underline{\operatorname{Hom}}(X,Y)$ parameterizing those maps $Z \times X \to Y$ whose specialization to every $z \in Z$ lies in $\operatorname{Hom}_{\mathcal{C}}(X,Y)$; this defines a bijection between conditions on morphisms $X \to Y$ and embedded substacks of $\underline{\operatorname{Hom}}(X,Y)$ (2.3.16).

Given a property of morphisms of topological stacks $\mathcal{P}$, we say that a condition $\mathcal{C}$ satisfies $\mathcal{P}$ when the morphism $\underline{\operatorname{Hom}}_{\mathcal{C}}(X,Y) \to \underline{\operatorname{Hom}}(X,Y)$ has $\mathcal{P}$. Concretely, this just means that the inclusion $Z_{\mathcal{C},f} \to Z$ has $\mathcal{P}$ for every map $f : Z \times X \to Y$.

2.3.45 Exercise. Show that $f(A) \subseteq V$ is a closed condition on maps $f : X \to Y$ for any subset $A \subseteq X$ and any closed subset $V \subseteq Y$.

2.3.46 Exercise. Show that $f|_A = \mathbf{1}_A$ is a closed condition on maps $f : X \to X$ for any subset $A \subseteq X$ provided X is Hausdorff.

2.3.47 Lemma. *For $K \subseteq X$ compact and $U \subseteq Y$ open, the condition $f(K) \subseteq U$ is open.*

Proof. Equivalently, we show that the condition $f(K) \cap V \neq \varnothing$ is closed for $V \subseteq Y$ closed. This condition may be alternatively stated as $f^{-1}(V) \cap K \neq \varnothing$. Given a map $F : Z \times X \to Y$, the subset $Z_F \subseteq Z$ of maps satisfying this condition is the image of $F|_{Z\times K}^{-1}(V)$ under the projection $Z \times K \to Z$. The inverse image $F|_{Z\times K}^{-1}(V)$ is closed, so its projection to Z is closed since $K \to *$ is universally closed (2.1.29). □

2.3.48 Lemma. *The diagonal of $\underline{\operatorname{Hom}}(X,Y)$ is an embedding* (2.1.7.3).

Proof. The diagonal of $\underline{\operatorname{Hom}}(X,Y)$ is the map $\underline{\operatorname{Hom}}(X,Y) \to \underline{\operatorname{Hom}}(X,Y \times Y)$ (2.3.42). Since the diagonal $Y \to Y \times Y$ is an embedding (2.1.31), it follows that $\underline{\operatorname{Hom}}(X,Y) = \underline{\operatorname{Hom}}_{\mathcal{C}}(X,Y \times Y)$ where $\mathcal{C}$ is the condition of having image contained in the diagonal. The inclusion of the subsheaf of maps satisfying any condition $\mathcal{C}$ is an embedding (2.3.44). □

2.3.49 Exercise. Show that if Y is Hausdorff, then the diagonal of $\underline{\operatorname{Hom}}(X,Y)$ is a closed embedding.

We now turn to representability of $\underline{\operatorname{Hom}}(X,Y)$. As remarked earlier, the set of maps $* \to \underline{\operatorname{Hom}}(X,Y)$ is simply the set of maps $X \to Y$. It follows that $\underline{\operatorname{Hom}}(X,Y)$ is representable iff there is a topology $\mathcal{T}$ on the set $\operatorname{Hom}(X,Y)$ such that a map $Z \times X \to Y$ is continuous iff the induced map $Z \to \operatorname{Hom}(X,Y)_{\mathcal{T}}$ is continuous.

2.3.50 Definition (Compact-open topology)**.** Let X and Y be topological spaces. The compact-open topology on the set $\operatorname{Hom}(X,Y)$ is the topology generated by declaring that, for all compact sets $K \subseteq X$ and open sets $U \subseteq Y$, the locus of maps $f : X \to Y$ satisfying $f(K) \subseteq U$ should be open. The resulting topological space is denoted $\operatorname{Hom}(X,Y)_{\mathrm{cptopen}}$.

For any map $Z \to \underline{\mathrm{Hom}}(X,Y)$, the induced map $Z \to \mathrm{Hom}(X,Y)_{\mathrm{cptopen}}$ is continuous by (2.3.47), which gives a tautological map $\underline{\mathrm{Hom}}(X,Y) \to \mathrm{Hom}(X,Y)_{\mathrm{cptopen}}$. Hence if $\underline{\mathrm{Hom}}(X,Y)$ is representable, necessarily by $\mathrm{Hom}(X,Y)_{\mathcal{T}}$ for some topology $\mathcal{T}$, then $\mathcal{T}$ is at least as fine as the compact-open topology.

⋆ **2.3.51 Proposition.** *If X is locally compact* (2.1.4)*, then the map $\underline{\mathrm{Hom}}(X,Y) \to \mathrm{Hom}(X,Y)_{\mathrm{cptopen}}$ is an isomorphism. In particular, $\underline{\mathrm{Hom}}(X,Y)$ is representable.*

Proof. We are to show that if $Z \to \mathrm{Hom}(X,Y)_{\mathrm{cptopen}}$ is continuous, then the resulting map $Z \times X \to Y$ is also continuous. What we must show is that if (z,x) is sent inside an open set $U \subseteq Y$, then a neighborhood of (z,x) is as well. Since X is locally compact, there is a compact neighborhood $K \subseteq X$ of x such that $z \times K$ is sent inside U. Since $Z \to \mathrm{Hom}(X,Y)_{\mathrm{cptopen}}$ is continuous in the compact-open topology, there is an open set $V \subseteq Z$ such that $V \times K$ is sent inside U. □

The basic mapping stack $\underline{\mathrm{Hom}}(-,-)$ (2.3.39) admits several important generalizations, such as the stack of sections of a fixed map $E \to X$ or maps between fibers $X_b \to Y_b$ of maps $X, Y \to B$. Here is the most general notion we will consider.

⋆ **2.3.52 Definition** (Parameterized stack of sections $\underline{\mathrm{Sec}}$)**.** Let $E \to X \to B$ be morphisms in a category C which has all pullbacks of $X \to B$. We define a presheaf $\underline{\mathrm{Sec}}_B(X,E)$ on C by the property that a map $Z \to \underline{\mathrm{Sec}}_B(X,E)$ from $Z \in \mathsf{C}$ is a map $Z \to B$ along with a map $X \times_B Z \to E$ over X.

$$\begin{tikzcd} & E \arrow[d] \\ X \times_B Z \arrow[ur] \arrow[r] \arrow[d] & X \arrow[d] \\ Z \arrow[r] & B \end{tikzcd} \tag{2.3.52.1}$$

For any functor $F : \mathsf{C} \to \mathsf{D}$, there is an induced morphism $\underline{\mathrm{Sec}}_B(X,E) \to F^*\underline{\mathrm{Sec}}_{F(B)}(F(X),F(E))$ obtained by applying F to diagrams (2.3.52.1), provided F preserves all pullbacks of $X \to B$.

Our present interest will be in the case of topological spaces, in which case $\underline{\mathrm{Sec}}_B(X,E)$ is evidently a sheaf.

2.3.53 Example. A point $* \to \underline{\mathrm{Sec}}_B(X,E)$ is a point $b \in B$ together with a section of $E_b \to X_b$.

2.3.54 Exercise. Show that a diagram

$$\begin{tikzcd} E \arrow[r] \arrow[d] & F \arrow[d] \\ X \arrow[r] \arrow[d] & Y \arrow[d] \\ B \arrow[r] & C \end{tikzcd} \tag{2.3.54.1}$$

induces a map $\underline{\mathrm{Sec}}_B(X,E) \to \underline{\mathrm{Sec}}_C(Y,F)$. Show that the tautological maps

$$\underline{\mathrm{Sec}}_{B'}(X \times_B B', E \times_B B') \xrightarrow{\sim} \underline{\mathrm{Sec}}_B(X,E) \times_B B' \tag{2.3.54.2}$$
$$\underline{\mathrm{Sec}}_B(X, E \times_X F) \xrightarrow{\sim} \underline{\mathrm{Sec}}_B(X,E) \times_B \underline{\mathrm{Sec}}(X,F) \tag{2.3.54.3}$$

are both isomorphisms.

2.3.55 Exercise. Show that for any embedding $E \to F$ (over X), the induced map $\underline{\mathrm{Sec}}_B(X,E) \to \underline{\mathrm{Sec}}_B(X,F)$ is also an embedding (compare (2.3.44)). Conclude that the diagonal of $\underline{\mathrm{Sec}}_B(X,E) \to B$ is an embedding (and so, in particular, representable).

2.3.56 Exercise. Let $s : B \to X$ be a section, and let $F \subseteq s^*E := E \times_X B$ be a closed substack. Show that the condition on $\underline{\mathrm{Sec}}_B(X,E)$ of sending s to F is a closed condition.

⋆ **2.3.57 Lemma.** *If $E \to F$ is a closed embedding (over X) and $X \to B$ is open, then $\underline{\mathrm{Sec}}_B(X,E) \to \underline{\mathrm{Sec}}_B(X,F)$ is a closed embedding. In partcular, if $E \to X$ is separated and $X \to B$ is open, then $\underline{\mathrm{Sec}}_B(X,E) \to B$ is separated.*

Proof. We saw earlier that $\underline{\mathrm{Sec}}_B(X,E) \to \underline{\mathrm{Sec}}_B(X,F)$ is an embedding (2.3.55). Fix a map $Z \to \underline{\mathrm{Sec}}_B(X,F)$, namely a diagram (2.3.52.1), and let us show that the locus of $z \in Z$ for which the specialization of the map $X \times_B Z \to F$ lands inside E is closed. The inverse image of $E \subseteq F$ is a closed subset K of $X \times_B Z$. Since the projection $X \times_B Z \to Z$ is open (being a pullback of $X \to B$), the locus of points $z \in Z$ whose inverse image $X \times_B z$ is contained in K is closed (being the complement of the image of the complement of K). □

2.3.58 Exercise. Let X be the locus $\{xy = 0\}$ (the union of the two axes in $\mathbb{R}^2$), and let $X \to B = \mathbb{R}$ be the projection to the x-coordinate. Show that $\underline{\mathrm{Hom}}_B(X,\mathbb{R}) \to B$ is not separated (a map $Z \to \underline{\mathrm{Hom}}_B(X,\mathbb{R})$ is a map $Z \to B$ and a map $X \times_B Z \to \mathbb{R}$).

2.3.59 Exercise. Argue as in (2.3.47) to show that if $E^\circ \subseteq E$ is open and $X \to B$ is universally closed, then $\underline{\mathrm{Sec}}_B(X,E^\circ) \to \underline{\mathrm{Sec}}_B(X,E)$ is an open embedding.

⋆ **2.3.60 Exercise** (Representability of $\underline{\mathrm{Sec}}_B(X,E)$)**.** Consider the 'compact-open' topology on $\mathrm{Sec}_B(X,E)$ (compare (2.3.50)) generated by declaring to be open the set of pairs $(b \in B, u : X_b \to E_b)$ with $b \in V$ and $u(K_b) \subseteq U_b$, for $V \subseteq B$ open, $U \subseteq E$ open, and $K \subseteq X \times_B V \to V$ universally closed. Conclude from (2.3.59) that there is a tautological map

$$\underline{\mathrm{Sec}}_B(X,E) \to \mathrm{Sec}_B(X,E)_{\mathrm{cptopen}}. \tag{2.3.60.1}$$

Now argue as in (2.3.51) to show that this map is an isomorphism provided $X \to B$ is *locally universally closed* in the sense that every point $x \in X$ has arbitrarily small neighborhoods $K \subseteq X$ which (in the subspace topology) are universally closed over some neighborhood of the image of x in B (note that this property is preserved under pullback).

2.3.61 Exercise. Fix maps $W \to A \to C \to B$, and consider the diagram

$$\begin{tikzcd}
& W \arrow[dd] \\
W \times_A (C \times_B \underline{\mathrm{Sec}}_B(C, A)) \arrow[ur] \arrow[dd] & \\
& A \arrow[dd] \\
C \times_B \underline{\mathrm{Sec}}_B(C, A) \arrow[ur] \arrow[dr] \arrow[dd] & \\
& C \arrow[dd] \\
\underline{\mathrm{Sec}}_B(C, A) \arrow[dr] & \\
& B
\end{tikzcd} \tag{2.3.61.1}$$

where the top and bottom squares are pullbacks and the map $C \times_B \underline{\mathrm{Sec}}_B(C, A) \to A$ is the tautological evaluation map. Show that the induced map

$$\underline{\mathrm{Sec}}_{\underline{\mathrm{Sec}}_B(C,A)}(C \times_B \underline{\mathrm{Sec}}_B(C, A), W \times_A (C \times_B \underline{\mathrm{Sec}}_B(C, A))) \to \underline{\mathrm{Sec}}_B(C, W) \tag{2.3.61.2}$$

is an isomorphism (compare universal properties).

Stability

Recall that a topological stack is called separated when its diagonal is proper, and that this is a generalization of the Hausdorff condition to topological stacks. Many topological stacks of interest, for instance the moduli stack of compact nodal Riemann surfaces, are non-separated. Rather, they contain an open substack of 'stable' points, which is instead the object of interest for many purposes. We now introduce a general structure which allows us to pick out this open 'stable locus' and deduce properties of it from properties of the ambient stack.

2.3.62 Exercise (Stable object)**.** Show that for an object X in a category C, the following are equivalent:

(2.3.62.1) Every morphism $Z \to X$ is a terminal object in the under-category $\mathsf{C}_{Z/}$.

(2.3.62.2) Every morphism $A \to B$ induces an isomorphism $\mathrm{Hom}(B, X) \to \mathrm{Hom}(A, X)$.

(2.3.62.3) For every diagram of solid arrows

$$\begin{tikzcd}
A \arrow[r] \arrow[dr] & B \arrow[d, dashed, "\exists!"] \\
& X
\end{tikzcd}$$

there exists a unique dotted arrow making the diagram commute.

We call an object X satisfying these conditions *stable*.

2.3.63 Exercise. Show that every morphism out of a stable object is a split monomorphism. Conclude that every morphism between stable objects is an isomorphism.

2.3.64 Exercise (Category with enough stable objects)**.** Show that for a category C, the following are equivalent:

(2.3.64.1) Every object admits a morphism to a stable object.

(2.3.64.2) Every under-category $\mathsf{C}_{Z/}$ has a terminal object, and for every morphism $Z \to Y$, the induced functor $\mathsf{C}_{Y/} \to \mathsf{C}_{Z/}$ sends terminal objects to terminal objects.

A category C satisfying these conditions is said to have *enough stable objects*.

2.3.65 Exercise (Stabilization)**.** For a category C with enough stable objects, let $i : \mathsf{C}^s \subseteq \mathsf{C}$ denote the full subcategory spanned by the stable objects (so C^s is a groupoid by (2.3.63)). Show that sending $Z \in \mathsf{C}$ to the target of a terminal object in $\mathsf{C}_{Z/}$ defines a functor $\mathrm{st} : \mathsf{C} \to \mathsf{C}^s$ with a natural transformation $\mathbf{1}_{\mathsf{C}} \to i \circ \mathrm{st}$ defining an adjunction (st, i) (hence that $\mathsf{C}^s \subseteq \mathsf{C}$ is a reflective subcategory (1.1.87)).

2.3.66 Exercise (Functor preserving stable objects)**.** Let C and D be categories with enough stable objects. Show that for a functor $F : \mathsf{C} \to \mathsf{D}$, the following are equivalent:

(2.3.66.1) F sends stable objects to stable objects.

(2.3.66.2) The induced functor $\mathsf{C}_{Z/} \to \mathsf{D}_{f(Z)/}$ sends terminal objects to terminal objects for every $Z \in \mathsf{C}$.

A functor F satisfying these conditions is said to *preserve stable objects*.

* *

We next study stable objects in the context of sheaves of categories. By a 'sheaf of categories' we mean a sheaf valued in the 2-category Cat.

2.3.67 Exercise. Show that the 'subcategory of isomorphisms' functor $\simeq: \mathsf{Cat} \to \mathsf{Grpd}$ is right adjoint to the inclusion $\mathsf{Grpd} \subseteq \mathsf{Cat}$, hence is continuous. Conclude that for any sheaf of categories $\vec{X}$, its subcategory of isomorphisms $X = (\vec{X})_{\simeq}$ is a sheaf of groupoids. Conclude that this functor

$$\simeq: \mathsf{Shv}(\mathsf{Top}, \mathsf{Cat}) \to \mathsf{Shv}(\mathsf{Top}, \mathsf{Grpd}) \tag{2.3.67.1}$$

is continuous.

We regard a sheaf of categories $\vec{X}$ as an 'enhancement' of its 'underlying sheaf of groupoids' $X = (\vec{X})_{\simeq}$ to be used to study X itself. A lift of a sheaf of groupoids X to a sheaf of categories $\vec{X}$ will be called *categorical structure* on X.

⋆ **2.3.68 Definition** (Topological stack of categories)**.** By a *topological stack of categories*, we mean a sheaf of categories on the category of topological spaces. Topological stacks of categories form the 2-category $\mathsf{Shv}(\mathsf{Top}, \mathsf{Cat})$.

2.3.69 Example. Vector bundles and linear maps is a topological stack of categories Vect, enhancing the topological stack $\mathsf{Vect}_{\simeq} = \bigsqcup_n */\mathrm{GL}_n\mathbb{R}$ of vector bundles and isomorphisms.

⋆ **2.3.70 Definition** (Stability structure)**.** A sheaf of categories $\vec{X}$ is called *pre-stable* iff it satisfies the following properties:

(2.3.70.1) Every $\vec{X}(Z)$ has enough stable objects.

(2.3.70.2) Every pullback $\vec{X}(Z) \to \vec{X}(Z')$ for $Z' \to Z$ preserves stable objects.

(2.3.70.3) (Isomorphism is an open condition) For every morphism $\alpha \to \beta$ in $\vec{X}(Z)$, there is an open subset $U \subseteq Z$ such that the pullback $i^*(\alpha \to \beta)$ under a map $i : Z' \to Z$ is an isomorphism iff $i(Z') \subseteq U$.

A *stability structure* on a sheaf of groupoids X is a pre-stable enhancement $\vec{X}$ of X.

⋆ **2.3.71 Exercise** (Stable locus)**.** Let $\vec{X}$ be a pre-stable categorical stack. Show that for every $\alpha \in \vec{X}(Z)$, there exists an open set $U \subseteq Z$ such that $i^*\alpha \in \vec{X}(Z')$ is stable iff $i(Z') \subseteq U$ (for any $i : Z' \to Z$). Conclude that the *stable locus* X^s defined by $X^s(Z) = \vec{X}(Z)^s$ is an open substack of X.

2.4 Smooth manifolds

We assume the reader has a foundational understanding of differential topology and smooth manifolds. The purpose of this section is to set notation and terminology and to recall arguments which will be adapted later to more novel settings.

Basic notions

⋆ **2.4.1 Definition** (Category of smooth manifolds Sm)**.** A *smooth manifold* is a pair (X, Φ) consisting of a topological space X and a collection Φ of pairs (U, φ) (called 'charts') where $U \subseteq \mathbb{R}^n$ is an open set and $\varphi : U \hookrightarrow X$ is an open embedding, such that for every pair $(U, \varphi), (U', \varphi') \in \Phi$, the 'transition map' $\varphi^{-1}\varphi' : (\varphi')^{-1}(\varphi(U)) \to U$ is smooth. A morphism of smooth manifolds $(X, \Phi) \to (Y, \Psi)$ is a map $X \to Y$ such that for every $(U, \varphi) \in \Phi$ and $(V, \psi) \in \Psi$, the composition $\psi^{-1} f \varphi : \varphi^{-1}(f^{-1}(\psi(V))) \to V$ is smooth. The category of smooth manifolds is denoted Sm. The underlying topological space of a smooth manifold M is denoted $|M|$.

2.4.2 Warning. The term 'smooth manifold' is usually taken to mean an object of Sm whose underlying topological space is Hausdorff and paracompact (2.1.42), since these are the objects of main interest to differential topology. As our current focus is more categorical and point set topological, it is more convenient to use the term 'smooth manifold' to refer to arbitrary objects of Sm. In later chapters, when we have a more differentiable topological focus, we will (explicitly) revert to the standard meaning of the term 'smooth manifold' (though the symbol Sm will continue to denote the category defined here). For now, it is logically clarifying to only include paracompact and Hausdorff assumptions when they are actually needed.

2.4.3 Definition (Open embedding)**.** A map $X \to Y$ in Sm is called an *open embedding* when it is an open embedding of topological spaces and sends charts $\mathbb{R}^n \supseteq U \hookrightarrow X$ to charts $U \hookrightarrow Y$. The notion of a *local isomorphism* in Sm is then defined as for topological spaces (2.1.10) with respect to this notion of open embedding. Open embeddings and local isomorphisms are preserved under pullback and closed under composition.

2.4.4 Inverse Function Theorem. *A map in* Sm *is a local isomorphism iff its derivative is an isomorphism at every point.*

2.4.5 Definition (Submersion)**.** A map in Sm is called a *submersion* (or *submersive*) when its derivative is surjective at every point (by (2.4.4), this is equivalent to being locally on the source a pullback of $\mathbb{R}^k \to *$). Submersivity is preserved under pullback, closed under composition, and local on the source.

2.4.6 Definition (Immersion)**.** A map in Sm is called an *immersion* (or *immersive*) when its derivative is injective at every point (by (2.4.4), this is equivalent to being locally on the source a submersive pullback of $* \to \mathbb{R}^k$). Immersivity is closed under composition and local on the source.

* *

The category Sm does not have all finite limits, and some finite limits which do exist are 'wrong'.

2.4.7 Example. The zero locus of a smooth function $f : \mathbb{R} \to \mathbb{R}$ is, by definition, the fiber product

$$\begin{tikzcd} f^{-1}(0) \arrow[r] \arrow[d] & * \arrow[d, "0"] \\ \mathbb{R} \arrow[r, "f"] & \mathbb{R} \end{tikzcd} \tag{2.4.7.1}$$

which we may take in either Sm or Top, resulting in two objects $f^{-1}(0)_{\mathsf{Sm}}$ and $f^{-1}(0)_{\mathsf{Top}}$ (which may or may not exist) and a comparison map $|f^{-1}(0)_{\mathsf{Sm}}| \to f^{-1}(0)_{\mathsf{Top}}$ between them.

Let us consider the smooth function $f(x) = x^n$ for a positive integer $n \geq 1$. The smooth zero locus $f^{-1}(0)_{\mathsf{Sm}}$ is a single point, with its unique structure as a smooth manifold. The topological zero locus $f^{-1}(0)_{\mathsf{Top}}$ is also a single point, with its unique topology, and the comparison map $|f^{-1}(0)_{\mathsf{Sm}}| \to f^{-1}(0)_{\mathsf{Top}}$ is an isomorphism.

Let us consider the smooth function $f(x) = e^{-1/x^2}\sin(1/x)$. The smooth zero locus $f^{-1}(0)_{\mathsf{Sm}}$ is representable: it is the zero set of f equipped with the discrete topology (which is a zero-dimensional manifold, an object of Sm). The topological zero locus $f^{-1}(0)_{\mathsf{Top}}$ is also representable, this time by the zero set of f equipped with the subspace topology inside $\mathbb{R}$. The tautological comparison map $|f^{-1}(0)_{\mathsf{Sm}}| \to f^{-1}(0)_{\mathsf{Top}}$ is evidently not an isomorphism. This difference reflects the fact that test objects in Sm cannot see how the zeroes of f converge to zero, while test objects in Top can.

Here is a class of 'good' finite limits.

2.4.8 Definition (Transverse diagram)**.** A pair of maps $M \to N \leftarrow Q$ in Sm is called *transverse* when at every point of the topological fiber product $|M| \times_{|N|} |Q|$, the map $TM \oplus TQ \to TN$ is surjective. In this case, the fiber product $M \times_N Q$ exists in Sm and has dimension $\dim M - \dim N + \dim Q$.

More generally, consider a finite diagram of smooth manifolds $D : J \to \mathsf{Sm}$ with 0-cells $(M_v)_v$, 1-cells $(f_e : M_{v(e)} \to M_{w(e)})_e$, and no 2-cells. Such a diagram is called *transverse* when at every point $p = (p_v)_v$ of its topological limit, the map

$$\bigoplus_v TM_v \xrightarrow{\bigoplus_e [\mathbf{1}_{TM_{w(e)}} - Tf_e]} \bigoplus_e TM_{w(e)} \tag{2.4.8.1}$$

is surjective. In this case, the limit of D exists in Sm and has dimension $\sum_v \dim M_v - \sum_e \dim M_{w(e)}$.

A *transverse limit* is the limit of a transverse diagram. Note that we have only defined transversality for diagrams of dimension ≤ 1. The generalization to diagrams of arbitrary dimension is given in (2.9.8).

2.4.9 Example. The zero locus $f^{-1}(0)$ (2.4.7) is a transverse limit precisely when $f(x) = 0$ implies $f'(x) \neq 0$.

* *

2.4.10 Definition (Tangent functor T)**.** The tangent functor $T : \mathsf{Sm} \to \mathsf{Sm}$ sends M to (the total space of) its tangent bundle TM and sends $f : M \to N$ to its derivative $Tf : TM \to TN$. The zero section $M \to TM$ and the projection $TM \to M$ are natural transformations $\mathbf{1} \Rightarrow T \Rightarrow \mathbf{1}$.

2.4.11 Exercise. Show that T sends vector bundles to vector bundles, in the following sense. A vector bundle is a triple $(V \to M, V \times \mathbb{R} \to V, V \times_M V \to V)$ which is locally (on M) isomorphic to the trivial family of $\mathbb{R}^k$ with its standard vector space structure. Show that applying T to such a triple yields another. Also show that T sends linear maps of vector bundles to the same.

⋆ **2.4.12 Definition** (Lie group)**.** A Lie group is a group object (1.1.128) in Sm.

2.4.13 Lemma. *Every Lie group is Hausdorff and paracompact.*

Proof. The inclusion of a point into any smooth manifold is a closed embedding, and a topological group whose identity is a closed point is Hausdorff (2.1.53). Smooth manifolds are locally compact, and a locally compact Hausdorff topological group is paracompact (2.1.54). □

Paracompactness and partitions of unity

In the context of smooth manifolds, bump functions (2.1.41) and partitions of unity (2.1.44) involve, by default, smooth functions (we will write *continuous* bump function or *continuous* partition of unity to indicate these objects on the underlying topological space of a smooth manifold). Since partitions of unity on paracompact locally compact Hausdorff spaces may be built out of any prescribed collection of 'regular' bump functions (2.1.51), so we just need to construct smooth bump functions.

2.4.14 Lemma. *Every smooth manifold has local bump functions (hence, if paracompact Hausdorff, partitions of unity* (2.1.51)*).*

Proof. The question reduces to the case of $\mathbb{R}^n$, where an explicit construction is immediate from the smooth function $\psi(x)$ below (say take $\psi(1 - \|x\|^2)$).

$$\psi(x) = \begin{cases} \exp(-1/x) & x > 0 \\ 0 & x \leq 0 \end{cases} \tag{2.4.14.1}$$

This function is smooth by explicit differentiation (for $x > 0$, its derivatives are linear combinations of $x^a e^{-1/x}$ for integers a). □

Non-linear averaging

Many results about smooth manifolds rely on averaging of real valued functions or, more generally, sections of a vector bundle. For others, we instead need a notion of average for a collection of nearby points in a smooth manifold. We now explain how to construct this sort of non-linear averaging operation.

⋆ **2.4.15 Definition** (Averaging on a manifold)**.** Let M be a smooth manifold which is paracompact and Hausdorff. We consider positive measures of unit total mass on M; call this set $\operatorname{Meas}(M)$. There is a tautological inclusion $M \hookrightarrow \operatorname{Meas}(M)$ sending a point of M to the delta measure at that point. Our goal is to define a 'smooth' retraction

$$\operatorname{avg} : \operatorname{Meas}(M) \to M \tag{2.4.15.1}$$

(a 'notion of average') on the set of measures of 'small' support (meaning, there will be an open cover $M = \bigcup_i V_i$, and $\operatorname{avg}(\mu)$ will be defined when $\operatorname{supp}\mu$ is contained in some V_i).

Let $U \subseteq M$ be an open set identified with a convex open set $U \subseteq \mathbb{R}^n$. Thus any measure μ supported inside U has an average $\operatorname{avg}_U(\mu) \in U$ defined by the linear structure on $\mathbb{R}^n$. We would like to interpolate between the map avg_U for measures supported deep inside U and the identity map for measures supported away from U. Given a smooth function of compact support $\eta : U \to [0, 1]$, such an interpolation can be given by

$$\operatorname{avg}_{U,\eta}(\mu) = \eta(\mu) \cdot \delta_{\operatorname{avg}_U(\mu)} + (1 - \eta(\mu))\mu, \tag{2.4.15.2}$$

where $\eta(\mu) = \int \eta\, d\mu$. This map $\operatorname{avg}_{U,\eta}$ is well behaved on the set of measures μ which are either supported inside U or supported inside $M \setminus \operatorname{supp}\eta$ (in which case $\operatorname{avg}_{U,\eta}(\mu) = \mu$). When $\operatorname{supp}\mu \subseteq \eta^{-1}(1)$, the average $\operatorname{avg}_{U,\eta}(\mu)$ is the single point $\operatorname{avg}_U(\mu)$.

We now define the averaging map avg as a composition of local averaging maps $\operatorname{avg}_{U,\eta}$. Choose an open cover $M = \bigcup_i U_i$ for open convex $U_i \subseteq \mathbb{R}^n$, and choose smooth functions of compact support $\eta_i : U_i \to [0, 1]$ such that $M = \bigcup_i \eta_i^{-1}(1)^\circ$. Define avg as the ordered composition of all $\operatorname{avg}_{U_i,\eta_i}$ with respect to an arbitrarily chosen total order of the set of indices i. This map $\operatorname{avg} : \operatorname{Meas}(M) \to \operatorname{Meas}(M)$ is defined on measures of sufficiently small support, and sends measures of sufficiently small support to delta measures (hence can be viewed as having target $M \subseteq \operatorname{Meas}(M)$).

In what sense is the map avg smooth? Let us declare a map $N \to \operatorname{Meas}(M)$ from a smooth manifold N to be smooth iff for every smooth function on $N \times M$, its fiberwise integral is a smooth function on N. A map $N \to M$ is thus smooth iff it is smooth as a map to $\operatorname{Meas}(M)$ landing in the subspace of delta measures. Now the map avg is smooth in the sense that composing a smooth map $N \to \operatorname{Meas}(M)$ with it yields a smooth map $N \to \operatorname{Meas}(M)$. Indeed, it suffices to show that each map $\operatorname{avg}_{U,\eta}$ is smooth in this sense, which follows from inspection.

2.4.16 Definition (Averaging on a manifold via vector fields)**.** Here is another construction of an averaging operation on a paracompact Hausdorff smooth manifold M satisfying the same key properties as (2.4.15). This construction is easier to make equivariant.

Choose a section $\xi : M \times M \to TM$ (the tangent bundle pulled back from the first factor) which vanishes on the diagonal $M \subseteq M \times M$ and whose derivative on the diagonal in the direction of the first factor is the identity map of TM (such a section ξ certainly exists locally, and can be patched using a partition of unity since M is paracompact Hausdorff). Given a measure $\mu \in \operatorname{Meas}(M)$, we may consider the average $\xi(\cdot, \mu) = \int_M \xi(\cdot, p)\, \mu(p) : M \to TM$. If $\operatorname{supp}\mu$ is sufficiently small, then $\xi(\cdot, \mu)$ will have a unique zero near $\operatorname{supp}\mu$ and this zero will be transverse. Sending μ to this canonical zero of $\xi(\cdot, \mu)$ defines a averaging map $\operatorname{avg}_\xi : \operatorname{Meas}(M) \to M$ (for measures of sufficiently small support) which is smooth in the same sense as (2.4.15).

If a group G acts smoothly on M and we choose ξ to be G-equivariant, then the resulting averaging operation avg_ξ is G-equivariant. If G is a compact Lie group, then we may produce a G-equivariant section ξ by beginning with any ξ_0 and averaging its G-translates with respect to an invariant measure on G.

The next result is fundamental, and we will meet many generalizations of it. The main ingredient in its proof is the averaging operation (2.4.15) above.

⋆ **2.4.17 Ehresmann's Theorem** ([25, 26]). *A proper submersion in* Sm *is trivial locally on the target.*

Proof. Let $M \to B$ be a proper submersion, and let us show that $M \to B$ is trivial in a neighborhood of a given point $0 \in B$. We are free to shrink B at will (that is, replace B with an open neighborhood of 0 and replace M with its inverse image).

Let M_0 denote the fiber of $M \to B$ over $0 \in B$. We first construct a retraction $M \to M_0$ after possibly shrinking B. We then show that such a retraction gives a local trivialization of $M \to B$ after further shrinking.

We claim that there exists a finite covering of M by open charts $U_i \times B \subseteq M$ for open sets $U_i \subseteq \mathbb{R}^k$. The source-local normal form for submersive maps provides such a chart near any point of M_0. By universal closedness of $M \to B$, there exists a finite collection of such charts which cover the inverse image of an open neighborhood of 0. We can thus shrink B so that they cover all of M.

In a given chart $U_i \times B \subseteq M$ there is an evident retraction to the fiber over $0 \in B$, namely projection to the U_i factor. These need not agree on overlaps. We will patch them together using a partition of unity and the averaging operation (2.4.15) on M_0.

Choose smooth compactly supported functions $\varphi_i : U_i \to \mathbb{R}_{\geq 0}$ which form a partition of unity on M_0. Since $M \to B$ is separated and $(\operatorname{supp}\varphi_i) \times B \to B$ is universally closed, it follows that $(\operatorname{supp}\varphi_i) \times B \to M$ is universally closed (2.1.40). It follows that the extension by zero of $\varphi_i \pi_{U_i}$ from $U_i \times B$ to M is smooth. These functions sum to unity on M_0, but may fail to elsewhere on M. The locus where their sum is > 0 is an open neighborhood of M_0, hence can be assumed to be all of M after shrinking B. Dividing by this sum produces a smooth partition of unity $\sum_i \psi_i \equiv 1$ on M.

Now we consider the map

$$M \to \mathrm{Meas}(M_0) \tag{2.4.17.1}$$

$$m \mapsto \sum_i \psi_i(m)\delta_{\pi_{U_i}(m)} \tag{2.4.17.2}$$

where we note that if $\psi_i(m) > 0$ then m lies inside the chart $U_i \times B \subseteq M$, so $\pi_{U_i}(m) \in M_0$ is defined. Composing this map with the averaging operation (2.4.15) on M_0 produces the desired retraction $M \to M_0$ in a neighborhood of M_0 (which becomes all of M after shrinking B).

Finally, let us argue that the existence of a retraction $M \to M_0$ implies triviality of $M \to B$ near 0. The induced map $M \to M_0 \times B$ over B is a local isomorphism in a neighborhood of M_0 (which by shrinking B is wlog all of M). There is a unique section of $M \to M_0 \times B$ over $M_0 \times 0$, and this section extends to a neighborhood of M_0 by (**??**). Any section of a local isomorphism is an open embedding (2.1.39). Further shrinking B means the image of this open embedding is all of M, thus it is a diffeomorphism. □

Local structure of mapping stacks

2.4.18 Lemma (Local structure of $\underline{\mathrm{Sec}}(M, Q)$)**.** *Let $\pi : Q \to M$ be a submersion. If M is paracompact Hausdorff, then any section $s : M \to Q$ extends to an open embedding $(s^*T_{Q/M}, 0) \to (Q, s)$ over M.*

Proof. We first construct a map $f : (Q, s) \to (s^*T_{Q/M}, 0)$ over M whose vertical derivative is the identity along the base section. For any $p \in M$, the source-local normal form for submersions provides such a map $f_p : (Q, s) \to (s^*T_{Q/M}, 0)$ over an open set $U_p \subseteq Q$ containing $s(p)$. Since M is paracompact Hausdorff, there exists a partition of unity $\sum_p \varphi_p \equiv 1$ (2.4.14)(2.1.51) subordinate to the open cover $M = \bigcup_p s^{-1}(U_p)$. Now the sum $f = \sum_p \varphi_p f_p : (Q, u) \to (u^*T_{Q/M}, 0)$ has the desired property and is defined over the open set $\bigcup_{I \subseteq M}(\bigcap_{p\in I} U_p \setminus \bigcup_{p\notin I} \pi^{-1}(\operatorname{supp}\varphi_p))$ (union over all finite subsets I), which contains the image of s.

Since the vertical derivative of $f : (Q, s) \to (s^*T_{Q/M}, 0)$ along the base section is the identity, it follows that f is a local isomorphism over a neighborhood of $s(M) \subseteq Q$. That is, for every point $p \in M$, there exists an open set $V_p \subseteq Q$ containing $s(p)$ over which f is an open embedding. It follows that f is an open embedding over the open set $\bigcup_{I\subseteq M}(\bigcap_{p\in I} V_p \setminus \bigcup_{p\notin I} \pi^{-1}(\operatorname{supp}\psi_p))$ for any choice of partition of unity $\sum_p \psi_p \equiv 1$ subordinate to the open cover $M = \bigcup_p s^{-1}(V_p)$ (indeed, it is certainly a local isomorphism over this locus, and it is also injective since injectivity can be checked fiberwise over M). Its inverse is thus an open embedding $i : (s^*T_{Q/M}, 0) \to (Q, s)$ defined in a neighborhood of the zero section.

Given an open embedding $i : (s^*T_{Q/M}, 0) \to (Q, s)$ defined in a neighborhood of the zero section, we can obtain a globally defined open embedding by pre-composing with a suitable open embedding of $s^*T_{Q/M}$ into itself (2.1.49). □

⋆ **2.4.19 Corollary** (Local structure of $\underline{\mathrm{Sec}}(M, Q)$)**.** *Let $Q \to M$ be a submersion. If M is compact Hausdorff, then the moduli stack $\underline{\mathrm{Sec}}(M, Q)$ is covered by the open substacks $\underline{\mathrm{Sec}}(M, Q^\circ) \subseteq \underline{\mathrm{Sec}}(M, Q)$ associated to open subsets $Q^\circ \subseteq Q$ for which $Q^\circ \to M$ can be equipped with the structure of a vector bundle.*

Proof. For any open subset $Q^\circ \subseteq Q$, the induced map $\underline{\mathrm{Sec}}(M, Q^\circ) \hookrightarrow \underline{\mathrm{Sec}}(M, Q)$ is an open embedding by (2.3.47) since M is compact. Since M is paracompact Hausdorff, every section $u : M \to Q$ extends to an open embedding $(u^*T_{Q/M}, 0) \to (Q, u)$ over M (2.4.18), so every point of $\underline{\mathrm{Sec}}(M, Q)$ is in the image of $\underline{\mathrm{Sec}}(M, Q^\circ)$ for an open $Q^\circ \subseteq Q$ which is the total space of a vector bundle over M. □

2.4.20 Lemma (Local structure of $\underline{\mathrm{Sec}}_B(M, Q)$)**.** *Let $Q \to M \to B$ be submersions. If $M \to B$ is proper, then for any $b \in B$ and any section $s : M_b \to Q_b$, there is (after replacing B with an open subset containing b) a trivialization $M = M_b \times B$ over B covered by an open embedding $s^*T_{Q/M} \times B \hookrightarrow Q$ identifying the zero section with s.*

$$\begin{tikzcd}[row sep=small]
 & & s^*T_{Q/M}\times B \arrow[rr, hook] \arrow[dd] & & Q \arrow[dd] \\
M_b \arrow[urr, "0"] \arrow[rrr, equal] \arrow[drr] \arrow[dd] & & & M_b \arrow[ur, "s"] \arrow[dr] \arrow[dd] & \\
 & & M_b\times B \arrow[rr, equal] \arrow[dd] & & M \arrow[dd] \\
* \arrow[rrr, equal] \arrow[drr, "b"'] & & & * \arrow[dr, "b"] & \\
 & & B \arrow[rr, equal] & & B
\end{tikzcd} \tag{2.4.20.1}$$

Proof. This is similar to (2.4.18). Since $M \to B$ is proper, Ehresmann (2.4.17) provides a local trivialization $M = M_b \times B$. As in (2.4.18), it suffices to construct a map $(Q, s) \to (s^*T_{Q/M}, 0)$ (over this choice of local trivialization) whose vertical derivative along s is the identity map. Such a map exists locally, hence globally using a partition of unity. □

⋆ **2.4.21 Corollary** (Local structure of $\underline{\mathrm{Sec}}_B(M, Q)$)**.** *Let $Q \to M \to B$ be submersions. If $M \to B$ is proper, then the moduli stack $\underline{\mathrm{Sec}}_B(M, Q)$ is covered by the open substacks $\underline{\mathrm{Sec}}_{B^\circ}(M^\circ, Q^\circ) \subseteq \underline{\mathrm{Sec}}(M, Q)$ associated to open subsets $B^\circ \subseteq B$ (let $M^\circ = M \times_B B^\circ$) and $Q^\circ \subseteq Q \times_B B^\circ$ for which $Q^\circ \to M^\circ \to B^\circ$ is isomorphic to a product $(Q_0 \to M_0 \to *) \times B^\circ$ where $Q_0 \to M_0$ is a vector bundle.*

Proof. This is similar to (2.4.19). Given an open subset $B^\circ \subseteq B$ (let $M^\circ = M \times_B B^\circ$) and an open subset $Q^\circ \subseteq Q \times_B B^\circ$, the induced map $\underline{\mathrm{Sec}}_{B^\circ}(M^\circ, Q^\circ) \to \underline{\mathrm{Sec}}_B(M, Q)$ is an open embedding since $M \to B$ is universally closed (2.3.59). That such open substacks where $Q^\circ \to M^\circ \to B^\circ$ has the form $(Q_0 \to M_0 \to *) \times B^\circ$ for $Q_0 \to M_0$ a vector bundle form a covering is the content of (2.4.20). □

Hadamard Lemma

⋆ **2.4.22 Hadamard's Lemma.** *If $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ vanishes on $0 \times \mathbb{R}^n$, then f has the form $x \cdot g(x, y_1, \ldots, y_n)$ for some smooth function g.*

Proof #1. If $f(0)=0$ then $f(x)=x\int_0^1 f'(xt)\,dt$. □

Proof #2. It suffices to show that $x^{-1}f(x)$ is of class C^k (k times continuously differentiable) for every $k<\infty$ under the assumption that $f(0)=0$. By subtracting off a polynomial from $f(x)$, we may in fact assume that $f(0)=f'(0)=\cdots=f^{(N)}(0)=0$ for some large $N<\infty$. This implies that $f^{(i)}(x)=O(x^{N+1-i})$ near $x=0$ for $0\le i\le N$. Now explicit differentiation shows that the ith derivative of $x^{-1}f(x)$ is $O(x^{N-i})$ near $x=0$ for $0\le i<N$, which implies $x^{-1}f(x)$ is of class C^{N-1}. □

2.4.23 Exercise. Conclude from Hadamard's Lemma (2.4.22) that if $f:\mathbb{R}^k\times\mathbb{R}^n\to\mathbb{R}$ vanishes on $0\times\mathbb{R}^n$ then $f=\sum_{i=1}^k x_ig_i$ for some functions g_i.

2.4.24 Exercise. Let E be a vector bundle over a paracompact Hausdorff smooth manifold M, and let $s:M\to E$ be a smooth section transverse to zero. Show that every function $f:M\to\mathbb{R}$ vanishing over $s^{-1}(0)$ is of the form $\lambda\cdot s$ for some smooth section $\lambda:M\to E^*$ (use (2.4.23) to prove it locally, and then patch together using a partition of unity).

2.4.25 Definition (Deformation to the tangent bundle)**.** We define a functor $\mathbb{P}:\mathsf{Sm}\to\mathsf{Sm}$ which sends a smooth manifold M to (the total space of) a submersion $\mathbb{P}(M)\to\mathbb{R}$ with fiber TM over 0 and fibers $M\times M$ over $\mathbb{R}\setminus 0$.

$$\begin{tikzcd} TM \arrow[r, hook] \arrow[d] & \mathbb{P}(M) \arrow[d] & M\times M\times(\mathbb{R}\setminus 0) \arrow[l, hook'] \arrow[d] \\ 0 \arrow[r, hook] & \mathbb{R} & \mathbb{R}\setminus 0 \arrow[l, hook'] \end{tikzcd} \tag{2.4.25.1}$$

This structure is functorial in the expected way: for a smooth map $f:M\to N$, the induced map $\mathbb{P}(f):\mathbb{P}(M)\to\mathbb{P}(N)$ is the product $f\times f\times\mathbf{1}$ over $\mathbb{R}\setminus 0$ and is the derivative Tf over 0. There is a functorial involution of $\mathbb{P}$ which swaps the two factors $M\times M$ over $\mathbb{R}\setminus 0$ and acts as negation on the fiber TM over 0.

Local coordinates for the functor $\mathbb{P}$ may be defined as follows. Fix any 'exponential' map $A:TM\to M$, meaning its vertical derivative along the zero section is the identity and its restriction to each fiber is an open embedding. Such an exponential map determines an open embedding $TM\times\mathbb{R}\hookrightarrow\mathbb{P}(M)$ in which $TM\times(\mathbb{R}\setminus 0)$ is glued to $M\times M\times(\mathbb{R}\setminus 0)$ via the map $(p,v,t)\mapsto(p,A_p(tv),t)$. To show that this recipe determines the desired functor $\mathbb{P}$, it suffices to show that for any other exponential map $B:TN\to N$ and any smooth map $f:M\to N$, the induced map $TM\times(\mathbb{R}\setminus 0)\to TN\times(\mathbb{R}\setminus 0)$ defined by conjugating $f\times f\times\mathbf{1}:M\times M\times\mathbb{R}\to N\times N\times\mathbb{R}$ by the relevant exponential maps extends smootly to $TM\times\mathbb{R}\to TN\times\mathbb{R}$.

$$\begin{tikzcd} TM\times\mathbb{R} \arrow[d, dashed] & TM\times(\mathbb{R}\setminus 0) \arrow[l, hook'] \arrow[r, hook] \arrow[d] & M\times M\times(\mathbb{R}\setminus 0) \arrow[d, "f\times f\times\mathbf{1}"] \\ TN\times\mathbb{R} & TN\times(\mathbb{R}\setminus 0) \arrow[l, hook'] \arrow[r, hook] & N\times N\times(\mathbb{R}\setminus 0) \end{tikzcd} \tag{2.4.25.2}$$

Concretely, this amounts to showing that the map $(p, v, t) \mapsto t^{-1}B^{-1}_{f(p)}(f(A_p(tv)))$ extends smoothly to $t = 0$. This follows from Hadamard's Lemma (2.4.22) since the map $B^{-1}_{f(p)}(f(A_p(tv)))$ is smooth and vanishes at $t = 0$. Existence of the claimed involution of $\mathbb{P}$ amounts to smoothness at $t = 0$ of the map $(p, v, t) \mapsto t^{-1}A^{-1}_{A_p(tv)}(p)$, which holds for the same reason.

Manifolds with boundary

The above discussion generalizes readily to the setting of manifolds-with-corners.

2.5 Smooth stacks

In (2.3), we studied stacks on the category of topological spaces. We now turn to stacks on the category of smooth manifolds, which we will call *smooth stacks*. As before, Yoneda gives a full faithul embedding $\mathsf{Sm} \subseteq \mathsf{Shv}(\mathsf{Sm})$, and it is helpful to regard smooth stacks as 'generalized smooth manifolds'. We will be particularly interested in the class of smooth stacks which admit a submersive atlas. The theory of such stacks is essentially equivalent to the theory of 'Lie groupoids' introduced by Ehresmann [27] and studied by many others since then. References include Heinloth [40].

The category of smooth manifolds Sm is a 'topological site' in the sense of (**??**). We can thus formulate the descent property and define stacks on Sm as in (2.3). Stacks $\mathsf{Shv}(\mathsf{Sm}) \subseteq \mathsf{P}(\mathsf{Sm})$ form a reflective subcategory, and the Yoneda embedding $\mathsf{Sm} \hookrightarrow \mathsf{Shv}(\mathsf{Sm})$ is fully faithful. If $\mathsf{Sm}^- \subseteq \mathsf{Sm}$ denotes the category of open subsets of Euclidean space and smooth maps between them (a full subcategory), then the restriction functor $\mathsf{Shv}(\mathsf{Sm}) \to \mathsf{Shv}(\mathsf{Sm}^-)$ is an equivalence (**??**). Since Sm^- is essentially small, there are fewer set-theoretic complications in comparison to the case of Top discussed in (**??**).

The ∞-category $\mathsf{Shv}(\mathsf{Sm})$ is complete, and the embedding $\mathsf{Sm} \hookrightarrow \mathsf{Shv}(\mathsf{Sm})$ preserves all limits which exist in Sm. The fact that the category Sm is not complete leads to some technical differences in comparison with the discussion of topological stacks in (2.3). Not all fiber products in Sm exist, so the class of all morphisms in Sm is not preserved under pullback, so we cannot define a notion of representability for general morphisms in $\mathsf{Shv}(\mathsf{Sm})$. Properties of morphisms in Sm which are preserved under pullback include submersions, local isomorphisms, and open embeddings; these notions extend to morphisms in $\mathsf{Shv}(\mathsf{Sm})$ in the usual way by pulling back to objects of Sm. The forgetful functor $\mathsf{Sm} \to \mathsf{Top}$ sends pullbacks of submersions to pullback squares in Top, so the intersection of submersion with any property of morphisms in Top preserved under pullback is a property of morphisms in Sm preserved under pullback (e.g. separated submersion, proper submersion, etc.).

Foundations

2.5.1 Exercise. Show that a submersion of smooth stacks $X \to Y$ factors uniquely as a surjective submersion $X \to V$ followed by an open embedding $V \to Y$. We call the open substack $V \subseteq Y$ the *image* of the submersion $X \to Y$.

2.5.2 Exercise (Relative tangent bundle of a submersion of smooth stacks)**.** Let $X \to Y$ be a submersion of smooth stacks, and let us define a vector bundle $T_{X/Y}$ over X. For any $Z \in \mathsf{Sm}$ with a map $Z \to X$, we consider the pullback $X \times_Y Z \to Z$ with its canonical section. We declare the pullback of $T_{X/Y}$ to Z to be the pullback of $T_{X \times_Y Z/Z}$ under the canonical section $Z \to X \times_Y Z$. Show that this assignment of a vector bundle over Z to every map $Z \to X$ is compatible with pullback, hence defines a vector bundle over X. Show that the relative tangent bundle is functorial, in the sense that for submersions $X \to Y$ and $X' \to Y'$, a commutative square $(X \to Y) \to (X' \to Y')$ induces a map from $T_{X/Y}$ to the pullback of $T_{X'/Y'}$ to X. Conclude that for a composition of submersions $X \to Y \to Z$, there are induced maps $T_{X/Y} \to T_{X/Z} \to T_{Y/Z}$ (the latter pulled back to X); moreover, show that this sequence is exact.

2.5.3 Exercise. Show that a submersion of smooth stacks $X \to Y$ is a local isomorphism iff $T_{X/Y} = 0$.

* *

An atlas on a smooth stack is defined as for topological stacks (2.3.23). We will be particularly interested in submersive atlases, which play the role of representable atlases from the theory of topological stacks.

2.5.4 Exercise. Show that $|\cdot|_! : \mathsf{Shv}(\mathsf{Sm}) \to \mathsf{Shv}(\mathsf{Top})$ preserves morphisms admitting local sections (use the fact that every morphism from a topological space Z to $|Y|_!$ locally factors through $|Z' \to Y'|_!$ for some smooth manifold Z').

⋆ **2.5.5 Definition** (Smooth (Lie) orbifold)**.** A *smooth orbifold* (resp. *smooth Lie orbifold*) is a smooth stack X such that for every $p \in X$ there exists a germ of open embedding $(V/G, 0) \to (X, p)$ where V/G is the quotient of a finite-dimensional real vector space V by a linear action of a finite (resp. compact Lie) group G.

2.5.6 Warning. As with smooth manifolds (2.4.2), orbifolds and Lie orbifolds are usually required to be separated (equivalently, have Hausdorff coarse space (2.3.26)(**??**)(2.3.22)) and have paracompact coarse space.

2.5.7 Lemma. *The quotient of a smooth manifold by a proper action of a finite group is a smooth orbifold.*

Proof. Properness of the action $G \curvearrowright M$ is equivalent to Hausdorffness of M (2.3.30).

Let $p \in M$ be arbitrary, and let us show that the stack quotient M/G is of the desired form (2.5.5) near p. The orbit $Gp \subseteq M$ is (as an abstract set) isomorphic to G/H for $H \subseteq G$ the stabilizer of p, and $Gp \subseteq M$ is discrete since M is Hausdorff.

Choose a germ (near Gp) of a retraction $r : M \to Gp$, and choose a germ of a section $s : M \to r^*TM$ vanishing over Gp whose derivative over Gp is the identity. By averaging, we may ensure that the section s is G-equivariant. By the inverse function theorem (2.4.4), the section s identifies a neighborhood of $Gp \subseteq M$ with a neighborhood of the zero section of $TM|_{Gp}$ (which is, in turn, identified with the total space (2.1.49); note that (2.1.49) works equivariantly by averaging the metric to make it G-invariant).

The quotient M/G near p is thus identified with the quotient $(TM|_{Gp})/G$. This quotient coincides with T_pM/H for $H \subseteq G$ the stabilizer of p (2.3.31), which is of the desired form. □

2.5.8 Lemma. *The quotient of a smooth manifold by a proper action of a compact Lie group is a smooth Lie orbifold.*

Proof. We follow the argument given just above in the case G is discrete (2.5.7).

Since the orbit $Gp = G/H$ may be positive-dimensional, there is no longer a unique (hence G-equivariant) germ of retraction $r : M \to Gp$. Rather, to construct such an equivariant retraction, we must begin with an arbitrary retraction (**??**) and make it G-equivariant by averaging with respect to an invariant measure on G using the equivariant averaging operation on manifolds (2.4.16).

We now choose a germ (near Gp) of section $s : M \to r^*(TM/TGp)$ vanishing along Gp whose derivative along Gp is the canonical map $TM \to TM/TGp$. By averaging, we may ensure that s is G-equivariant, so it provides a G-equivariant identification between a neighborhood of $Gp \subseteq M$ and the total space $(TM|_{Gp})/TGp$.

The quotient M/G near p is thus identified with $((TM|_{Gp})/TGp)/G = (T_pM/T_pGp)/H$, which is of the desired form. ☐

* *

2.5.9 Exercise (Tangent cohomology of a smooth stack with submersive atlas)**.** Let X be a smooth stack with submersive atlas. For any submersion $u : U \to X$, consider the two-term complex

$$u^*TX = [\overset{-1}{T_{U/X}} \to \overset{0}{TU}] \tag{2.5.9.1}$$

of vector bundles on U. We will eventually identify this two-term complex with the pullback of a two-term complex of vector bundles on X denoted TX, but for now the notation u^*TX is purely motivational.

Show that for any pair of submersions $V, U \to X$ with a map $V \to U$ over X, the induced map from v^*TX to the pullback of u^*TX to V is a quasi-isomorphism (pull the situation back to a submersive atlas $W \twoheadrightarrow X$). Conclude that the fiberwise cohomology of these two-term complexes u^*TX descends to X, in the sense that for every $p \in X$ there are well-defined vector spaces $T_p^{-1}X$ and T_p^0X (that is, functors $T^iX : \operatorname{Hom}(*, X) \to \mathsf{Vect}_{\mathbb{R}}^{\mathsf{fin}}$ for $i = -1, 0$) together with, for every submersion $U \to X$ and every point $p \in U$, an exact sequence

$$0 \to T_p^{-1}X \to (T_{U/X})_p \to T_pU \to T_p^0X \to 0 \tag{2.5.9.2}$$

compatible with maps of submersions over X. In (2.5.31) below, we will refine this discussion to construct a two-term complex of vector bundles TX on X with cohomology T^iX.

2.5.10 Example. If X is a smooth manifold, then $T^{-1}X = 0$ and $T_p^0X = T_pX$ is the fiber at p of the tangent bundle of X in the usual sense.

2.5.11 Exercise. Show that for a Lie group G, we have $T^0\mathbb{B}G = 0$ and $T^{-1}\mathbb{B}G = \mathfrak{g}$ is the Lie algebra of G equipped with the conjugation action.

2.5.12 Exercise. Show that for a Lie group G acting on a smooth manifold M, there is an exact sequence

$$0 \to T_p^{-1}(M/G) \to \mathfrak{g} \to T_pM \to T_p^0(M/G) \to 0 \tag{2.5.12.1}$$

at points $p \in M$.

2.5.13 Exercise. Show that for a smooth stack X with submersive atlas, the condition that $T_x^{-1}X = 0$ is open in x.

2.5.14 Corollary. *For any submersive atlas $U \twoheadrightarrow X$ and any $x \in X$, the map $x \times_X U \to U$ is, locally on the source, a submersion onto a submanifold of codimension $\dim T_x^0 X$ with fibers of dimension $\dim T_x^{-1} X$.*

Proof. The derivative of $x \times_X U \to U$ is $T_{U/X} \to TU$, whose kernel and cokernel have constant rank $\dim T_x^{-1} X$ and $\dim T_x^0 X$, respectively. ☐

2.5.15 Corollary. *The automorphism stack $\underline{\mathrm{Aut}}(x)$ of a point x of a smooth stack X with submersive atlas is a Lie group with Lie algebra $T_x^{-1} X$.*

Proof. The automorphism stack $\underline{\mathrm{Aut}}(x)$ is always a group object (**??**), so to show it is a Lie group, it suffices to show it is representable. Choose an atlas $U \twoheadrightarrow X$ and a lift of x to a point $u \in U$. Then $\underline{\mathrm{Aut}}(x) = x \times_X x = u \times_X u$ is the fiber of $u \times_X U \to U$ over u. By (2.5.14), this fiber is a smooth manifold whose tangent space is the kernel of $T_{U/X} \to TU$, namely $T_x^{-1} X$. ☐

2.5.16 Definition (Minimal submersion)**.** Given a smooth stack X, a submersion $U \to X$ from a smooth manifold U is called *minimal at $u \in U$* when the map $T_{U/X} \to TU$ vanishes at u (compare (2.5.9.2)).

2.5.17 Lemma (Existence of a minimal atlas)**.** *For every point x of a smooth stack X with submersive atlas, there exists an atlas $U \twoheadrightarrow X$ which is minimal at some lift $u \in U$ of x.*

Proof. Suppose $U \to X$ is a submersion and $V \to U$ is a map of smooth manifolds. We claim that $V \to X$ is a submersion iff $TV \oplus T_{U/X} \to TU$ is surjective. Submersivity of $V \to X$ can be checked after pulling back to an atlas $W \twoheadrightarrow X$, and such pullback also preserves the surjectivity condition in question. We are thus reduced to the situation that X is itself a smooth manifold, in which case the equivalence is immediate.

With this fact in hand, we can now conclude. Begin with an arbitrary atlas $U \twoheadrightarrow X$ and a lift $u \in U$ of x. Let $V \subseteq U$ be a locally closed submanifold passing through u chosen so that $TV \subseteq TU$ is a complement of the image of $T_{U/X} \to TU$ at u (and so that $TV \oplus T_{U/X} \to TU$ is everywhere surjective). Thus $V \to X$ is a submersion at u, and it remains to show that it is minimal at u.

The map $[T_{V/X} \to TV] \to [T_{U/X} \to TU]$ is a quasi-isomorphism of two-term complexes (both calculate TX (2.5.9)). Together with the fact that $TV \subseteq TU$ is a complement of the image of $T_{U/X} \to TU$ at u, this implies that the map $T_{V/X} \to TV$ vanishes at u, as desired. ☐

2.5.18 Lemma (Proper atlas from proper diagonal)**.** *Let X be a smooth stack with proper diagonal, and let $U \to X$ be a submersion which is minimal at $p \in U$. For every sufficiently small open neighborhood $p \in V \subseteq U$, we have $p \times_X p = p \times_X V$ and the map $V \to X$ is proper over an open substack of X containing the image of p.*

Proof. Given the purely topological result (2.3.32), it suffices to show that $p \times_X p \subseteq p \times_X U$ is open, which is equivalent to minimality of $U \to X$ at p (2.5.14). ☐

Zung's Theorem

We now come to the fundamental 'local linearization' result for smooth stacks with submersive atlas and proper diagonal. It was conjectured by Weinstein [107, 108] and proved by Zung [112] (see also Crainic–Struchiner [17] and Hoyo–Fernandes [18]). An analogous result in algebraic geometry was proven later by Alper–Hall–Rydh [8].

⋆ **2.5.19 Theorem** (Zung [112])**.** *A smooth stack with submersive atlas and proper diagonal is a Lie orbifold* (2.5.5)*.*

Proof. Let X be a smooth stack with submersive atlas and proper diagonal. Let $x \in X$, and let $G = x \times_X x$ be its automorphism Lie group (2.5.15). Since X has proper diagonal, G is compact.

Fix a submersion $U \to X$ from a smooth manifold U which is minimal at a lift $u \in U$ of x. By replacing U with an open neighborhood of u, we can ensure that $u \times_X U \to U$ has image $\{u\}$ and that $U \to X$ is proper over an open substack of X containing x (2.5.18).

We have constructed a proper submersion $U \to X$ from a smooth manifold U over a neighborhood of x. It suffices to equip it with the structure of a principal G-bundle. Indeed, this implies that $X = U/G$, which is a Lie orbifold (2.5.8).

A 'pseudo-principal G-bundle' structure on $U \to X$ is simply a map $\phi : U \times_X U \to G$ (2.5.20). A principal G-bundle structure on $U \to X$ is the same as a pseudo-principal G-bundle structure for which ϕ is a groupoid homomorphism (meaning the two maps $U \times_X U \times_X U \to G$ given by $(x, y, z) \mapsto \phi(x, y)\phi(y, z)$ and $\phi(x, z)$ coincide) and the restriction of ϕ to $u \times_X U \to G$ is a diffeomorphism for every $u \in U$ (2.5.21). We will first construct a pseudo-principal G-bundle structure on $U \to X$ and then correct it to a principal G-bundle structure.

Since $U \to X$ and X are separated, it follows that U is separated (Hausdorff). The map $U \times_X U \to U$ is separated (pullback of $U \to X$), so $U \times_X U$ is also Hausdorff. Now $G = u \times_X u = u \times_X U \subseteq U \times_X U$ is a smooth submanifold (it is a fiber of the submersion $U \times_X U \to U$). There is thus a retraction $U \times_X U \to G$ defined in a neighborhood of $G = u \times_X u = u \times_X U$ (**??**). The complement of this neighborhood is a closed subset of $U \times_X U$, hence has closed image in X by properness of $U \to X$. This image does not contain x, since the inverse image of x is the image of $u \times_X U \to U$, which is just u. Thus after replacing X with an open substack containing x, we conclude that $U \to X$ is a pseudo-principal G-bundle.

By construction, this pseudo-principal G-bundle structure on $U \to X$ satisfies the condition for being a principal G-bundle over $x \in X$. It would thus suffice to functorially 'correct' pseudo-principal G-bundle structures to principal G-bundle structures, at least over an open subset containing the locus where they are already principal. Such a functorial correction is defined in (2.5.23) below, depending on an additional piece of data, namely that of a smooth positive fiberwise density on $U \to X$, namely a smooth positive section of $|\det T^*_{U/X}|$ (where $\det : \mathrm{GL}_n(\mathbb{R}) \to \mathrm{GL}_1(\mathbb{R})$ and $|\cdot| : \mathrm{GL}_1(\mathbb{R}) = \mathbb{R}^\times \to \mathbb{R}_{>0}$ are group homomorphisms applied to principal bundles) over U; simply choose one arbitrarily. □

2.5.20 Definition (Pseudo-principal G-bundle)**.** Let G be a compact Lie group. A *pseudo-principal G-bundle* is a proper submersion $P \to X$ together with a smooth map $\phi : P \times_X P \to G$. We denote the stack of pseudo-principal G-bundles by $\mathbb{P}G$.

2.5.21 Exercise. Every principal G-bundle is a pseudo-principal G-bundle: the map ϕ is defined by the property $\phi(x,y)y = x$; this defines a map of smooth stacks $\mathbb{B}G \to \mathbb{P}G$. Show that $\operatorname{Hom}(Z, \mathbb{B}G) \to \operatorname{Hom}(Z, \mathbb{P}G)$ is fully faithful. Show that a psuedo-principal G-bundle $P \to X$ and $\phi : P \times_X P \to G$ comes from a principal G-bundle iff ϕ is a groupoid homomorphism (meaning $\phi(x,y)\phi(y,z) = \phi(x,z)$ for $(x,y,z) \in P \times_X P \times_X P$) and its restriction to $p \times_X P \to G$ is a diffeomorphism for every $p \in P$.

2.5.22 Definition (Measured submersion)**.** A submersion $Q \to B$ equipped with a smooth fiberwise density will be called a *measured submersion*. We denote the stacks of measured principal G-bundles and measured pseudo-principal G-bundles by $\widetilde{\mathbb{B}}G$ and $\widetilde{\mathbb{P}}G$, respectively.

2.5.23 Proposition (Zung [112])**.** *Let G be a compact Lie group. The map $\widetilde{\mathbb{B}}G \to \widetilde{\mathbb{P}}G$ lands inside an open substack $\widetilde{\mathbb{P}}^\circ G \subseteq \widetilde{\mathbb{P}}G$ which has a retraction $\widetilde{\mathbb{P}}^\circ G \to \widetilde{\mathbb{B}}G$ over the stack of measured proper submersions.*

$$\begin{tikzcd} & \widetilde{\mathbb{P}}^\circ G \arrow[dl, dashed, bend right] \arrow[d, dashed, hook] \\ \widetilde{\mathbb{B}}G \arrow[r] \arrow[ur, dashed] & \widetilde{\mathbb{P}}G \end{tikzcd} \tag{2.5.23.1}$$

Proof. Let F be a compact Hausdorff smooth manifold equipped with a positive smooth density μ. Recall that a function $\phi : F \times F \to G$ is called a groupoid homomorphism when $\phi(x,y)\phi(y,z) = \phi(x,z)$ (2.5.21). We will define, for an open locus of $(\phi, \mu) \in C^\infty(F \times F, G) \times C^\infty(F, \Omega_F^{>0})$, a groupoid homomorphism $R(\phi, \mu) : F \times F \to G$ which we call the 'rectification' of ϕ with respect to μ, so that if ϕ is a groupoid homomorphism then $R(\phi, \mu)$ is defined and equals ϕ. Applying this operation fiberwise to a measured pseudo-principal G-bundle $(Q \to B, \phi, \mu)$ produces a measured principal G-bundle $(Q^\circ \to B^\circ, R(\phi, \mu), \mu)$, where $Q^\circ = Q \times_B B^\circ$ and $B^\circ \subseteq B$ is the open subset where $R(\phi, \mu)$ is defined. This defines the desired open substack $\widetilde{\mathbb{P}}^\circ G \subseteq \widetilde{\mathbb{P}}G$ with retraction $\widetilde{\mathbb{P}}^\circ G \to \widetilde{\mathbb{B}}G$. Our goal is thus to define the rectification operation $R(\phi, \mu)$ with the aforementioned properties.

Let us begin with some general definitions and estimates. For a function $f : F \to G$ taking values in a small neighborhood of the identity, we define its expectation

$$\mathbb{E}_x[f(x)] = \exp(\mathbb{E}_x[\log f(x)]) \tag{2.5.23.2}$$

with respect to μ using the exponential map $\exp : \mathfrak{g} = T_{\mathbf{1}}G \to G$ with inverse (near the identity) denoted log. Expectation is thus conjugation invariant: $\mathbb{E}_x[af(x)a^{-1}] = a\mathbb{E}_x[f(x)]a^{-1}$. When G is non-abelian, we do not have $\mathbb{E}_x[f(x)g(x)] = \mathbb{E}_x f(x) \mathbb{E}_x g(x)$, rather we have an estimate

$$|\mathbb{E}_x[f(x)g(x)] - \mathbb{E}_x f(x) \mathbb{E}_x g(x)| \leq \mathrm{const} \cdot \sup |f| \cdot \sup |g|, \tag{2.5.23.3}$$

where $|a|$ for $a \in G$ means $|\log a|$ for some fixed conjugation invariant norm $|\cdot| : \mathfrak{g} \to \mathbb{R}_{\geq 0}$ (which exists since G is compact). To prove this estimate, it suffices to bound both the quantities

$$|\exp \mathbb{E}_x \log f(x)g(x) - \exp \mathbb{E}_x[\log f(x) + \log g(x)]| \tag{2.5.23.4}$$
$$|\exp(\mathbb{E}_x \log f(x) + \mathbb{E}_x \log g(x)) - (\exp \mathbb{E}_x \log f(x))(\exp \mathbb{E}_x \log g(x))| \tag{2.5.23.5}$$

by $\mathrm{const} \cdot \sup |f| \cdot \sup |g|$, and these bounds follow from the estimates

$$|\log(XY) - (\log X + \log Y)| \leq \mathrm{const} \cdot |X||Y|, \tag{2.5.23.6}$$
$$|\exp(X + Y) - \exp X \exp Y| \leq \mathrm{const} \cdot |X||Y|, \tag{2.5.23.7}$$

respectively. As a special case of (2.5.23.3), we also have the estimate

$$|\mathbb{E}_x(a \cdot f(x)) - a \cdot \mathbb{E}_x f(x)| \leq \mathrm{const} \cdot |a| \cdot \sup |f| \tag{2.5.23.8}$$

(and the same with a on the right).

To measure how close a given map $\phi : F \times F \to G$ is to being a groupoid homomorphism, we consider the 'error' function $E(\phi) : F \times F \times F \to G$ given by

$$E(\phi)(a, b, c) = \phi(a, b)\phi(b, c)\phi(a, c)^{-1}. \tag{2.5.23.9}$$

Now the rectification $R(\phi, \mu)$ will be defined by iterating the 'averaging' operation $\phi \mapsto \hat{\phi}$ given by

$$\hat{\phi}(a, b) = \phi(a, b)\mathbb{E}_x[\phi(a, b)^{-1}\phi(a, x)\phi(b, x)^{-1}], \tag{2.5.23.10}$$

whose domain is, by definition, those ϕ with $\sup |E(\phi)| < \varepsilon$, for some fixed small $\varepsilon > 0$ (note that this is an open condition on ϕ since F is compact (2.3.47)). The argument of the expectation in (2.5.23.10) may be written as $\phi(a, b)^{-1}E(\phi)(a, b, x)^{-1}\phi(a, b)$, so the condition $\sup |E(\phi)| < \varepsilon$ implies that this expectation is defined (provided our fixed $\varepsilon > 0$ is chosen to be sufficiently small). Moreover this expression shows that

$$\sup |\hat{\phi} - \phi| \leq \mathrm{const} \cdot \sup |E(\phi)|. \tag{2.5.23.11}$$

The key to showing favorable asymptotic behavior of the iteration $\phi \mapsto \hat{\phi}$ is to show that the error $E(\phi)$ is rapidly decreasing.

Let us bound $E(\hat{\phi})$ in terms of $E(\phi)$ following [112, Lemma 2.12]. The product $\hat{\phi}(a, b)\hat{\phi}(b, c)$ is given by

$$\phi(a, b)\mathbb{E}_x[\phi(a, b)^{-1}\phi(a, x)\phi(b, x)^{-1}]\phi(b, c)\mathbb{E}_x[\phi(b, c)^{-1}\phi(b, x)\phi(c, x)^{-1}] \tag{2.5.23.12}$$

whereas $\hat{\phi}(a, c) = \phi(a, c)\mathbb{E}_x[\phi(a, c)^{-1}\phi(a, x)\phi(c, x)^{-1}]$. To estimate the difference between these two expressions, we appeal to the approximate homomorphism property of expectation (2.5.23.3)(2.5.23.8). The first expression $\hat{\phi}(a, b)\hat{\phi}(b, c)$ can be written, using conjugation invariance of expectation, as

$$\begin{aligned} &\phi(a, b)\phi(b, c)\mathbb{E}_x[(\phi(a, b)\phi(b, c))^{-1}\phi(a, x)\phi(b, x)^{-1}\phi(b, c)] \\ &\qquad\qquad \times \mathbb{E}_x[\phi(b, c)^{-1}\phi(b, x)\phi(c, x)^{-1}]. \end{aligned} \tag{2.5.23.13}$$

We can now apply (2.5.23.3) to conclude that this expression differs by at most a constant times $(\sup|E(\phi)|)^2$ from

$$\phi(a,b)\phi(b,c)\mathbb{E}_x[(\phi(a,b)\phi(b,c))^{-1}\phi(a,x)\phi(c,x)^{-1}]. \tag{2.5.23.14}$$

This expression is in turn related to $\hat{\phi}(a,c)$ by substituting $\phi(a,c)$ for $\phi(a,b)\phi(b,c)$ (in both places at once!), which by (2.5.23.8) again incurs an error of at most a constant times $(\sup|E(\phi)|)^2$. We have thus shown the 'quadratic decay estimate'

$$\sup|E(\hat{\phi})| \leq \text{const}\cdot(\sup|E(\phi)|)^2. \tag{2.5.23.15}$$

This estimate implies that once $\sup|E(\phi)|$ is sufficiently small, it then decreases super-exponentially as we iterate the operation $\phi \mapsto \hat{\phi}$. This decay implies that the iteration converges uniformly by (2.5.23.11).

We now define the rectification R. A pair (ϕ,μ) is in the domain of R when the iteration

$$\phi_0 = \phi, \tag{2.5.23.16}$$

$$\phi_i = (\phi_{i-1})^\wedge \quad \text{for } i > 0, \tag{2.5.23.17}$$

is defined for all $i \geq 0$ and the error decays to zero

$$\sup|E(\phi_i)| \xrightarrow{i\to\infty} 0. \tag{2.5.23.18}$$

The quadratic decay estimate (2.5.23.15) implies that this is an open condition in (ϕ,μ). Combining the quadratic decay estimate with the fact that the error controls the increments of the iteration (2.5.23.11), we see that the error decay property (2.5.23.18) also implies uniform convergence of ϕ_i as $i \to \infty$. We may thus define

$$R(\phi,\mu) = \lim_{i\to\infty}\phi_i. \tag{2.5.23.19}$$

Since $\phi_i \to R(\phi,\mu)$ uniformly, the error decay property (2.5.23.18) implies that $E(R(\phi,\mu)) = 0$, which means $R(\phi,\mu)$ is a groupoid homomorphism. It is evident that $R(\phi,\mu) = \phi$ whenever ϕ is a groupoid homomorphism.

What we have shown so far is that for smooth $\phi : B\times F\times F \to G$ and $\mu : B\times F \to \Omega_F^{>0}$ (for any smooth manifold B), the rectification $R(\phi,\mu)$ is continuous on its domain of definition, which is $B^\circ \times F\times F$ for some open set $B^\circ \subseteq B$.

It remains to show that $R(\phi,\mu)$ is in fact smooth. We will show that $\lim_{i\to\infty}\phi_i$ converges in the smooth topology (of local uniform convergence of all derivatives) on the total space $B^\circ\times F\times F$. We will proceed slightly differently from Zung [112, Lemma 2.13].

First, we need to slightly generalize the basic setup. Rather than assuming that F is compact, we instead fix a compact submanifold $F_0 \subseteq F$. The function ϕ remains defined on $F\times F$, but the measure μ now lives on F_0, and the averaging (2.5.23.2) takes place over F_0. The quadratic decay estimate (2.5.23.15) holds by the same argument. Now we declare a pair (ϕ,μ) to be in the domain of R when there exists a neighborhood of F_0 inside F over which the iteration ϕ_i is defined for all i and the error decays to zero (in other words, we regard F as a germ near F_0). The domain of R is open for the same reason as before. That is, for smooth $\phi : B \times F \times F \to G$ and $\mu : B \times F_0 \to \Omega^{>0}_{F_0}$, the subset $B^\circ \subseteq B$ where $R(\phi, \mu)$ is defined is open, and $R(\phi, \mu)$ is a continuous function on an (unspecified) open subset of $B^\circ \times F \times F$ containing $B^\circ \times F_0 \times F_0$.

We now return to the question of smooth convergence of $R(\phi, \mu) = \lim_{i\to\infty} \phi_i$, now in the above generalized setup. We claim that for every $k \geq 0$, the limit $R = \lim_i \phi_i$ converges in C^k over the open set where it converges in C^0. The case $k = 0$ is vacuous, and for $k \geq 1$ we will use induction via the tangent functor T (2.4.10). Given a pair $\phi : B \times F \times F \to G$ and $\mu : B \times F_0 \to \Omega^{>0}_{F_0}$, we may obtain a pair $T\phi : TB \times TF \times TF \to TG$ and $\mu : TB \times F_0 \to \Omega^{>0}_{F_0}$ by applying the tangent functor T to ϕ and pulling back μ under the projection $TB \to B$ (this operation $(\phi, \mu) \mapsto (T\phi, \mu)$ is what compels the generalization in the previous paragraph). Note that TG is itself a Lie group (the functor T sends group objects to group objects since it preserves finite products). Now the key point is that applying T commutes with the averaging operation, in the sense that $T\hat{\phi} = (T\phi)^\wedge$. Indeed, this holds by functoriality of T and the fact that $\exp_{TG} = T \exp_G$. Thus applying the claim at a given k to the iteration $T\phi_i = (T\phi)_i$ implies the claim at $k+1$ for the iteration ϕ_i, so the claim holds for all $k \geq 0$ by induction. □

Artin morphisms

Here is an analogue for smooth stacks of the notion of an n-Artin morphism of topological stacks (2.3.33). Due to the fact that Sm does not have all pullbacks, the definition involves an extra submersivity condition, and some care is needed in generalizing the basic results about Artin morphisms of topological stacks.

⋆ **2.5.24 Definition** (n-Artin morphism)**.** A morphism of smooth stacks $X \to Y$ is called *n-Artin* (for integers $n \geq 0$) when for every map $U \to Y$ from a smooth manifold U, the pullback $X \times_Y U$ admits an $(n-1)$-Artin atlas $W \twoheadrightarrow X \times_Y U$ with $W \to U$ submersive (in which case *every* $(n-1)$-Artin atlas $W \twoheadrightarrow X \times_Y U$ is submersive over U (2.5.26)(2.5.27)) (this is an inductive definition, the base case being that a morphism is (-1)-Artin iff it is an isomorphism). It is immediate that n-Artin morphisms are preserved under pullback.

2.5.25 Exercise. Let Y be representable. Show that a morphism of smooth stacks $X \to Y$ is n-Artin iff X has an $(n-1)$-Artin atlas submersive over Y.

2.5.26 Lemma. *n-Artin morphisms of smooth stacks are closed under composition.*

Proof. The same argument given for topological stacks (2.3.36) applies, provided we note that in the relevant diagram (2.3.36.1), submersivity of both $W \to V \to U$ implies submersivity of their composition. □

2.5.27 Lemma. *A morphism of smooth manifolds is Artin iff it is a submersion.*

Proof. A submersion of smooth manifolds is 0-Artin by definition. For the converse, let $X \to Y$ be an Artin morphism of smooth manifolds. By definition, this means that there exists an Artin atlas $U \twoheadrightarrow X$ with $U \to Y$ submersive. Since $U \twoheadrightarrow X$ has local sections, submersivity of $U \to Y$ implies submersivity of $X \to Y$. $\square$

2.5.28 Lemma. *Left Kan extension* $|\cdot|_! : \mathsf{Shv}(\mathsf{Sm}) \to \mathsf{Shv}(\mathsf{Top})$ *preserves n-Artin morphisms and pullbacks of n-Artin morphisms.*

Proof. We proceed by induction on n, the base case $n = -1$ being trivial.

Let $X \to Y$ be an n-Artin morphism of smooth stacks, and let us show that $|\cdot|_!$ preserves all of pullbacks of $X \to Y$ and sends $X \to Y$ to an n-Artin morphism. According to (**??**), the left Kan extension $|\cdot|_!$ preserves pullbacks of $X \to Y$ iff it preserves all pullbacks of $X \times_Y Z \to Z$ along $Z' \to Z$ for $Z, Z' \in \mathsf{Sm}$. Moreover, if $|\cdot|_!$ sends each pullback $X \times_Y Z \to Z \in \mathsf{Sm}$ to an n-Artin morphism, then it sends $X \to Y$ to an n-Artin morphism, since every morphism from a topological space to $|Y|_!$ factors locally through $|Z \to Y|_!$ for some $Z \in \mathsf{Sm}$. We have thus reduced to the case that $Y \in \mathsf{Sm}$ and to pullbacks to $Y' \in \mathsf{Sm}$.

Since $X \to Y$ is n-Artin and $Y \in \mathsf{Sm}$, there exists an $(n-1)$-Artin atlas $W \twoheadrightarrow X$ (and the composition $W \to Y$ is submersive (2.5.27)). Now $|\cdot|_!$ preserves $(n-1)$-Artin morphisms by the induction hypothesis, and $|\cdot|_!$ preserves atlases since it sends submersive maps to representable maps (2.8.49) and preserves admitting local sections (2.5.4). Thus $|W|_! \to |X|_!$ is an $(n-1)$-Artin atlas, hence $|X|_! \to |Y|_!$ is n-Artin. To show that $|\cdot|_!$ preserves pullbacks of $X \to Y$ along maps from smooth manifolds $Y' \to Y$, apply $|\cdot|_!$ to the pullback diagram

$$\begin{tikzcd} W' \arrow[r] \arrow[d, two heads] & W \arrow[d, two heads] \\ X' \arrow[r] \arrow[d] & X \arrow[d] \\ Y' \arrow[r] & Y \end{tikzcd} \tag{2.5.28.1}$$

and note that $|\cdot|_!$ of the upper square is a fiber square by the induction hypothesis since $W \to X$ is $(n-1)$-Artin. The composite square is a submersive pullback (2.5.26)(2.5.27) hence is preserved by $|\cdot|_!$ (2.8.49). It thus follows from the exceptional case of fiber product cancellation (1.1.58)(**??**)(**??**) that $|\cdot|_!$ preserves the bottom fiber square since $|W \to X|_!$ admits local sections. $\square$

Tangent complexes

A naive notion of the tangent space of a smooth stack is the following:

2.5.29 Definition (Tangent space of a smooth stack)**.** The tangent space functor $T : \mathsf{Sm} \to \mathsf{Vect} \rtimes \mathsf{Sm}$ and the total space functor $\mathrm{tot} : \mathsf{Vect} \rtimes \mathsf{Sm} \to \mathsf{Sm}$ induce left Kan extension functors on stacks.

$$\mathsf{Shv}(\mathsf{Sm}) \xrightarrow{T_!} \mathsf{Shv}(\mathsf{Vect} \rtimes \mathsf{Sm}) \xrightarrow{\mathrm{tot}_!} \mathsf{Shv}(\mathsf{Sm}) \tag{2.5.29.1}$$

It will be more useful to have a notion of the *tangent complex* of a smooth stack X which is a complex of vector bundles $TX \in \mathsf{Perf}^{\leq 0}(X)$. Such a tangent complex cannot exist for arbitrary smooth stacks:

2.5.30 Example. Consider the pushout $X = \operatorname{colim}(\mathbb{R} \leftarrow * \rightarrow \mathbb{R}^2)$ in $\mathsf{Shv}(\mathsf{Sm})$. There is a map $\gamma : [0,1] \to X$ sending the endpoints of the interval to points of $\mathbb{R}$ and $\mathbb{R}^2$ (respectively) not identified by the colimit. Now $X = \mathbb{R}$ and $X = \mathbb{R}^2$ in a neighborhood of the endpoints of γ, so for any notion of tangent bundle for smooth stacks which is compatible with restriction to open substacks and agrees with the usual notion of tangent bundle for smooth manifolds, the pullback $\gamma^* TX$ must be the trivial bundle $\underline{\mathbb{R}}$ near one endpoint of the interval and be $\underline{\mathbb{R}}^2$ near the other endpoint. There is no finite complex of finite-dimensional vector bundles on $[0,1]$ with this property, since the Euler characteristic is locally constant.

It turns out that there is a good notion of the relative tangent complex $T_{X/Y} \in \mathsf{Perf}^{\leq 0}(X)$ for morphisms of smooth stacks $X \to Y$ which are Artin (2.5.24). We now state the axioms of this theory of tangent complexes and show that they characterize it uniquely.

⋆ **2.5.31 Definition** (Relative tangent complex of an Artin morphism)**.** The *relative tangent complex functor* associates to each Artin morphism of smooth stacks $X \to Y$ an object $T_{X/Y} \in \mathsf{Perf}^{\leq 0}(X)$ (the sheafified ∞-category of complexes of vector bundles supported in non-positive cohomological degree (**??**)) and to each square

$$\begin{tikzcd} X' \arrow[r] \arrow[d, "\text{Artin}"'] & X \arrow[d, "\text{Artin}"] \\ Y' \arrow[r] & Y \end{tikzcd} \tag{2.5.31.1}$$

associates a morphism $T_{X'/Y'} \to (X' \to X)^* T_{X/Y}$. More formally, it is a section of the cartesian fibration over $\mathsf{Fun}(\Delta^1, \mathsf{Shv}(\mathsf{Sm}))_{\mathsf{Artin}}$ (the full subcategory of $\mathsf{Fun}(\Delta^1, \mathsf{Shv}(\mathsf{Sm}))$ spanned by Artin morphisms) obtained via pullback under $\mathrm{ev}_0 : \mathsf{Fun}(\Delta^1, \mathsf{Shv}(\mathsf{Sm}))_{\mathsf{Artin}} \to \mathsf{Shv}(\mathsf{Sm})$ from the cartesian fibration $\mathsf{Perf}^{\leq 0} \rtimes \mathsf{Shv}(\mathsf{Sm}) \to \mathsf{Shv}(\mathsf{Sm})$ encoding the functor $\mathsf{Perf}^{\leq 0} : \mathsf{Shv}(\mathsf{Sm}) \to \mathsf{Cat}_\infty$.

$$\begin{tikzcd} \mathrm{ev}_0^* \mathsf{Perf}^{\leq 0} \rtimes \mathsf{Fun}(\Delta^1, \mathsf{Shv}(\mathsf{Sm}))_{\mathsf{Artin}} \arrow[d, shift right] \\ \mathsf{Fun}(\Delta^1, \mathsf{Shv}(\mathsf{Sm}))_{\mathsf{Artin}} \arrow[u, bend right, "(X \to Y) \mapsto T_{X/Y}"'] \end{tikzcd} \tag{2.5.31.2}$$

The tangent complex functor T satisfies the following axioms:

(2.5.31.3) (Compatibility with pullback) T sends pullback squares to cartesian morphisms. In other words, if a square (2.5.31.1) is a pullback square, then its associated map $T_{X'/Y'} \to (X' \to X)^* T_{X/Y}$ is an isomorphism.

(2.5.31.4) (Exactness) T sends identities to zero and compositions to exact triangles. In other words, $T_{X/X} = 0$ for all X, and for any composition of Artin morphisms $X \to Y \to Z$, the induced triangle in $\mathsf{Perf}(X)$ is exact (??).

$$\begin{tikzcd} (X \to Y) \arrow[r] \arrow[d] & (X \to Z) \arrow[d] \\ (Y \to Y) \arrow[r] & (Y \to Z) \end{tikzcd} \overset{T}{\rightsquigarrow} \begin{tikzcd} T_{X/Y} \arrow[r] \arrow[d] & T_{X/Z} \arrow[d] \\ 0 = (X \to Y)^* T_{Y/Y} \arrow[r] & (X \to Y)^* T_{Y/Z} \end{tikzcd}$$

We will see below that a functor T satisfying these axioms is determined uniquely by its restriction to submersions of smooth manifolds $\mathsf{Fun}(\Delta^1, \mathsf{Sm})_{\mathsf{Artin}}$ (2.5.32). Taking this restriction to be the usual relative tangent space $T_{X/Y} = \ker(TX \to (X \to Y)^* TY)$, we obtain the *relative tangent complex functor* for Artin morphisms of smooth stacks.

2.5.32 Theorem. *The restriction functor*

$$\begin{array}{c} \mathsf{Fun}(\mathsf{Fun}(\Delta^1, \mathsf{Shv}(\mathsf{Sm}))_{\mathsf{Artin}}, \mathrm{ev}_0^* \mathsf{Perf})_{\mathsf{ex,cart}} \\ \downarrow \\ \mathsf{Fun}(\mathsf{Fun}(\Delta^1, \quad \mathsf{Sm} \quad)_{\mathsf{Artin}}, \mathrm{ev}_0^* \mathsf{Perf})_{\mathsf{ex,cart}} \end{array} \tag{2.5.32.1}$$

is an equivalence of ∞-categories, where the subscripts ex *and* cart *indicate those functors which are exact* (2.5.31.4) *and cartesian* (2.5.31.3)*, respectively. Moreover, it identifies the full subcategories of sections landing inside* $\mathsf{Perf}^{\leq 0} \subseteq \mathsf{Perf}$*.*

Before embarking on the proof, we sketch the basic idea. It will suffice to give a functorial procedure for determining the value of an exact and cartesian functor T on an Artin morphism $X \to Y$ in terms of its values on submersions of smooth manifolds. Let us describe how to determine the value of T on an n-Artin morphism $X \to Y$ in terms of its values on $(n-1)$-Artin morphisms. Since T is cartesian (2.5.31.3), the pullback of $T_{X/Y}$ to $X \times_Y Z$ equals $T_{X \times_Y Z/Z}$ for any map of smooth stacks $Z \to Y$. The ensemble of all $T_{X \times_Y Z/Z}$ defines an object of $\lim_{Z \in (\mathsf{Sm} \downarrow Y)} \mathsf{Perf}(X \times_Y Z)$, which is the image of $T_{X/Y}$ under the pullback map $\mathsf{Perf}(X) \xrightarrow{\sim} \lim_{Z \in (\mathsf{Sm} \downarrow Y)} \mathsf{Perf}(X \times_Y Z)$, which is an equivalence since Perf is continuous (??) (recall that $\operatorname{colim}_{Z \in (\mathsf{Sm} \downarrow Y)} Z \to Y$ is an isomorphism (1.4.193) hence so its its pullback along $X \to Y$ (2.2.16)); this reduces us to treating the case that Y is a smooth manifold. Since Y is a smooth manifold and $X \to Y$ is n-Artin, there exists an $(n-1)$-Artin atlas $W \twoheadrightarrow X$ for which the composition $W \twoheadrightarrow X \to Y$ is a submersion (2.5.24). Since T is exact (2.5.31.4), the pullback of $T_{X/Y}$ under any Artin morphism $W \to X$ is the cone of the map $T_{W/Y} \to T_{W/X}$, so when W is a smooth manifold and $W \to X$ is $(n-1)$-Artin, this describes the pullback of $T_{X/Y}$ to W in terms of the values of T on $(n-1)$-Artin morphisms. Finally, recall that since X has an $(n-1)$-Artin atlas, it is the colimit of all $(n-1)$-Artin maps to it from smooth manifolds (??), so $T_{X/Y}$ is determined by the ensemble of its pullbacks under all such maps (??).

Proof. □

2.6 Log topological spaces

A log structure on a topological space is a marking which, roughly speaking, specifies how functions are 'allowed to vanish'. Log structures originated in algebraic geometry in work of Fontaine and Illusie, with further development by Kato [59].

Basic notions

2.6.1 Definition (Monoid)**.** A *monoid* shall mean an $\mathbb{R}_{\geq 0}$-linear abelian monoid. That is, a monoid M is set with an associative, commutative, and unital operation $+ : M \times M \to M$ (unit $0 \in M$) along with a bi-linear operation $\cdot : (\mathbb{R}_{\geq 0}, +) \times (M, +) \to (M, +)$ satisfying $1 \cdot m = m$ and $(r \cdot s) \cdot m = r \cdot (s \cdot m)$. These operations may also be denoted multiplicatively $\cdot : M \times M \to M$ (unit $1 \in M$) and $(r, m) \mapsto m^r : \mathbb{R}_{\geq 0} \times M \to M$.

2.6.2 Definition (Sheaves of continuous functions)**.** For any topological space X, let C_X denote the sheaf on X of continuous maps to $\mathbb{R}$, and let $C_X^{>0} \subseteq C_X^{\geq 0} \subseteq C_X$ denote the subsheaves of functions taking values in $\mathbb{R}_{>0} \subseteq \mathbb{R}_{\geq 0} \subseteq \mathbb{R}$, respectively.

⋆ **2.6.3 Definition** (Log topological space)**.** Let X be a topological space. A *pre-log structure* on X is a sheaf of (commutative) monoids $\mathcal{O}_X^{\geq 0}$ on X together with a map of sheaves of monoids $\mathcal{O}_X^{\geq 0} \to C_X^{\geq 0}$, where the monoid operation on $C_X^{\geq 0}$ is multiplication of functions. We consider the subsheaf $\mathcal{O}_X^{>0} \subseteq \mathcal{O}_X^{\geq 0}$ defined as the pullback

$$\begin{tikzcd}
\mathcal{O}_X^{>0} \arrow[r] \arrow[d] & C_X^{>0} \arrow[d] \\
\mathcal{O}_X^{\geq 0} \arrow[r] & C_X^{\geq 0}
\end{tikzcd} \tag{2.6.3.1}$$

and a *log structure* is a pre-log structure for which the map $\mathcal{O}_X^{>0} \to C_X^{>0}$ is an isomorphism. A *log topological space* is a topological space equipped with a log structure. A map of log topological spaces $(f, f^\flat) : (X, \mathcal{O}_X^{\geq 0}) \to (Y, \mathcal{O}_Y^{\geq 0})$ is a continuous map $f : X \to Y$ together with a map $f^\flat : f^*\mathcal{O}_Y^{\geq 0} \to \mathcal{O}_X^{\geq 0}$ such that the following diagram commutes.

$$\begin{tikzcd}
f^*\mathcal{O}_Y^{\geq 0} \arrow[r, "f^\flat"] \arrow[d] & \mathcal{O}_X^{\geq 0} \arrow[d] \\
f^*C_Y^{\geq 0} \arrow[r, "f^*"] & C_X^{\geq 0}
\end{tikzcd} \tag{2.6.3.2}$$

It is sometimes helpful to think in terms of 'log coordinates' $\log : \mathbb{R}_{\geq 0} \xrightarrow{\sim} \mathbb{R}_{\geq -\infty}$. In these coordinates, the sheaf $\mathcal{O}_X^{\geq 0}$ becomes an enlargement of the sheaf of real-valued functions to (possibly) include some functions taking the value $-\infty$ at some points. The category of log topological spaces is denoted LogTop.

2.6.4 Remark. If $\mathcal{O}_X^{\geq 0} \to C_X^{\geq 0}$ is injective, then there is at most one log map $X \to Y$ lifting a given continuous map $X \to Y$.

2.6.5 Exercise (Adjoint functors $(\mathrm{init}, |\cdot|, \mathrm{triv}, \square)$)**.** Every topological space X has a 'trivial' log structure $\mathcal{O}_X^{\geq 0} = C_X^{>0}$, which is the default way to view X as a log topological space. Show that this defines a full faithful embedding $\mathrm{triv} : \mathsf{Top} \to \mathsf{LogTop}$ which is right adjoint to the forgetful functor $|\cdot| : \mathsf{LogTop} \to \mathsf{Top}$. In practice, one is usually interested in log structures which are 'finite extensions' of the trivial log structure.

Show that the left adjoint to the forgetful functor $|\cdot|$ equips a topological space X with the 'initial' log structure $\mathcal{O}_X^{\geq 0} = C_X^{\geq 0}$. This log structure is too 'wild' to be of much direct interest.

Finally, show that sending a log topological space X to its largest open subset over which its log structure is trivial defines a functor $\square : \mathsf{LogTop} \to \mathsf{Top}$ which is right adjoint to triv. This may be called the 'non-degenerate locus' of X.

2.6.6 Exercise (Log structure associated to a pre-log structure)**.** Show that the inclusion of log structures on X into pre-log structures on X has a left adjoint given by sending ${}^{\mathrm{pre}}\mathcal{O}_X^{\geq 0}$ to the colimit $\mathcal{O}_X^{\geq 0}$ of ${}^{\mathrm{pre}}\mathcal{O}_X^{\geq 0} \leftarrow {}^{\mathrm{pre}}\mathcal{O}_X^{>0} \to C_X^{>0}$. Show that a section of $\mathcal{O}_X^{\geq 0}$ is given locally by a product of sections $(a, b) \in {}^{\mathrm{pre}}\mathcal{O}_X^{\geq 0} \times C_X^{>0}$ (that is, show ${}^{\mathrm{pre}}\mathcal{O}_X^{\geq 0} \times C_X^{>0} \to \mathcal{O}_X^{\geq 0}$ is surjective), and show that two such pairs (a, b) and (a', b') determine the same section of $\mathcal{O}_X^{\geq 0}$ iff there exist (locally) sections $m, m' \in {}^{\mathrm{pre}}\mathcal{O}_X^{>0}$ with $(am, |m|^{-1}b) = (a'm', |m'|^{-1}b')$.

2.6.7 Exercise (Log structure from a function)**.** Let X be a topological space, and let $f : X \to \mathbb{R}_{\geq 0}$ be a continuous function with $Z := f^{-1}(0)$. There is an induced pre-log structure $\mathbb{Z}_{\geq 0} \to C_X^{\geq 0}$ given by $n \mapsto f^n$ for $n > 0$ and $0 \mapsto 1$; denote by $\mathcal{O}_X^{\geq 0}$ the associated log structure. Show that a global section of $\mathcal{O}_X^{\geq 0}$ consists of a function $g : X \to \mathbb{R}_{\geq 0}$, a locally constant function $n : Z \to \mathbb{Z}_{\geq 0}$, and functions $h_k : (X \setminus Z) \cup n^{-1}(k) \to \mathbb{R}_{>0}$ such that $f^k h_k = g|_{(X \setminus Z) \cup n^{-1}(k)}$. Show that $\mathcal{O}_X^{\geq 0} \to C_X^{\geq 0}$ is injective iff $Z^\circ = \varnothing$. Show that there are maps

$$0 \to \mathcal{O}_X^{>0} \to \mathcal{O}_X^{\geq 0} \to (i_Z)_* \mathbb{Z}_{\geq 0} \to 0 \tag{2.6.7.1}$$

which form a 'short exact sequence', in the sense that $\mathcal{O}_X^{\geq 0} \to (i_Z)_* \mathbb{Z}_{\geq 0}$ is an epimorphism of underlying sheaves of sets (i.e. every section of $(i_Z)_* \mathbb{Z}_{\geq 0}$ is locally the image of a section of $\mathcal{O}_X^{\geq 0}$; compare (**??**)) and its fibers are $\mathcal{O}_X^{>0}$-torsors (i.e. any two sections of $\mathcal{O}_X^{\geq 0}$ with the same image in $(i_Z)_* \mathbb{Z}_{\geq 0}$ are related by a unique section of $\mathcal{O}_X^{>0}$).

Given a pair (X, Z) consisting of a topological space X and a closed subset $Z \subseteq X$, one might also attempt to consider the log structure given by those non-negative functions on X whose zero locus is contained in Z. Like the initial log structure $\mathcal{O}_X^{\geq 0} = C_X^{\geq 0}$ (2.6.5), this log structure is too wild to be of much use.

2.6.8 Remark (Log structure from a Cartier divisor)**.** The construction above (2.6.7) defines a map from $C_X^{\geq 0}$ to the sheaf of log structures on open subsets of X. Since multiplication by a positive function determines an isomorphism of the associated log structures, this map descends to the groupoid quotient $C_X^{\geq 0}/C_X^{>0}$. A section of $C_X^{\geq 0}/C_X^{>0}$ is called a *Cartier divisor.*

2.6.9 Exercise (Standard log structure $'\mathbb{R}_{\geq 0}$)**.** We denote by $'\mathbb{R}_{\geq 0}$ the topological space $\mathbb{R}_{\geq 0}$ equipped with the log structure associated to the identity function by the construction in (2.6.7). Show that $'\mathbb{R}_{\geq 0}$ has the following universal property: maps $X \to {}'\mathbb{R}_{\geq 0}$ are in natural bijection with global sections of $\mathcal{O}_X^{\geq 0}$ for log topological spaces X. What are the global sections of this log structure on $'\mathbb{R}_{\geq 0}$? (Equivalently, what are the log maps $'\mathbb{R}_{\geq 0} \to {}'\mathbb{R}_{\geq 0}$?)

2.6.10 Definition (Pullback log structure)**.** Given a map of topological spaces $X \to Y$ and a log structure $\mathcal{O}_Y^{\geq 0}$ on Y, the pullback log structure $f^{\#}\mathcal{O}_Y^{\geq 0}$ on X is the log structure associated to the pre-log structure $f^*\mathcal{O}_Y^{\geq 0}$ (ordinary sheaf pullback). That is, $f^{\#}\mathcal{O}_Y^{\geq 0}$ is the sheaf pushout (i.e. the sheafification of the presheaf pushout) of $f^*\mathcal{O}_Y^{\geq 0} \leftarrow f^*\mathcal{O}_Y^{>0} \to \mathcal{O}_X^{>0}$ (compare (2.6.6)). A map of log topological spaces $(X, \mathcal{O}_X^{\geq 0}) \to (Y, \mathcal{O}_Y^{\geq 0})$ can be equivalently defined as a map of topological spaces $X \to Y$ together with a map $f^{\#}\mathcal{O}_Y^{\geq 0} \to \mathcal{O}_X^{\geq 0}$ of log structures on X.

2.6.11 Example. The log structure associated to a continuous function $f : X \to \mathbb{R}_{\geq 0}$ by the construction (2.6.7) is precisely $f^{\#}$ of the log structure on $'\mathbb{R}_{\geq 0}$.

2.6.12 Exercise. Let $f : X \to Y$ be continuous, and let $\mathcal{O}_Y^{\geq 0}$ be a log structure on Y. Show that its pullback $f^{\#}\mathcal{O}_Y^{\geq 0}$ satisfies the following universal property. For any log map $g : (Z, \mathcal{O}_Z^{\geq 0}) \to (Y, \mathcal{O}_Y^{\geq 0})$ and continuous map $h : Z \to X$ satisfying $g = f \circ h$, there exists a unique refinement of h to a log map $(Z, \mathcal{O}_Z^{\geq 0}) \to (X, f^{\#}\mathcal{O}_Y^{\geq 0})$ such that $g = f \circ h$ as log maps.

2.6.13 Exercise (Limits of log topological spaces)**.** Show that the limit of a diagram of log topological spaces $(X_\alpha, \mathcal{O}_{X_\alpha}^{\geq 0})$ is given by the limit of underlying topological spaces X_α equipped with the colimit of the pullbacks of $\mathcal{O}_{X_\alpha}^{\geq 0}$.

2.6.14 Definition (Strict log map)**.** A map of log topological spaces $(X, \mathcal{O}_X^{\geq 0}) \to (Y, \mathcal{O}_Y^{\geq 0})$ is called *strict* when the map $f^{\#}\mathcal{O}_Y^{\geq 0} \to \mathcal{O}_X^{\geq 0}$ is an isomorphism.

2.6.15 Definition (Embedding of log topological spaces)**.** An *embedding* of log topological spaces is a strict log map which is an embedding of underlying topological spaces.

2.6.16 Exercise. Show that strictness is preserved under pullback, hence so is the property of being an embedding.

2.6.17 Definition (Ghost sheaf)**.** For a log topological space X, the sheaf of monoids $\mathcal{Z}_X = \mathcal{O}_X^{\geq 0}/\mathcal{O}_X^{>0}$ is called the *ghost sheaf* of X.

2.6.18 Example. Let X be equipped with the log structure (2.6.7) associated to a function $f : X \to \mathbb{R}_{\geq 0}$ with zero set Z. The short exact sequence (2.6.7.1) shows that $\mathcal{Z}_X = (i_Z)_*\mathbb{Z}_{\geq 0}$.

2.6.19 Exercise. Show that two sections $f, g \in \mathcal{O}_X^{\geq 0}$ have the same image in $\mathcal{Z}_X$ iff there exist local expressions $f = ug$ for various local $u \in \mathcal{O}_X^{>0}$. Show that the only invertible section of the ghost sheaf $\mathcal{Z}_X$ is the identity.

2.6.20 Exercise. Show that a log map $f : X \to Y$ induces a map $f^{\flat\flat} : f^*\mathcal{Z}_Y \to \mathcal{Z}_X$ as a quotient of $f^\flat$. Show that if f is strict then $f^{\flat\flat}$ is an isomorphism (note that $\mathcal{Z} = \mathcal{O}^{\geq 0}/\mathcal{O}^{>0}$ is the pushout of $0 \leftarrow \mathcal{O}^{>0} \to \mathcal{O}^{\geq 0}$ and use the fact that f^* preserves colimits).

2.6.21 Definition (Quasi-integral)**.** A log topological space $(X, \mathcal{O}_X^{\geq 0})$ *quasi-integral* when the action of $\mathcal{O}_X^{>0}(U)$ on $\mathcal{O}_X^{\geq 0}(U)$ is free for every open $U \subseteq X$.

2.6.22 Exercise. Show that $(X, \mathcal{O}_X^{\geq 0})$ is quasi-integral iff the sequence

$$0 \to \mathcal{O}_X^{>0} \to \mathcal{O}_X^{\geq 0} \to \mathcal{Z}_X \to 0 \qquad (2.6.22.1)$$

is exact in the sense that any two sections of $\mathcal{O}_X^{\geq 0}$ with the same image in $\mathcal{Z}_X$ differ by a unique section of $\mathcal{O}_X^{>0}$.

2.6.23 Exercise (Checking strictness via ghost sheaves)**.** For a log map $f : X \to Y$, show that if $f^{\flat\flat} : f^*\mathcal{Z}_Y \to \mathcal{Z}_X$ is an isomorphism and X is quasi-integral, then f is strict.

2.6.24 Exercise (Cancellative)**.** A monoid M is called *cancellative* when $x + a = y + a$ implies $x = y$ for all elements $x, y, a \in M$. Show that the inclusion of cancellative monoids into all monoids has a left adjoint $M \mapsto M^c$ whose unit map $M \twoheadrightarrow M^c$ is surjective and sends $m, m' \in M$ to the same element of M^c iff $m + a = m' + a$ for some $a \in M$. Conclude that cancellative monoids are closed under all limits inside all monoids, and show that they are also closed under all directed colimits.

2.6.25 Definition (Substrict)**.** A morphism of log topological spaces $f : X \to Y$ is called *substrict* when every section of $\mathcal{O}_X^{\geq 0}$ is locally a product of a section of $C_X^{>0}$ and a section of $\mathcal{O}_Y^{\geq 0}$ (in other words, the morphism of sheaves of monoids $f^\#\mathcal{O}_Y^{\geq 0} \to \mathcal{O}_X^{\geq 0}$ is surjective (**??**)).

2.6.26 Exercise. Show that if $X \to Y$ has a retraction, then it is substrict. Conclude that every relative diagonal $X \to X \times_Y X$ is substrict.

Log topological stacks

We denote by $\mathsf{Shv}(\mathsf{LogTop})$ the ∞-category of log topological stacks, whose definition is directly analogous to that of topological stacks $\mathsf{Shv}(\mathsf{Top})$ (2.3) and smooth stacks $\mathsf{Shv}(\mathsf{Sm})$ (2.5) (see also (2.8)).

2.6.27 Definition. Recall the functors $|\cdot|, \square : \mathsf{LogTop} \to \mathsf{Top}$ and $\mathrm{triv}, \mathrm{init} : \mathsf{Top} \to \mathsf{LogTop}$ between the categories of topological spaces and log topological spaces (2.6.5). They each induce pullback and left Kan extension functors between the ∞-categories of topological stacks and log topological stacks (2.8.36). The adjunctions $(\mathrm{init}, |\cdot|, \mathrm{triv}, \square)$ induce adjunctions $(\mathrm{init}^*, |\cdot|^*, \mathrm{triv}^*, \square^*)$ between their associated pullback functors (1.1.104)(1.1.101), hence identifications:

$$|\cdot|_! = \mathrm{init}^* : \mathsf{Shv}(\mathsf{LogTop}) \to \mathsf{Shv}(\mathsf{Top}) \qquad (2.6.27.1)$$

$$\mathrm{triv}_! = |\cdot|^* : \mathsf{Shv}(\mathsf{Top}) \to \mathsf{Shv}(\mathsf{LogTop}) \qquad (2.6.27.2)$$

$$\square_! = \mathrm{triv}^* : \mathsf{Shv}(\mathsf{LogTop}) \to \mathsf{Shv}(\mathsf{Top}) \qquad (2.6.27.3)$$

2.6.28 Definition (Point vs non-degenerate point)**.** A *non-degenerate point* of a log topological stack X is a morphism of log topological stacks $* \to X$. A *point* of a log topological stack X a point of its associated topological stack $|X|_!$ (equivalently, it a morphism of log topological stacks $*_{\mathrm{init}} \to X$ (2.6.27)).

Log mapping stacks

Following (2.3.39), we studied topological mapping stacks. We noted that a point of a topological mapping stack is just a continuous map or section in the relevant sense, and we recalled that under mild hypotheses, these stacks are represented by a natural topology on their set of points (2.3.51)(2.3.60).

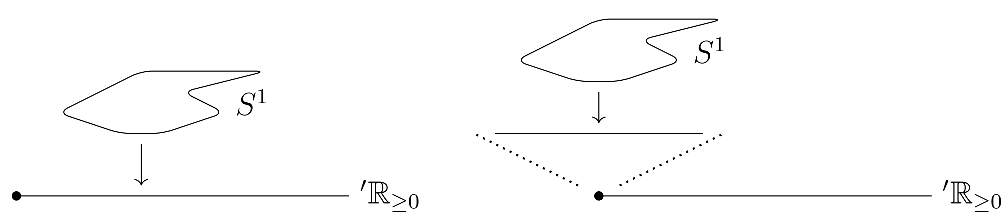
We now make a similar study of mapping stacks between log topological spaces. The main new feature in the log setting is the difference between points and non-degenerate points of log topological stacks (2.6.28). While a *non-degenerate* point of a log topological mapping stack is the same as a continuous map or section in the relevant sense, a *point* is more general. Points of log mapping stacks may be regarded as 'degenerated' maps/sections, and they capture the sort of 'target degenerations' relevant for enumerative theories of pseudo-holomorphic curves.

2.6.29 Exercise (Points of $\underline{\operatorname{Hom}}(X,Y)$)**.** Let X and Y be log topological spaces. Recall the mapping stack $\underline{\operatorname{Hom}}(X,Y)\in\mathsf{Shv}(\mathsf{LogTop})$ (1.1.134) (that is, a map $Z\to\underline{\operatorname{Hom}}(X,Y)$ is a map $X\times Z\to Y$). A non-degenerate point of $\underline{\operatorname{Hom}}(X,Y)$ is a map $X\to Y$, while a point of $\underline{\operatorname{Hom}}(X,Y)$ is a map $X\times *_{\mathrm{init}}\to Y$ (2.6.28). Compute the structure sheaf of $X\times *_{\mathrm{init}}$ to be

$$\mathcal{O}^{\geq 0}_{X\times *_{\mathrm{init}}}=\mathcal{O}^{\geq 0}_X\sqcup\mathcal{O}^{\geq 0}_X/\mathbb{R}_{>0}, \tag{2.6.29.1}$$

with the map to $C^{\geq 0}_X$ sending the second term to zero.

Conclude that a point of $\underline{\operatorname{Hom}}(X,{}'\mathbb{R}_{\geq 0})$ is a partition $X=U\sqcup V$ into disjoint open subsets, a map of log topological spaces $f:U\to{}'\mathbb{R}_{\geq 0}$, and the data g on V of local maps to ${}'\mathbb{R}_{\geq 0}$ which agree up to multiplication by constants in $\mathbb{R}_{>0}$ on overlaps (where the map on underlying topological spaces $|X|\to\mathbb{R}_{\geq 0}$ corresponding to such a point is given by $|f|$ on $|U|$ and sends V to $0\in\mathbb{R}_{\geq 0}$).



A non-degenerate point of $\underline{\operatorname{Hom}}(S^1,{}'\mathbb{R}_{\geq 0})$ A point of $\underline{\operatorname{Hom}}(S^1,{}'\mathbb{R}_{\geq 0})$

(2.6.29.2)

2.6.30 Exercise. Let us compute the points of the mapping stack $\underline{\operatorname{Hom}}_B(C,X)$ when $C\to B$ is the multiplication map ${}'\mathbb{R}^2_{\geq 0}\to{}'\mathbb{R}_{\geq 0}$ and $X={}'\mathbb{R}_{\geq 0}$. Over $\mathbb{R}_{>0}\subseteq{}'\mathbb{R}_{\geq 0}=B$, the family $C\to B$ is trivial with fiber $\mathbb{R}$, and we described the points of $\underline{\operatorname{Hom}}(\mathbb{R},{}'\mathbb{R}_{\geq 0})$ in (2.6.29). It thus suffices to describe the points of $\underline{\operatorname{Hom}}_B(C,X)$ lying over $0\in{}'\mathbb{R}_{\geq 0}=B$. Such a point is a map ${}'\mathbb{R}^2_{\geq 0}\times_{{}'\mathbb{R}_{\geq 0}}\times *_{\mathrm{init}}\to{}'\mathbb{R}_{\geq 0}$.

The stalk of the structure sheaf of ${}'\mathbb{R}^2_{\geq 0}\times_{{}'\mathbb{R}_{\geq 0}}\times *_{\mathrm{init}}$ at the node $(0,0)$ is given by

$$\mathcal{O}^{\geq 0}_{{}'\mathbb{R}^2_{\geq 0}\times_{{}'\mathbb{R}_{\geq 0}}\times *_{\mathrm{init}}}=x^{\mathbb{R}_{\geq 0}}y^{\mathbb{R}_{\geq 0}}C^{>0}_{\{xy=0\}}/(xy=\underline{0}), \tag{2.6.30.1}$$

where $\underline{0} \in \mathcal{O}^{\geq 0}_{*_{\mathrm{init}}} = \mathbb{R}_{\geq 0}$ satisfies $\underline{0}^a = \underline{0}$ (any $a > 0$) and $r\underline{0} = \underline{0}$ (any $r > 0$). Elements of this stalk are thus of exactly one of the following four forms:

(2.6.30.2) $f(x,y)$ for continuous $f : \{xy = 0\} \to \mathbb{R}_{>0}$.

(2.6.30.3) $x^a f(x,y)$ for continuous $f : \{xy = 0\} \to \mathbb{R}_{>0}$ and $a > 0$.

(2.6.30.4) $y^b f(x,y)$ for continuous $f : \{xy = 0\} \to \mathbb{R}_{>0}$ and $b > 0$.

(2.6.30.5) $x^a y^b f(x,y)\underline{0}$ for continuous $f : \{xy = 0\} \to \mathbb{R}_{>0}$ and $a, b > 0$, modulo $(a,b) \sim (a',b')$ when $a - b = a' - b'$ and modulo $f \sim rf$ for any $r > 0$.

2.6.31 Lemma. *Let X be a log topological space for which $|X|$ is locally compact. For any strict map of log topological spaces $Y' \to Y$, the induced morphism of log topological stacks $\underline{\mathrm{Hom}}(X,Y') \to \underline{\mathrm{Hom}}(X,Y)$ is strict representable (in fact, it is a pullback of a morphism of topological spaces equipped with trivial log structures).*

Proof. If $Y' \to Y$ is strict, then the following is a pullback square:

$$\begin{array}{ccc} \mathrm{Hom}(-,Y') & \longrightarrow & \mathrm{Hom}(|-|,|Y'|) \\ \downarrow & & \downarrow \\ \mathrm{Hom}(-,Y) & \longrightarrow & \mathrm{Hom}(|-|,|Y|) \end{array} \tag{2.6.31.1}$$

Now pulling back under $X \times -$ and using the fact that $|\cdot|$ commutes with products, we conclude that $\underline{\mathrm{Hom}}(X,Y') \to \underline{\mathrm{Hom}}(X,Y)$ is a pullback of $|\cdot|^*(\underline{\mathrm{Hom}}(|X|,|Y'|) \to \underline{\mathrm{Hom}}(|X|,|Y|))$. Since $|X|$ is locally compact, both $\underline{\mathrm{Hom}}(|X|,|Y'|)$ and $\underline{\mathrm{Hom}}(|X|,|Y|)$ are topological spaces (2.3.51), and applying $|\cdot|^*$ equips them with the trivial log structure by the adjunction $(|\cdot|, \mathrm{triv})$ (2.6.5). ☐

2.6.32 Proposition. *Let $C \to B$ be a map of log topological spaces, and assume:*

(2.6.32.1) *$C \to B$ is open.*

(2.6.32.2) *$C \to B$ is strict.*

*If $W' \to W$ is a substrict (2.6.25) closed embedding, then $\underline{\mathrm{Sec}}_B(C,W') \to \underline{\mathrm{Sec}}_B(C,W)$ is proper (**??**). In particular, if $W \to C$ is separated, then $\underline{\mathrm{Sec}}_B(C,W) \to B$ is separated.*

Proof. Since $W' \to W$ is a monomorphism (any substrict injection of log topological spaces is a monomorphism), it follows that $\underline{\mathrm{Sec}}_B(C,W') \to \underline{\mathrm{Sec}}_B(C,W)$ is also a monomorphism.

To check that $\underline{\mathrm{Sec}}_B(C,W') \to \underline{\mathrm{Sec}}_B(C,W)$ is proper (2.3.20), it suffices to check that it is universally closed (2.3.17) (since it is a monomorphism, its diagonal is an isomorphism). To show the subswarm lifting property for ($|\cdot|_! = \mathrm{init}^*$ applied to) $\underline{\mathrm{Sec}}_B(C,W') \to \underline{\mathrm{Sec}}_B(C,W)$, we will show the lifting property

$$\begin{tikzcd} S^*_{\mathrm{init}} \times_B C \arrow[r] \arrow[d] & W' \arrow[d] \\ S_{\mathrm{init}} \times_B C \arrow[r] \arrow[ur, dashed] & W \end{tikzcd} \tag{2.6.32.3}$$

for any map from a limit pointed topological space S to B (in fact, the subswarm lifting property asks that such a lift exist after possibly pulling back under a map of limit pointed topological spaces $T \to S$, but this additional flexibility is unnecessary here).

We solve the lifting problem (2.6.32.3) in two steps. The first step is to perform the lift topologically, and the second step is to extend the pullback map on log structure sheaves. In other words, we consider the factorization $W' \to W'_W \to W$ where W'_W denotes W' equipped with the pullback of the log structure on W (thus $W' \to W'_W$ is a homeomorphism and $W'_W \to W$ is strict).

To solve the topological lifting problem (that is, to lift along $W'_W \to W$), simply note that $S^* \times_B C \subseteq S \times_B C$ is dense since $C \to B$ is open, so the map $S \times_B C \to W$ lands inside the closed subset $|W'| \subseteq |W|$ since its restriction to $S^* \times_B C$ does so.

It remains to address the descent problem for the pullback map on log structure sheaves (that is, to lift along $W' \to W'_W$).

$$\begin{tikzcd}
\mathcal{O}_{\overline{W}}^{\geq 0} \arrow[r, two heads] \arrow[d] & \mathcal{O}_{\overline{W'}}^{\geq 0} \arrow[d] \arrow[dl, dashed] \\
\mathcal{O}_{\overline{S_{\mathrm{init}} \times_B C}}^{\geq 0} \arrow[r] & \mathcal{O}_{\overline{S^*_{\mathrm{init}} \times_B C}}^{\geq 0}
\end{tikzcd} \tag{2.6.32.4}$$

Since $W' \to W$ is substrict, it suffices to show that $\mathcal{O}_{\overline{S_{\mathrm{init}} \times_B C}}^{\geq 0} \to j_* j^* \mathcal{O}_{\overline{S_{\mathrm{init}} \times_B C}}^{\geq 0}$ is injective, where $j : S^* \times_B C \hookrightarrow S \times_B C$ is the open embedding.

Now let us describe $\mathcal{O}_{\overline{C \times_B S_{\mathrm{init}}}}^{\geq 0}$ using the fact that $C \times_B S_{\mathrm{init}} \to S_{\mathrm{init}}$ is strict (2.6.32.2). A section may be described locally by a pair $f \in C_{\overline{S}}^{\geq 0}$ and $g \in C_{\overline{C \times_B S}}^{>0}$, and two such pairs (f, g) and (f', g') represent the same section iff $f = \alpha f'$ and $g = \alpha^{-1} g'$ for some $\alpha \in C_{\overline{S}}^{>0}$ (locally). Now suppose we have two pairs (f, g) and (f', g') representing two sections of $\mathcal{O}_{\overline{C \times_B S_{\mathrm{init}}}}^{\geq 0}$, and suppose they are equivalent over $C \times_B S^*$. This means the quotient g/g' is pulled back from S over $C \times_B S^*$. Since $C \times_B S^* \subseteq C \times_B S$ is dense (this follows from openness of $C \times_B S \to S$), this implies that g/g' is pulled back from S over $C \times_B S$. This defines a function $\alpha \in C_{\overline{S}}^{>0}$ with $f = \alpha f'$ over S^*, which implies the same over S since $S^* \subseteq S$ is dense, and hence that (f, g) and (f', g') are equivalent over $C \times_B S$. We conclude that $\mathcal{O}_{\overline{S_{\mathrm{init}} \times_B C}}^{\geq 0} \to j_* j^* \mathcal{O}_{\overline{S_{\mathrm{init}} \times_B C}}^{\geq 0}$ is injective as desired.

□

2.7 Log smooth manifolds

A log smooth manifold is a log topological space (2.6) equipped with an atlas of charts from open subsets of real affine toric varieties $X_P = \mathrm{Hom}(P, \mathbb{R}_{\geq 0})$ for polyhedral cones P, with transition functions which are smooth in a certain sense. This key notion of 'log smoothness' arises from a certain notion of tangent bundle for the local models X_P, namely the b-tangent bundle of Melrose [79, 80, 81] or the log tangent bundle as it is called in algebraic geometry. Log smooth manifolds were introduced by Joyce [52], who also proposed their application to moduli spaces of solutions of non-linear elliptic partial differential equations on families of degenerating manifolds. There is also closely related work of Parker [91].

Our goal here is to set up basic differential topology for log smooth manifolds.

Real affine toric varieties

⋆ **2.7.1 Definition** (Polyhedral cone)**.** A (real) polyhedral cone $P \subseteq \mathbb{R}^n$ is a subset defined by finitely many inequalities of the form $\sum_i a_i x_i \geq 0$. A map of polyhedral cones $P \to Q$ is the restriction of a linear map (the embedding into $\mathbb{R}^n$ is thus irrelevant).

2.7.2 Remark (Integral vs real polyhedral cones)**.** One could work with integral polyhedral cones (subsets $P \subseteq \mathbb{Z}^n$ defined by finitely many inequalities of the form $\sum_i a_i x_i \geq 0$ for $a_i \in \mathbb{Z}$) instead of real polyhedral cones. The resulting geometric theory would be very similar, but somewhat more rigid. The additional flexibility afforded by real polyhedral cones is needed to describe the elliptic partial differential equations, solutions thereof, and moduli spaces of solutions, which we will study later. It is for this reason that we choose to work here with real polyhedral cones.

2.7.3 Exercise. Let P be a polyhedral cone. Show that the groupification P^{gp} is a finite-dimensional real vector space, and that the map $P \to P^{\mathrm{gp}}$ identifies P with a polyhedral cone in P^{gp}.

2.7.4 Exercise. Show that every polyhedral cone P admits a surjection from some $\mathbb{R}^n_{\geq 0}$.

2.7.5 Exercise. Show that the category of polyhedral cones is an additive category (**??**).

2.7.6 Exercise (Face)**.** A *face* $F \subseteq P$ of a polyhedral cone P is a subset of the form $F = P \cap \ell^{-1}(0)$ for some linear functional $\ell \in (P^{\mathrm{gp}})^*$ with the property that $\ell(p) \geq 0$ for all $p \in P$. For any map of polyhedral cones $P \to Q$, the inverse image of a face of Q is evidently a face of P. Show that a subset $F \subseteq P$ is a face iff

(2.7.6.1) $0 \in F$ and $a, b \in F \iff a + b \in F$.

⋆ **2.7.7 Definition** (Real affine toric varieties X_P)**.** Let P be a polyhedral cone. We consider the real affine toric variety

$$X_P = \mathrm{Hom}((P, +), (\mathbb{R}_{\geq 0}, \times)), \tag{2.7.7.1}$$

which we equip with the compact-open topology (2.3.50) and with the log structure associated to the pre-log structure $P \to C^{\geq 0}_{X_P}$ (the tautological 'evaluation' map). This log structure $\mathcal{O}^{\geq 0}_{X_P} \subseteq C^{\geq 0}_{X_P}$ consists of those functions on X_P which locally take the form $x(p)g(x)$ for some $p \in P$ and $g \in C^{>0}_{X_P}$. A *log topological manifold* is a log topological space locally isomorphic to open subsets of various X_P.

2.7.8 Example. If $P = \mathbb{R}$, then $X_P = \mathbb{R}_{>0}$ with the trivial log structure (2.6.5). If $P = \mathbb{R}_{\geq 0}$, then $X_P = {}'\mathbb{R}_{\geq 0}$ is the half-line $\mathbb{R}_{\geq 0}$ with its the standard log structure (2.6.9), namely the sheaf of continuous functions which locally have the form $f(x)x^a$ for some real number $a \geq 0$ and a continuous positive function f. If $P = \mathbb{R}^n_{\geq 0}$, then $X_P = {}'\mathbb{R}^n_{\geq 0}$, namely $\mathbb{R}^n_{\geq 0}$ equipped with the sheaf of continuous functions locally of the form $f(x)\prod_{i=1}^n x_i^{a_i}$ for real $a_i \geq 0$ and continuous positive f.

2.7.9 Exercise (Monomial maps $X_P \to X_Q$)**.** Show that a map of polyhedral cones $Q \to P$ induces a map $X_P \to X_Q$ (such maps are called *monomial*). Show that if $Q \twoheadrightarrow P$ is surjective then $X_P \hookrightarrow X_Q$ is a closed embedding.

2.7.10 Exercise (Presentation of X_P)**.** Show that a surjection $\varphi : \mathbb{R}^n_{\geq 0} \twoheadrightarrow P$ (which always exists (2.7.4)) presents $X_P \subseteq {}'\mathbb{R}^n_{\geq 0}$ as the subset cut out by finitely many conditions of the form $\prod_{i=1}^n x_i^{a_i} = 1$ for real $a_i \geq 0$ (corresponding to a finite set of generators of $\ker \varphi$). For instance, presenting $P = \mathbb{R}$ as $\mathbb{R}^2_{\geq 0}/(1,1)$ corresponds to realizing $X_P = \mathbb{R}_{>0}$ as the locus $\{xy = 1\} \subseteq {}'\mathbb{R}^2_{\geq 0}$.

2.7.11 Definition ($\mathbb{R}_{\geq 0}$-linear log structure)**.** An $\mathbb{R}_{\geq 0}$-linear monoid is a monoid M equipped with a bilinear operation $\mathbb{R}_{\geq 0} \times M \to M$; maps of $\mathbb{R}_{\geq 0}$-linear monoids are monoid maps respecting the $\mathbb{R}_{\geq 0}$-linear structure. For example, a real polyhedral cone has a canonical $\mathbb{R}_{\geq 0}$-linear structure, and in this way real polyhedral cones form a full subcategory of $\mathbb{R}_{\geq 0}$-linear monoids.

For X any topological space, $C^{\geq 0}_X$ is a sheaf of $\mathbb{R}_{\geq 0}$-linear monoids. By taking the definition of a log structure and replacing the category of monoids with that of $\mathbb{R}_{\geq 0}$-linear monoids, we obtain the notion of an $\mathbb{R}_{\geq 0}$-linear log structure. The foundations of log topological spaces following (2.6.3) carry over as written. The map $P \to C^{\geq 0}_{X_P}$ is $\mathbb{R}_{\geq 0}$-linear, so it determines an $\mathbb{R}_{\geq 0}$-linear log structure on X_P. (This discussion of $\mathbb{R}_{\geq 0}$-linearity is due to our use of real polyhedral cones as opposed to integral polyhedral cones (2.7.2).)

2.7.12 Exercise (Universal properties of X_P)**.** Show that:

(2.7.12.1) Maps $Z \to X_P$ from topological spaces Z are in natural bijection with $\mathbb{R}_{\geq 0}$-linear maps of monoids $P \to C^{\geq 0}_Z$.

(2.7.12.2) Maps $Z \to X_P$ from $\mathbb{R}_{\geq 0}$-linear log topological spaces Z are in natural bijection with $\mathbb{R}_{\geq 0}$-maps of monoids $P \to \mathcal{O}^{\geq 0}_Z$.

Conclude that the natural map $X_{P \oplus Q} \to X_P \times X_Q$ (induced by the embeddings $P \to P \oplus Q \leftarrow Q$) is an isomorphism of $\mathbb{R}_{\geq 0}$-linear log topological spaces (that is, the contravariant functor $P \mapsto X_P$ sends coproducts to products).

⋆ **2.7.13 Example** (Log coordinates)**.** The map $P \to P^{\mathrm{gp}}$ to the groupification induces $X_{P^{\mathrm{gp}}} \to X_P$, which is a dense open embedding denoted $X_P^{\square} \subseteq X_P$ called the 'non-degenerate locus' (consisting of 'non-degenerate points'), whose complement $X_P^{\infty} = X_P \setminus X_P^{\square}$ is called the 'ideal locus' (consisting of 'ideal points'). The non-degenerate locus $X_P^{\square} \subseteq X_P$ is also the set $\operatorname{Hom}((P,+),(\mathbb{R}_{>0},\times))$. Applying the logarithm map $\log : (\mathbb{R}_{>0},\times) \xrightarrow{\sim} (\mathbb{R},+)$ yields an isomorphism

$$(P^{\mathrm{gp}})^* = X_P^{\square} \subseteq X_P \tag{2.7.13.1}$$

referred to as *log coordinates* on $X_P^{\square}$. In log coordinates, monomial maps are linear.

2.7.14 Exercise. Show that $X_P^{\square} \subseteq X_P$ is dense. Conclude that the map $\mathcal{O}_{X_P}^{\geq 0} \to C_{X_P}^{\geq 0}$ is injective and that X_P is cancellative (2.6.24). It was observed in (2.6.4) that this implies that for any log topological space Z, log map $X_P \to Z$ is a continuous map with a *property*. Show that a log map from X_P to an $\mathbb{R}_{\geq 0}$-linear log topological space is automatically $\mathbb{R}_{\geq 0}$-linear.

⋆ **2.7.15 Exercise** (Asymptotically cylindrical structures as log structures)**.** Let U and V be open subsets of $\mathbb{R}^k$, and consider maps $U \times {}'\mathbb{R}_{\geq 0} \to V \times {}'\mathbb{R}_{\geq 0}$. Show that such a map necessarily takes the non-degenerate locus $U \times \mathbb{R}_{>0}$ to the non-degenerate locus $V \times \mathbb{R}_{>0}$. Show that near the ideal locus $U \times 0$, such a map locally takes the form (using log coordinates $x = e^s$)

$$(u,s) \mapsto (f(u) + o(1), a \cdot s + b(u) + o(1)) \tag{2.7.15.1}$$

where $f : U \to V$, $a \geq 0$, $b : U \to \mathbb{R}$, and $o(1)$ indicates a quantity approaching zero as $s \to -\infty$, uniformly over compact subsets of U.

⋆ **2.7.16 Definition** (Stratification of X_P)**.** Each space X_P is stratified by the set of faces of P. Namely, to a monoid homomorphism $x : P \to \mathbb{R}_{\geq 0}$ we associate the face $F_x = x^{-1}(\mathbb{R}_{>0}) \subseteq P$ (note that $x^{-1}(\mathbb{R}_{>0})$ satisfies the criterion (2.7.6.1) for being a face). Given a face $F \subseteq P$, there is an embedding of topological spaces $X_F \subseteq X_P$ given by extension by zero on $P \setminus F$ (but note this is not an embedding of log topological spaces). The stratum of X_P associated to F is $X_F^{\square}$. Restriction along the inclusion $F \subseteq P$ defines a morphism of log topological spaces $X_P \to X_F$ which is a topological retraction.

2.7.17 Remark. It is known that X_P and P are homeomorphic as stratified spaces (a reference is [88, Theorem 1.4]). We will only ever use elementary special cases of this result, such as for $P = \mathbb{R}^n_{\geq 0}$.

2.7.18 Exercise (Rational monomial maps $X_P \to X_Q$)**.** Let $Q \to P^{\mathrm{gp}}$ be a map of polyhedral cones, and consider the union of the strata $X_F^{\square} \subseteq X_P$ for the faces $F \subseteq P$ for which $Q \to P^{\mathrm{gp}}$ lands inside $P + F^{\mathrm{gp}} \subseteq P^{\mathrm{gp}}$. Show that this is an open subset of X_P, and that $Q \to P^{\mathrm{gp}}$ defines a map $X_P \to X_Q$ on this open subset (such maps are called *rational monomial*).

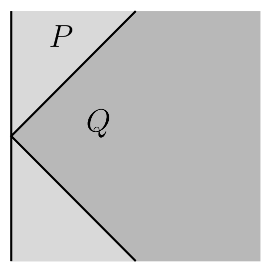
2.7.19 Definition (Sharp)**.** A polyhedral cone P is called *sharp* when its minimal stratum is $\{0\}$ (that is, when P contains no nonzero invertible elements). The quotient of a polyhedral cone P by its minimal stratum $P_0 \subseteq P$ is denoted $P^{\#}$, which is always sharp. The functor $P \mapsto P^{\#}$ is left adjoint to the inclusion of sharp polyhedral cones into all polyhedral cones.

2.7.20 Example (Local structure of X_P)**.** Let $x \in X_P$, and let $F_x \subseteq P$ index the stratum $X^{\square}_{F_x} \subseteq X_P$ containing x, namely $F_x = x^{-1}(\mathbb{R}_{>0})$. Then x lies in the open subset $X_{P+F_x^{\mathrm{gp}}} \subseteq X_P$, in which it lies on the minimal stratum, namely $X_{F_x^{\mathrm{gp}}} = X^{\square}_{F_x}$. We define $P_x = P/F_x^{\mathrm{gp}}$, so there is a short exact sequence

$$0 \to F_x^{\mathrm{gp}} \to P + F_x^{\mathrm{gp}} \to P_x \to 0. \tag{2.7.20.1}$$

The polyhedral cone P_x is sharp and is the stalk $\mathcal{Z}_{X_P,x}$ of the ghost sheaf $\mathcal{Z}_{X_P} = \mathcal{O}^{\geq 0}_{X_P}/\mathcal{O}^{>0}_{X_P}$ (2.6.17) at x. The polyhedral cone P_x controls the local structure of X_P near x: a choice of splitting of (2.7.20.1) induces an isomorphism $X_{P_x} \times X_{F_x^{\mathrm{gp}}} = X_{P+F_x^{\mathrm{gp}}} \subseteq X_P$.

2.7.21 Exercise. Consider $P = \mathbb{R}_{\geq 0} \times \mathbb{R} = \{x \geq 0\} \times \{y\}$ and $Q = \{y \leq |x|\} \subseteq P$.



(2.7.21.1)

Let $f : X_P \to X_Q$ denote the restriction map, and let $p = 0 \in X_P$ be the basepoint $p(0, y) = 1$ and $p(x, y) = 0$ for $x > 0$. The short exact sequence (2.7.20.1) for $f(p) \in X_Q$ maps naturally to that for $p \in X_P$. Show that this results in the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] \arrow[d] & Q \arrow[r] \arrow[d] & Q \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathbb{R} \arrow[r] & P \arrow[r] & \mathbb{R}_{\geq 0} \arrow[r] & 0
\end{tikzcd} \tag{2.7.21.2}$$

Describe the map f geometrically.

2.7.22 Exercise. Let $f : X_P \to X_Q$ and let $x \in X_P$. Show that $f = f_{\mathrm{rat}} \cdot g$ in a neighborhood of x for some rational monomial map $f_{\mathrm{rat}} : X_P \to X_Q$ and some map $g : X_P \to X^{\square}_Q$, where $\cdot$ denotes multiplication in X_Q. Moreover, show that this pair (f_{rat}, g) is unique up to a natural action of $\mathrm{Hom}(Q, F_x^{\mathrm{gp}})$.

2.7.23 Example (Recovering local coordinates on a log topological manifold)**.** Let M be a log topological manifold, and let $p \in M$. The action of $\mathcal{O}^{>0}_M$ on $\mathcal{O}^{\geq 0}_M$ is free since $X^{\square}_P \subseteq X_P$ is dense (2.7.14). It follows that the forgetful map $\mathcal{O}^{\geq 0}_{M,p} \to \mathcal{Z}_{M,p}$ (2.6.17) has a section (**??**). Such a choice of section is equivalently the data of a germ

$$(M, p) \to (X_{\mathcal{Z}_{M,p}}, 0) \tag{2.7.23.1}$$

(where $0 \in X_{\mathcal{Z}_{M,p}}$ is the map $\mathcal{Z}_{M,p} \to \mathbb{R}_{\geq 0}$ sending everything other than the identity to zero) whose action on ghost sheaves (2.6.17) at the basepoint is the identity map of $\mathcal{Z}_{M,p}$. The target is stratified by the faces of $\mathcal{Z}_{M,p}$ (2.7.16), which determines by pullback a germ of stratification of M near p. This stratification is described more intrinsically as follows: a point q near p is sent to the face of $\mathcal{Z}_p$ consisting of those functions which do not vanish at q. It thus recovers the canonical local stratification of M by the face poset of $\mathcal{Z}_{M,p}$ (2.7.20). It hence induces an isomorphism on ghost sheaves, so is a strict log map (2.6.23).

Now let $f : M \to N$ be a map of log manifolds, and let $x \in M$. The map of sharp polyhedral cones $f_x^{\flat\flat} : \mathcal{Z}_{N,f(x)} \to \mathcal{Z}_{M,x}$ satisfies $(f_x^{\flat\flat})^{-1}(0) = 0$ (if $f_x^{\flat\flat}(z) = 0$, then $z(f(x)) = (f_x^{\flat\flat}z)(x) > 0$, implying $z = 0$) but need not be injective (2.7.21). For any point y in a neighborhood of x, we can consider the associated faces $F_y \subseteq \mathcal{Z}_{M,x}$ and $F_{f(y)} \subseteq \mathcal{Z}_{N,f(x)}$ (the functions which do not vanish at y and $f(y)$, respectively), which satisfy $(f_x^{\flat\flat})^{-1}(F_y) = F_{f(y)}$. The map $f_x^{\flat\flat}$ induces a map $\mathcal{Z}_{N,f(y)} = \mathcal{Z}_{N,f(x)}/F_{f(y)}^{\mathrm{gp}} \to \mathcal{Z}_{M,x}/F_x^{\mathrm{gp}} = \mathcal{Z}_{M,y}$ which is precisely $f_y^{\flat\flat}$ (compare (2.7.20)).

If $f_x^{\flat\flat}$ is injective, then in the diagram

$$\begin{tikzcd} \mathcal{O}_{M,x}^{\geq 0} \arrow[r] & \mathcal{Z}_{M,x} \\ \mathcal{O}_{N,f(x)}^{\geq 0} \arrow[r] \arrow[u, "f_x^{\flat}"] & \mathcal{Z}_{N,f(x)} \arrow[u, "f_x^{\flat\flat}"'] \end{tikzcd} \tag{2.7.23.2}$$

any section of $\mathcal{O}_{N,f(x)}^{\geq 0} \to \mathcal{Z}_{N,f(x)}$ extends to a section of $\mathcal{O}_{M,x}^{\geq 0} \to \mathcal{Z}_{M,x}$ (**??**). The result is a diagram of germs

$$\begin{tikzcd} (M,x) \arrow[r] \arrow[d, "f"'] & (X_{\mathcal{Z}_{M,x}}, 0) \arrow[d, "f_x^{\flat\flat}"] \\ (N,f(x)) \arrow[r] & (X_{\mathcal{Z}_{N,f(x)}}, 0) \end{tikzcd} \tag{2.7.23.3}$$

in which the horizontal maps are strict.

2.7.24 Exercise. Show that for a map of log smooth manifolds $f : X \to Y$ and a point $x \in X$, the following are equivalent:

(2.7.24.1) $f_x^{\flat\flat}$ is an isomorphism.
(2.7.24.2) $f_{x'}^{\flat\flat}$ is an isomorphism for all x' in a neighborhood of x.
(2.7.24.3) f is strict in a neighborhood of x.

⋆ **2.7.25 Definition** (Depth)**.** The *depth* of a log topological manifold M at a point x is the dimension of the sharp polyhedral cone $\mathcal{Z}_{M,x}$ (2.7.20). The depth of M itself is the maximum depth over all its points.

2.7.26 Example. The depth of X_P is the dimension of $P^{\#} = P/P_0$ where $P_0 \subseteq P$ denotes the minimal stratum.

2.7.27 Example.

(2.7.27.1) Depth 0 is equivalent to being locally modelled on open subsets of $\mathbb{R}^k$.

(2.7.27.2) Depth ≤ 1 is equivalent to being locally modelled on open subsets of $\mathbb{R}^k \times {}'\mathbb{R}_{\geq 0}$.
(2.7.27.3) Depth ≤ 2 is equivalent to being locally modelled on open subsets of $\mathbb{R}^k \times {}'\mathbb{R}^2_{\geq 0}$.
(2.7.27.4) In depth 3, there are infinitely many local models (indeed, there infinitely many isomorphism classes of sharp polyhedral cones of dimension three).

Decay conditions

It is sometimes useful to further restrict the allowable morphisms between the local models X_P (2.7.7) by imposing certain decay conditions in neighborhoods of strata. For example, in the case of asymptotically cylindrical structures (2.7.15), this means $o(1)$ is replaced by some stronger condition like $O(s^{-N})$ for every $N < \infty$. For any decay condition δ (2.7.28), we obtain a category of log topological manifolds with δ decay. Decay is most relevant in conjunction with log smoothness, which we will introduce after the discussion of decay (2.7.36).

⋆ **2.7.28 Definition** (Decay condition)**.** Consider a set δ of functions $\mathbb{R} \to \mathbb{R}_{\geq 0}$ with the following properties:
(2.7.28.1) $0 \in \delta$.
(2.7.28.2) If $f \in \delta$ then $f(x) \to 0$ as $x \to \infty$.
(2.7.28.3) If $f \in \delta$ and $g(x) \leq f(x)$ as $x \to \infty$, then $f \in \delta$.
(2.7.28.4) If $f \in \delta$, then $x \mapsto 2 \cdot \max_{2a+1 \geq x} f(a)$ is also in δ.
We call such a class of functions δ a *decay condition*. Note that condition (2.7.28.3) implies that membership of f in δ depends only on the germ of f at infinity, and hence we may (and frequently will) discuss membership in δ for functions which are only defined on sufficiently large arguments. Note that the axioms imply that if $f, g \in \delta$ then $f + g \in \delta$.

Examples of decay conditions include:
(2.7.28.5) $f(x) = o(1)$ as $x \to \infty$ (meaning $f(x) \to 0$ as $x \to \infty$).
(2.7.28.6) $f(x) = O(x^{-N})$ as $x \to \infty$ (meaning $f(x) \leq Mx^{-N}$ for some $M < \infty$ as $x \to \infty$).
(2.7.28.7) $f(x) = O(x^{-N})$ as $x \to \infty$ for all $N < \infty$ ('super-polynomial decay').
(2.7.28.8) $f(x) = O(e^{-\varepsilon x})$ as $x \to \infty$ for some $\varepsilon > 0$ ('exponential decay').
The condition that $f(x) = O(e^{-x})$ as $x \to \infty$ is not a decay condition, since it violates (2.7.28.4).

Given functions $f, g : W \to \mathbb{R}_{\geq 0}$ (any set W), we will write

$$f(w) = O(\delta(g(w)) \tag{2.7.28.9}$$

to mean that the function $x \mapsto \sup_{g(w) \geq x} f(w)$ lies in δ.

⋆ **2.7.29 Definition** (Log map with decay)**.** Let P be a polyhedral cone. We say that a continuous function $f : X_P \to \mathbb{R}$ (possibly defined on just an open subset) is of class $C^{0,1,\delta}$ for some decay condition δ (2.7.28) when it satisfies the following conditions:
(2.7.29.1) The restriction of f to every locally closed stratum $X_F^\square \subseteq X_P$ ($F \subseteq P$ a face) is locally Lipschitz in log coordinates (that is, every point of $X_F^\square$ has a neighborhood over

which $|f(x) - f(y)| \le M|x-y|$ for some $M < \infty$, where distance in $X_F^\square$ is measured using the vector space structure on $X_F^\square = (F^{\mathrm{gp}})^*$).

(2.7.29.2) We have
$$|f(x) - f(x_F)| = O(\delta(s_F(x)))$$
(in the sense of (2.7.28.9)) in a neighborhood of every point of $X_F^\square \subseteq X_P$ ($F \subseteq P$ a face), where $x \mapsto x_F$ denotes the pullback retraction $X_P \to X_F \subseteq X_P$ (concretely, take $x : P \to \mathbb{R}_{\ge 0}$ and modify it to be zero on the complement of F) and where $s_F : X_P \to \mathbb{R} \cup \{\infty\}$ is given by $-\log \max_i x(p_i)$ for some finite list $p_1, \ldots, p_r \in P \setminus F$ which generate P/F in the sense that the induced map $\mathbb{R}_{\ge 0}^r \to P/F$ is surjective (the function s_F is well defined up to commensurability, in the sense that any two such finite lists $p_1, \ldots, p_r$ and $p_1', \ldots, p_{r'}'$ give rise to functions s_F and s_F' satisfying $M^{-1}s_F \le s_F' \le Ms_F$ for some $M < \infty$, and hence the meaning of the bound on $|f(x) - f(x_F)|$ above is well defined (2.7.28.4)).

If $f_1, \ldots, f_n : X_P \to \mathbb{R}$ are of class $C^{0,1,\delta}$, then so is $F(f_1, \ldots, f_n)$ for any locally Lipschitz function $F : \mathbb{R}^n \to \mathbb{R}$; in particular, we may declare a map from X_P to a smooth manifold to be of class $C^{0,1,\delta}$ when it is so with respect to some covering family of (equivalently, all) local coordinate charts on the target. A log map $X_P \to X_Q$ is said to be of class $C^{0,1,\delta}$ iff when we express it (locally) as the product (2.7.22) of a rational monomial map $X_P \to X_Q$ and a map $X_P \to X_Q^\square$, the latter is of class $C^{0,1,\delta}$ (this is independent of the choice of decomposition since every rational monomial map $f : X_P \to X_Q^\square$ is of class $C^{0,1,\delta}$ since $f(x) = f(x_F)$ for every face $F \subseteq P$ in the domain of f).

2.7.30 Lemma. *Log maps of class $C^{0,1,\delta}$ are closed under composition.*

Proof. Let maps $X_P \to X_Q \to X_R$ be of class $C^{0,1,\delta}$, and let us show their composition is as well. The map $X_Q \to X_R$ is the product of a monomial map $X_Q \to X_R$ and a $C^{0,1,\delta}$ map $X_Q \to X_R^\square = (R^{\mathrm{gp}})^*$. It suffices to show that the composition of $X_P \to X_Q$ with each of these maps is of class $C^{0,1,\delta}$ (indeed, a product of $C^{0,1,\delta}$ maps is $C^{0,1,\delta}$). The composition of a $C^{0,1,\delta}$ map $X_P \to X_Q$ with a monomial map $X_Q \to X_R$ is of class $C^{0,1,\delta}$ by inspection (post-composition with a monomial map out of X_Q preserves factorization of $X_P \to X_Q$ into a monomial map $X_P \to X_Q$ and a $C^{0,1,\delta}$ map $X_P \to X_Q^\square = (Q^{\mathrm{gp}})^*$). To show that a composition of $C^{0,1,\delta}$ maps $X_P \to X_Q \to X_R^\square$ is again $C^{0,1,\delta}$, note that we may assume the target $X_R^\square = (R^{\mathrm{gp}})^*$ is just $\mathbb{R}$ (treat each linear coordinate separately).

We have thus formally reduced to the case of a composition $X_P \to X_Q \to \mathbb{R}$, which we will now treat by explicit computation. Denote by $f : X_Q \to \mathbb{R}$ the second map, and express the first map $X_P \to X_Q$ as the product of the monomial map $\mu^* : X_P \to X_Q$ associated to a map of polyhedral cones $\mu : Q \to P$ and a map $g : X_P \to X_Q^\square$. We assume f and g are of class $C^{0,1,\delta}$, and we wish to show that $f \circ (g\mu^*)$ is as well. The stratum-wise Lipschitz property (2.7.29.1) for the composition $f \circ (g\mu^*)$ follows immediately from the same property of f and g. Now let us show that the composition $f \circ (g\mu^*)$ satisfies the decay property (2.7.29.2). Let $F \subseteq P$ be a face. To bound $|f(g(x)\mu^*(x)) - f(g(x_F)\mu^*(x_F))|$, apply the triangle inequality with the third point $f((g(x)\mu^*(x))_{\mu^{-1}F})$. The quantity $|f(g(x)\mu^*(x)) - f((g(x)\mu^*(x))_{\mu^{-1}F})|$ is bounded by $O(\delta(s_{\mu^{-1}F}(g(x)\mu^*(x))))$ by the decay property (2.7.29.2) of f. Now the multiplicative factor of $g(x)$ becomes an additive perturbation after applying the logarithms involved in $s_{\mu^{-1}F}$, which is then annihilated by δ by (2.7.28.4), giving a bound of $O(\delta(s_{\mu^{-1}F}(\mu^*(x))))$. We have $s_{\mu^{-1}F}(\mu^*(x)) \geq s_F(x)$ (up to multiplicative constant), so we obtain the desired bound of $O(\delta(s_F(x)))$. To bound $|f(g(x_F)\mu^*(x_F)) - f((g(x)\mu^*(x))_{\mu^{-1}F})|$, note that both arguments of f lie in the stratum $X^\square_{\mu^{-1}F} \subseteq X_Q$, so since f is stratum-wise locally Lipschitz (2.7.29.1), this quantity is bounded by a constant times the distance $|g(x_F)\mu^*(x_F) - (g(x)\mu^*(x))_{\mu^{-1}F}| = |(g(x_F) - g(x))\mu^*(x_F)|$. The factor $\mu^*(x_F)$ has no effect since we are measuring distance in log coordinates $X^\square_{\mu^{-1}F} = ((\mu^{-1}F)^{\mathrm{gp}})^*$, so we are left with $|g(x_F) - g(x)|$, which is $O(\delta(s_F(x)))$ as desired since g is $C^{0,1,\delta}$. $\square$

Log smoothness

Our next major topic is differentiability on log manifolds. This discussion is a reformulation of Melrose's notions of b-tangent bundle, b-differential operators, etc.

⋆ **2.7.31 Definition** (Tangent space of X_P)**.** The tangent bundle $TX_P \to X_P$ is the trivial vector bundle with fiber $(P^{\mathrm{gp}})^*$. Over the non-degenerate locus of X_P (which is a smooth manifold), we identify this with the tangent bundle in the usual sense using log coordinates $X^\square_P = (P^{\mathrm{gp}})^*$.

2.7.32 Example. The vector field $x\partial_x$ is an everywhere non-vanishing section of $T'\mathbb{R}_{\geq 0}$. More generally, $x_1\partial_{x_1}, \ldots, x_n\partial_{x_n}$ is a basis for $T'\mathbb{R}^n_{\geq 0}$. A basis for the tangent space of $U \times {}'\mathbb{R}_{\geq 0}$ ($U \subseteq \mathbb{R}^n$ open) is given, in log coordinates $x = e^s$ on ${}'\mathbb{R}_{\geq 0}$, by $\partial_{u_1}, \ldots, \partial_{u_n}, \partial_s$.

⋆ **2.7.33 Definition** (Cotangent cone of X_P)**.** The *cotangent cone* of X_P at a point x is the polyhedral cone $T^{\circledast}_x X_P = P + F^{\mathrm{gp}}_x$, whose groupification is the *cotangent space* $T^*_x X_P$.

2.7.34 Example. A general section of the cotangent cone of $T^{\circledast\prime}\mathbb{R}^n_{\geq 0}$ takes the form $\sum_i a_i(x)\frac{dx_i}{x_i}$ where $a_i(x) \geq 0$ over the locus where $x_i = 0$.

⋆ **2.7.35 Definition** (Log (co)tangent short exact sequence)**.** The short exact sequence (2.7.20.1) associated to a point $x \in X_P$ can be viewed as a sequence of cotangent cones

$$0 \to T^*_x X_{F_x} = T^{\circledast}_x X_{F_x} \to T^{\circledast}_x X_P \to \mathcal{Z}_x \to 0. \tag{2.7.35.1}$$

Dualizing gives a short exact sequence of tangent spaces

$$0 \to (\mathcal{Z}^{\mathrm{gp}}_x)^* \to T_x X_P \to T_x X_{F_x} \to 0. \tag{2.7.35.2}$$

Note that the direction of these maps is opposite to the situation of a manifold-with-boundary: a tangent vector to X_P at x determines a tangent vector to the stratum $X_{F_x} \subseteq X_P$ containing x, rather than the other way around.

2.7.36 Definition (Differentiability of maps $X_P \to X_Q$)**.** A map $f : X_P \to \mathbb{R}_{>0}$ is said to be differentiable at $x \in X_P$ when its restriction to $X^\square_{F_x} = (F^{\mathrm{gp}}_x)^*$ is differentiable at

x. The derivative of f at x is then the composition of the map $T_xX_P \to T_xX_{F_x}^\square$ (namely $(P^{\mathrm{gp}})^* \to (F_x^{\mathrm{gp}})^*$) with the derivative of f restricted to $X_{F_x}^\square$ at x; dually, this is an element of $T_x^*F_x = F_x^{\mathrm{gp}} \subseteq P + F_x^{\mathrm{gp}} = T_x^\circledast X_P$. We say f is continuously differentiable when its derivative $df : X_P \to T^\circledast X_P$ is continuous.

A map $f : X_P \to {}'\mathbb{R}_{\geq 0}$ is said to be differentiable at $x \in X_P$ when $f = p \cdot g$ (near x) for $p \in P$ and $g : X_P \to \mathbb{R}_{>0}$ differentiable at x. The derivative of f at x is the sum of $p \in P \subseteq P + F_x^{\mathrm{gp}} = T_x^\circledast X_P$ and the derivative of g at x; this is independent of the choice of decomposition $f = p \cdot g$.

A map $f : X_P \to X_Q$ is said to be differentiable at $x \in X_P$ iff for every $q \in Q$, the composite $q \circ f : X_P \to {}'\mathbb{R}_{\geq 0}$ is differentiable at x. In this case, the induced map $Q \to T_x^\circledast X_P$ sends $F_{f(x)}$ to invertible elements $F_x^{\mathrm{gp}} \subseteq P + F_x^{\mathrm{gp}} = T_x^\circledast X_P$ since for $q \in F_{f(x)}$ the composite $q \circ f$ is nonzero at x. The map $Q \to T_x^\circledast X_P$ thus extends to $Q + F_{f(x)}^{\mathrm{gp}} = T_{f(x)}^\circledast X_Q$, and the resulting map $T_x^\circledast f : T_{f(x)}^\circledast X_Q \to T_x^\circledast X_P$ is called the derivative of f at x. This derivative respects the short exact sequences (2.7.35), namely the following diagram commutes.

$$\begin{tikzcd} 0 \arrow[r] & T_x^* X_{F_x} \arrow[r] & T_x^\circledast X_P \arrow[r] & \mathcal{Z}_x \arrow[r] & 0 \\ 0 \arrow[r] & T_{f(x)}^* X_{F_{f(x)}} \arrow[r] \arrow[u, "T_x^*(f|_{X_{F_x}})"] & T_{f(x)}^\circledast X_Q \arrow[r] \arrow[u, "T_x^\circledast f"] & \mathcal{Z}_{f(x)} \arrow[r] \arrow[u, "f_x^{\flat\flat}"] & 0 \end{tikzcd} \tag{2.7.36.1}$$

A map $f : X_P \to X_Q$ is called *continuously differentiable* iff it is differentiable at every point and the map $Tf : TX_P \to TX_Q$ given on each fiber by the derivative is continuous. In this case, Tf is a log map since $TX_P \to X_P$ is strict. The adjectives 'k times continuously differentiable' ('class C^k') and 'smooth' ('class C^∞') are now defined by iterating T (where we regard TX_P as a real affine toric variety in the evident way $TX_P = X_P \times (P^{\mathrm{gp}})^* = X_P \times X_{P^{\mathrm{gp}}} = X_{P \oplus P^{\mathrm{gp}}}$).

A map $X_P \to X_Q$ is said to be of class $C^{k,1,\delta}$ for $k \geq 1$ when it is continuously differentiable and its derivative is of class $C^{k-1,1,\delta}$ (where the base case $C^{0,1,\delta}$ is (2.7.29)). A map is said to be of class $C^{\infty,\delta}$ ('smooth with δ decay') when it is of class $C^{k,1,\delta}$ for all $k < \infty$ (note that the stratum-wise Lipschitz property (2.7.29.1) is redundant in this case, since it is implied by continuous differentiability).

2.7.37 Example. The derivative of a monomial map $X_P \to X_Q$ is the map $(P^{\mathrm{gp}})^* \to (Q^{\mathrm{gp}})^*$ induced by $Q \to P$. In particular, the derivative of a monomial map is again a monomial map; hence monomial maps are smooth.

2.7.38 Example. The diagonal map $\mathbb{R}_{\geq 0} \to \mathbb{R}^2_{\geq 0}$ of polyhedral cones induces the monomial map

$$f : {}'\mathbb{R}^2_{\geq 0} \to {}'\mathbb{R}_{\geq 0} \tag{2.7.38.1}$$
$$(x, y) \mapsto xy = \lambda \tag{2.7.38.2}$$

which is thus a log smooth map. The vector fields $\{x\partial_x, y\partial_y\}$ and $\lambda\partial_\lambda$ form bases of the tangent spaces of the source and target, respectively (note that, in particular, these vector fields are *nonzero* even where $x = 0$, $y = 0$, or $\lambda = 0$). The derivative of f sends $x\partial_x \mapsto \lambda\partial_\lambda$ and $y\partial_y \mapsto \lambda\partial_\lambda$. Its kernel (i.e. the vertical tangent bundle) is thus of constant rank one, spanned by $x\partial_x - y\partial_y$. Lifting $\lambda\partial_\lambda$ to $\frac{1}{2}(x\partial_x + y\partial_y)$ may be viewed as defining a 'connection'.

2.7.39 Example. Write $f : X_P \to X_Q$ near $x \in X_P$ as $f = f_{\mathrm{rat}} \cdot g$ as in (2.7.22). There is a unique such decomposition for which the derivative of $g|_{X^{\square}_{F_x}}$ at x vanishes. For this decomposition, f_{rat} is the rational monomial map associated to the derivative $T^{\circledast}_x f : Q + F^{\mathrm{gp}}_{f(x)} \to P + F^{\mathrm{gp}}_x$ at x.

2.7.40 Lemma (Chain Rule)**.** *If $f : X_P \to X_Q$ is differentiable at x and $g : X_Q \to X_R$ is differentiable at $f(x)$, then their composition $X_P \to X_R$ is differentiable at x, and its derivative is the composition of the derivatives of f and g.*

Proof. By the definition of differentiability of maps with target X_R, it suffices to treat the case $X_R = {}'\mathbb{R}_{\geq 0}$. We are thus in the situation of a pair of maps $f : X_P \to X_Q$ and $g : X_Q \to {}'\mathbb{R}_{\geq 0}$. By the definition of $T^{\circledast}_{f(x)} g$ as a sum, we are reduced to two cases, namely $g(f(x)) > 0$ or $g \in Q$. When $g \in Q$, the relation $T^{\circledast}(g \circ f) = T^{\circledast} g \circ T^{\circledast} f$ is the definition of $T^{\circledast} f$. When $g(f(x)) > 0$, we are reduced to the chain rule for the restrictions $f|_{X_{F_x}} : X_{F_x} \to X_{F_{f(x)}}$ and $g|_{X_{F_{f(x)}}}$. □

⋆ **2.7.41 Definition** (Log smooth manifold)**.** A *log smooth manifold* is a log topological space equipped with an atlas of charts from open subsets of various X_P whose transition functions are smooth. The category of log smooth manifolds and smooth maps is denoted LogSm.

2.7.42 Exercise (Log smooth manifolds via the structure sheaf)**.** For any log smooth manifold M, let $\mathcal{A}^{\geq 0}_M \subseteq \mathcal{O}^{\geq 0}_M$ denote the subsheaf of functions to ${}'\mathbb{R}_{\geq 0}$ which are smooth. Prove that $X_P \to X_Q$ is smooth iff it pulls back functions in $\mathcal{A}^{\geq 0}_{X_Q}$ to functions in $\mathcal{A}^{\geq 0}_{X_P}$. Conclude that a log smooth manifold is equivalently a log topological space M with a subsheaf $\mathcal{A}^{\geq 0}_M \subseteq \mathcal{O}^{\geq 0}_M$ which is locally isomorphic to $(X_P, \mathcal{A}^{\geq 0}_{X_P})$. Conclude that a log smooth manifold is also a topological space M with a subsheaf $\mathcal{A}^{\geq 0}_M \subseteq C^{\geq 0}_M$ which is locally isomorphic to $(X_P, \mathcal{A}^{\geq 0}_{X_P})$.

2.7.43 Exercise (Normalization)**.** For a log smooth manifold X, define a map of topological spaces $\tilde{X} \to X$ by taking $(X_P)^\sim = \bigsqcup_{F \subseteq P} X_F$ (mapping to X_P by 'extension by zero' (2.7.16)) and gluing. Equip $\tilde{X}$ with the log structure given by the subsheaf $\mathcal{O}^{\geq 0}_{\tilde{X}} \subseteq \mathrm{im}(\mathcal{O}^{\geq 0}_{\tilde{X}} \to C^{\geq 0}_{\tilde{X}})$ of functions whose zero set is nowhere dense, and show that $(X_P)^\sim = \bigsqcup_{F \subseteq P} X_F$ as log topological spaces (note that the map of topological spaces $\tilde{X} \to X$ is *not* a log map). Make the same definition with log smooth functions (2.7.42), and show that $(X_P)^\sim = \bigsqcup_{F \subseteq P} X_F$ as log smooth manifolds, hence $\tilde{X}$ is a log smooth manifold for all X. Finally, show that $X \mapsto \tilde{X}$ is a functor from the category of log smooth manifolds to itself (first argue it is a functor to log topological manifolds, and then show that the maps are log smooth by inspecting the rings of log smooth functions).

⋆ **2.7.44 Definition** (Log (co)tangent short exact sequence)**.** In view of the functoriality of the log (co)tangent short exact sequences (2.7.36.1), they make sense on log smooth manifolds. That is, for any point x of a log smooth manifold M, there is a short exact sequence

$$0 \to T^*_x M_x \to T^{\circledast}_x M \to \mathcal{Z}_{M,x} \to 0, \tag{2.7.44.1}$$

where $M_x \subseteq M$ denotes the local stratum containing x. A smooth map of log smooth manifolds $f : M \to N$ induces a map of such short exact sequences.

$$\begin{tikzcd} 0 \arrow[r] & T^*_x M_x \arrow[r] & T^{\circledast}_x M \arrow[r] & \mathcal{Z}_{M,x} \arrow[r] & 0 \\ 0 \arrow[r] & T^*_{f(x)} N_{f(x)} \arrow[r] \arrow[u, "T^*_x(f|_{M_x})"] & T^{\circledast}_{f(x)} N \arrow[r] \arrow[u, "T^{\circledast}_x f"] & \mathcal{Z}_{N,f(x)} \arrow[r] \arrow[u, "f^{\flat\flat}_x"] & 0 \end{tikzcd} \tag{2.7.44.2}$$

Recall that the map $f^{\flat\flat}_x$ determines $f^{\flat\flat}_y$ for all y in a neighborhood of x (2.7.23).

2.7.45 Definition. There is an evident short exact sequence

$$0 \to \mathcal{A}^{>0}_M \to \mathcal{A}^{\geq 0}_M \to \mathcal{Z}_M \to 0 \tag{2.7.45.1}$$

for log smooth manifolds M, which is functorial under log smooth maps $f : M \to N$. By taking logarithmic derivatives, this sequence maps to the log cotangent short exact sequence (2.7.44), resulting in a diagram with exact rows and columns.

$$\begin{tikzcd} \ker(\mathcal{A}^{>0}_{M,x} \to T^*_x M_x) \arrow[r, equal] \arrow[d] & \ker(\mathcal{A}^{\geq 0}_{M,x} \to T^{\circledast}_x M) \arrow[d] & \\ \mathcal{A}^{>0}_{M,x} \arrow[r] \arrow[d] & \mathcal{A}^{\geq 0}_{M,x} \arrow[r] \arrow[d] & \mathcal{Z}_{M,x} \arrow[d, equal] \\ T^*_x M_x \arrow[r] & T^{\circledast}_x M \arrow[r] & \mathcal{Z}_{M,x} \end{tikzcd} \tag{2.7.45.2}$$

In particular, we have a short exact sequence

$$0 \to \ker(\mathcal{A}^{>0}_{M,x} \to T^*_x M_x) \to \mathcal{A}^{\geq 0}_{M,x} \to T^{\circledast}_x M \to 0 \tag{2.7.45.3}$$

expressing $T^{\circledast}_x M$ as germs of smooth functions to ${}'\mathbb{R}_{\geq 0}$ modulo those functions with vanishing derivative at x.

⋆ **2.7.46 Exercise** (Asymptotically cylindrical structures as log structures)**.** As a continuation of (2.7.15), show that a log map $U \times {}'\mathbb{R}_{\geq 0} \to V \times {}'\mathbb{R}_{\geq 0}$ of class C^k takes the form

$$(u, s) \mapsto (f(u), a \cdot s + b(u)) + o(1)_{C^k} \tag{2.7.46.1}$$

for $f \in C^k$, $b \in C^k$, and $o(1)_{C^k}$ indicating a function of class C^k whose derivatives of order up to k approach zero as $s \to -\infty$, uniformly over compact subsets of U.

2.7.47 Exercise. Show that the map ${}'\mathbb{R}_{\geq 0} \to \mathbb{R}$ given by $x \mapsto x$ is smooth.

2.7.48 Exercise (Smooth functions ${}'\mathbb{R}^2_{\geq 0} \to \mathbb{R}$)**.** Show that a function ${}'\mathbb{R}^2_{\geq 0} \to \mathbb{R}$ is smooth iff it is given in log coordinates $(x, y) = (e^s, e^t)$ by $(s, t) \mapsto a + b(s) + c(t) + o(1)_{C^\infty}$ as $\min(s, t) \to -\infty$ (uniformly over sets on which $\max(s, t)$ is bounded) for $a \in \mathbb{R}$, $b(s) = o(1)_{C^\infty}$ as $s \to -\infty$, and similarly for $c(t)$.

⋆ **2.7.49 Exercise** (Real blow-up of a normal crossings divisor)**.** Consider the category of *complex manifolds equipped with a normal crossings divisor*, denoted by $\mathsf{Cpx}^{\mathsf{ncd}}$. An object $(X, D) \in \mathsf{Cpx}^{\mathsf{ncd}}$ consists of a complex analytic manifold X and a subset $D \subseteq X$ locally isomorphic to $\{z_1 \cdot \cdots \cdot z_m = 0\} \subseteq \mathbb{C}^n$ for some $n \geq m \geq 0$. A morphism $(X, D) \to (X', D')$ is a complex analytic map $f : X \to X'$ with $f(X \setminus D) \subseteq X' \setminus D'$ (that is, $f^{-1}(D') \subseteq D$). Define a log structure on a complex (analytic) manifold X to consist of a sheaf of monoids $\mathcal{O}_X^{\log}$ with a map $\mathcal{O}_X^{\log} \to \mathcal{O}_X$ (the sheaf of analytic functions to $\mathbb{C}$, regarded as a monoid under multiplication) which is an isomorphism over $\mathcal{O}_X^\times \subseteq \mathcal{O}_X$ (the subsheaf of invertible functions); given this basic definition, the rest of the framework of log structures on topological spaces (2.6.3) is extended to the setting of complex manifolds without change, in particular yielding a category of log complex manifolds LogCpx. Show that there is a fully faithful functor $\mathsf{Cpx}^{\mathsf{ncd}} \to \mathsf{LogCpx}$ which associates to (X, D) the log structure $\mathcal{O}_{(X,D)}^{\log} \subseteq \mathcal{O}_X$ given by the sheaf of morphisms from (X, D) to $(\mathbb{C}, 0)$ in $\mathsf{Cpx}^{\mathsf{ncd}}$.

Now consider the standard real blow-up model:

$$ {}'\mathbb{R}_{\geq 0} \times S^1 \to \mathbb{C} \tag{2.7.49.1} $$
$$ (r, \theta) \mapsto re^{i\theta} \tag{2.7.49.2} $$

Show that this map is log smooth. Show that every map $(\mathbb{C}^n, \{z_1 \cdot \cdots \cdot z_m = 0\}) \to (\mathbb{C}, 0)$ in $\mathsf{Cpx}^{\mathsf{ncd}}$ lifts uniquely to a log smooth map $({}'\mathbb{R}_{\geq 0} \times S^1)^m \times \mathbb{C}^{n-m} \to {}'\mathbb{R}_{\geq 0} \times S^1$ (express the morphism in question as a product $z_1^{a_1} \cdot \cdots \cdot z_m^{a_m} g(z_1, \ldots, z_n)$ for integers $a_1, \ldots, a_m \geq 0$, and use the fact that compositions of log smooth maps are log smooth). Conclude that every map $(\mathbb{C}^n, \{z_1 \cdot \cdots \cdot z_m = 0\}) \to (\mathbb{C}^{n'}, \{z_1 \cdot \cdots \cdot z_{m'} = 0\})$ in $\mathsf{Cpx}^{\mathsf{ncd}}$ lifts uniquely to a log smooth map $({}'\mathbb{R}_{\geq 0} \times S^1)^m \times \mathbb{C}^{n-m} \to ({}'\mathbb{R}_{\geq 0} \times S^1)^{m'} \times \mathbb{C}^{n'-m'}$. Conclude that this recipe on local charts defines a functor $\mathsf{Cpx}^{\mathsf{ncd}} \to \mathsf{LogSm}$, which we call the *real blow-up* functor.

⋆ **2.7.50 Lemma.** *A log smooth manifold has local bump functions (hence, if paracompact Hausdorff, partitions of unity* (2.1.51)*).*

Proof. It suffices to consider the case of a local model X_P. A surjection $\mathbb{R}^n_{\geq 0} \to P$ determines a closed embedding $X_P \hookrightarrow {}'\mathbb{R}^n_{\geq 0}$ (2.7.9) which is smooth (2.7.37), so it suffices to exhibit bump functions on ${}'\mathbb{R}^n_{\geq 0}$. The tautological inclusion map ${}'\mathbb{R}^n_{\geq 0} \to \mathbb{R}^n$ is also smooth (2.7.47), so we reduce further to the case of $\mathbb{R}^n$, which was treated earlier (2.4.14). □

2.7.51 Lemma. *Let M be a log smooth manifold. A smooth function to $\mathbb{R}$ defined on a closed union of strata of M extends locally to M (hence also globally if M is paracompact Hausdorff).*

Proof. Global extension follows from local extension given partitions of unity (2.7.50), so we focus on local extension.

For the local extension question, it suffices (by induction on strata) to treat extension from the ideal locus $X_P^\infty = X_P \setminus X_P^\square$ to X_P. Let $f : X_P \setminus X_P^\square \to \mathbb{R}$ be log smooth, and consider the function $\bar{f} : X_P \to \mathbb{R}$ given by

$$\bar{f}(x) = \sum_{F \subsetneq P} (-1)^{\dim P - \dim F - 1} f(x_F) \tag{2.7.51.1}$$

where $x_F \in X_F \subseteq X_P$ denotes the restriction of x to $F \subseteq P$. Since $x \mapsto x_F$ is a monomial map, it is log smooth, so each function $(x \mapsto f(x_F)) : X_P \to \mathbb{R}$ is log smooth, hence so is their alternating sum $\bar{f}$. It remains to show that $\bar{f}|_{X_P \setminus X_P^\square} = f$. To check that $f - \bar{f} = 0$ over a proper face $G \subsetneq P$, it suffices to show that the formal sum of faces $\sum_{F \subseteq P} (-1)^{\dim P - \dim F}[G \cap F]$ vanishes, which we verify in (2.7.53) below. □

2.7.52 Lemma. *For any polyhedral cone P, we have $\sum_{F \subseteq P} (-1)^{\dim F} = 0$ provided P has at least one proper face.*

Proof. The compactly supported Euler characteristic $\chi_c(X) = \sum_i (-1)^i \dim H^i_c(X)$ (defined whenever $H^*_c(X)$ is finite-dimensional) is additive under open-closed decompositions: $\chi_c(X) = \chi_c(Z) + \chi_c(X \setminus Z)$ for $Z \subseteq X$ closed, by the long exact sequence in cohomology (provided it is defined for any two of X, Z, and $X \setminus Z$, which implies it is defined for the third). Additivity of the compactly supported Euler characteristic implies that $\chi_c(P) = \sum_{F \subseteq P} \chi_c(F^\circ)$. It thus suffices to show that $\chi_c(P^\circ) = (-1)^{\dim P}$ (always) and that $\chi_c(P) = 0$ if P has at least one proper face.

To see that $\chi_c(P^\circ) = (-1)^{\dim P}$, note that P° is a non-empty convex open subset of the vector space P^{gp}, and as such it is homeomorphic to P^{gp} (**??**), which has compactly supported cohomology $H^*_c(P^{\mathrm{gp}})$ free of rank one concentrated in degree $\dim P$.

To see that $\chi_c(P) = 0$, it suffices to construct a proper map $H : P \times \mathbb{R}_{\geq 0} \to P$ which is the identity on $P \times 0$ (this implies $H^*_c(P) = 0$). If P has at least one proper face, such a map may be given by $H(x,t) = x + tz$ for any fixed $z \in P^\circ$. Properness of H follows from the fact that z is in the interior of P and P is contained in a half-space inside P^{gp}. □

2.7.53 Corollary. *Let P be a polyhedral cone, and let $A \subseteq G \subsetneq P$ be faces. We have $\sum_{\substack{F \subseteq P \\ F \cap G = A}} (-1)^{\dim F} = 0$.*

Proof. We proceed by induction on $\dim G - \dim A$. Note that the result for (A, G, P) is equivalent to the result for $(0, G/A, P/A)$, so we may assume wlog that $A = 0$. The summation of $(-1)^{\dim F}$ over all the faces $F \subseteq P$ is zero by (2.7.52) (note P has a proper face since $G \subsetneq P$ by assumption). Now let us partition this sum over $F \subseteq P$ according to the intersection $F \cap G$. The sum over those F with intersection $F \cap G = A' \subseteq G$ for any fixed nonzero $A' \subseteq G$ vanishes by the induction hypothesis. The sum over those F with zero intersection $F \cap G = 0$ thus also vanishes, as desired. □

⋆ **2.7.54 Definition** (Averaging on a log manifold)**.** Let M be a paracompact Hausdorff log smooth manifold, and let $\mathrm{Meas}(M)$ denote the set of positive measures on M of unit total mass. As in (2.4.15), we seek to construct an 'averaging' operation

$$\mathrm{avg} : \mathrm{Meas}(M) \to M \tag{2.7.54.1}$$

on the set of measures of 'sufficiently small' support.

Let us call an open set $U \subseteq X_P$ *convex* when its intersection with every stratum $X_F^\square \subseteq X_P$ (for faces $F \subseteq P$) is convex in log coordinates $X_F^\square = (F^{\mathrm{gp}})^*$. For a measure μ on U supported inside $U \cap X_F^\square$ for some face $F \subseteq P$, the average $\mathrm{avg}_U(\mu) \in U$ is defined via the linear structure on $(F^{\mathrm{gp}})^* = X_F^\square$. Given a smooth function of compact support $\eta : U \to [0,1]$, we may define the cutoff average $\mathrm{avg}_{U,\eta}$ as in (2.4.15.2).

Now consider a cover $M = \bigcup_i U_i$ by open sets identified with convex open sets $U_i \subseteq X_P$. To see that such an open cover exists, it suffices to show that every point of X_P has arbitrarily small convex open neighborhoods. By embedding $X_P \hookrightarrow {}'\mathbb{R}^n_{\geq 0}$ via a surjection $\mathbb{R}^n_{\geq 0} \twoheadrightarrow P$ (2.7.10), we may reduce to the case of points of ${}'\mathbb{R}^n_{\geq 0}$ and hence, since convexity is preserved by products, to the case of ${}'\mathbb{R}_{\geq 0}$ where the result is obvious. We may now follow the manifold case (2.4.15) and define the global average (2.7.54.1) as a composition of local averages $\mathrm{avg}_{U_i,\eta_i}$. This composition is defined on measures of 'sufficiently small' support, which now means, for some fine open cover $M = \bigcup_i U_i$ by convex open sets $U_i \subseteq X_{P_i}$, that the support of μ is contained in $U_i \cap X_F^\square$ for some i and some face $F \subseteq P_i$.

The averaging map avg is smooth in the following sense. Consider families of measures $N \to \mathrm{Meas}(M)$ parameterized by a log smooth manifold N which, locally on N, are of the form of a finite sum $\sum_i w_i \delta_{p_i}$ for some smooth functions $w_i : N \to [0,1]$ and $p_i : N \to M$. Now each local averaging operation, hence also the global averaging operation, preserves families of this form (inspection).

Log inverse function theorem

We now discuss the inverse function theorem for log smooth manifolds. Recall that each point of a log smooth manifold has a cotangent polyhedral cone (2.7.33), which carries more information than its groupification the cotangent space. For this reason, the correct statement of the inverse function theorem involves cotangent cones rather than (co)tangent spaces.

2.7.55 Exercise (Failure of naive log inverse function theorem)**.** Consider the map $\mathbb{R}^2_{\geq 0} \to \mathbb{R}^2_{\geq 0}$ given by $f(x,y) = (x^2y, xy^2)$. Show that f induces an isomorphism on tangent spaces yet is not a local homeomorphism (in fact, it is not even open).

2.7.56 Exercise (Log inverse function theorem hypothesis)**.** Use the log cotangent short exact sequence (2.7.44) to show that for $f : X \to Y$ a smooth map of log smooth manifolds, the following are equivalent:

(2.7.56.1) $T^\circledast_x f : T^\circledast_{f(x)} Y \to T^\circledast_x X$ is an isomorphism (of polyhedral cones).

(2.7.56.2) $f_x^{\flat\flat} : \mathcal{Z}_{Y,f(x)} \to \mathcal{Z}_{X,x}$ and $T_x f : T_x X \to T_{f(x)} Y$ are isomorphisms.

(2.7.56.3) $f_x^{\flat\flat}:\mathcal{Z}_{Y,f(x)}\to\mathcal{Z}_{X,x}$ and $T_xf|_{X_x}:T_xX_x\to T_{f(x)}Y_{f(x)}$ are isomorphisms.
Use (2.7.24) to observe that these conditions are open.

⋆ **2.7.57 Log Inverse Function Theorem.** *Let $f:M\to N$ be C^k for $k\geq 1$. If $T_x^{\circledast}f$ is an isomorphism of polyhedral cones, then f is a local log homeomorphism at x, its local inverse is also C^k, and $T(f^{-1})=(Tf)^{-1}$.*

Proof. We first treat the case $k=1$ and then deduce the general case by induction.

The assertion is local, so we may consider a C^1 map $f:(X_P,p)\to(X_Q,f(p))$ of real affine toric varieties for which $T_p^{\circledast}f$ is an isomorphism of polyhedral cones. By replacing P with $P+F_p^{\mathrm{gp}}$ and replacing Q with $Q+F_{f(p)}^{\mathrm{gp}}$, we may assume wlog that p and $f(p)$ are on the minimal strata of X_P and X_Q, respectively. We thus have $T_p^{\circledast}X_P=P$ and $T_{f(p)}^{\circledast}X_Q=Q$, so the derivative of f at p is a map $Q\to P$, which we have assumed to be an isomorphism. Identifying Q with P via this isomorphism and translating so that $p=0\in X_P$ and $f(p)=0\in X_Q$, our map now takes the form

$$f:(X_P,0)\to(X_P,0) \tag{2.7.57.1}$$
$$x\mapsto u(x)x \tag{2.7.57.2}$$

for some C^1 map $u:X_P\to X_P^{\square}$ whose derivative vanishes at $0\in X_P$ (compare (2.7.39)).

Consider $u:X_P\to X_P^{\square}$ in log coordinates $X_P^{\square}=(P^{\mathrm{gp}})^*$ (2.7.13) on (the non-degenerate locus of) the source and target. The first derivative of u in such coordinates approaches zero as $x\to 0\in X_P$; that is, we have

$$u(x)=\mathrm{const}+o(1)_{C^1} \tag{2.7.57.3}$$

in log coordinates in the limit $x\to 0\in X_P$. Now the key point is simply that every map of the form $\mathbf{1}+\mathrm{const}+o(1)_{C^1}$ on $\mathbb{R}^n$ has an inverse of the same form. Thus f is a diffeomorphism from each stratum $X_F^{\square}\subseteq X_P$ (faces $F\subseteq P$) to itself, in a neighborhood of $0\in X_P$. In particular, f is a continuous bijection in a neighborhood of zero, which implies it is a local homeomorphism there since X_P is locally compact Hausdorff. Now $f^{\flat\flat}$ is an isomorphism in a neighborhood of 0 (2.7.56), which implies f is strict (2.6.23), hence is an isomorphism of log topological spaces.

Now let us show that the inverse f^{-1} is continuously differentiable with derivative $T(f^{-1})=(Tf)^{-1}$. Note that Tf is an isomorphism of vector bundles covering an isomorphism of log topological spaces, so it is itself an isomorphism of log topological spaces, hence has an inverse $(Tf)^{-1}$. It thus suffices to show that f^{-1} is differentiable with derivative $(Tf)^{-1}$ at any given point. We may wlog just treat the case of the basepoint $0\in X_P$ itself, where the desired assertion follows from the fact that f^{-1} has the form $\mathbf{1}+\mathrm{const}+o(1)_{C^1}$ on each stratum in log coordinates. We have thus proven the case $k=1$.

We may now derive the case of general $k\geq 1$ from the case $k=1$ using induction. Suppose f is C^k and $T_p^{\circledast}f$ is an isomorphism of polyhedral cones. Since $k\geq 1$, the inverse f^{-1} exists and is C^1 with derivative $T(f^{-1})=(Tf)^{-1}$. We wish to show that f^{-1} is C^k, equivalently that $T(f^{-1})=(Tf)^{-1}$ is C^{k-1}. This follows from the induction hypothesis and the fact that

Tf is C^{k-1}, provided we show that the derivative of Tf is an isomorphism of polyhedral cones. The cotangent cone of TM along the zero section $M \subseteq TM$ is the direct sum of $T^{\circledast}M$ (cotangent to the zero section) and T^*M (cotangent to the fibers). The derivative of Tf respects this decomposition and is an isomorphism on each piece, hence is an isomorphism over the zero section of TM. It is thus an isomorphism over a neighborhood of the zero section, hence is so everywhere by scaling equivariance. □

2.7.58 Example (Recovering local coordinates on a log smooth manifold)**.** Let $p \in M$ be a point of a log smooth manifold, and let us show how to construct a germ of diffeomorphism

$$(M, p) \to X_{T_p^{\circledast}M} \tag{2.7.58.1}$$

using the log inverse function theorem (2.7.57). The map $\mathcal{A}_{M,p}^{\geq 0} \to T_p^{\circledast}M$ is a torsor for the subspace of $\mathcal{A}_{M,p}^{\geq 0}$ consisting of those functions whose first derivative at p vanishes (2.7.45.3). It follows that there exists a section $T_p^{\circledast}M \to \mathcal{A}_{M,p}^{\geq 0}$ (**??**). Such a section is equivalently a germ (2.7.58.1) which induces the 'identity' map on cotangent cones. It is thus a local diffeomorphism by the log inverse function theorem (2.7.57).

We will see a relative version of this argument in (2.7.62).

Submersions

We now study submersions of log smooth manifolds. As can be expected from the form of the inverse function theorem for log smooth manifolds (2.7.57), arbitrary submersions of log smooth manifolds are not so well behaved. Instead, we will see that a more useful notion ('exact submersion') is obtained by imposing additional conditions on cotangent cones.

⋆ **2.7.59 Definition** (Submersion)**.** A map of log smooth manifolds $f : M \to N$ is called a *submersion* at $p \in M$ when its derivative $T_pM \to T_{f(p)}N$ is surjective (equivalently, $T_{f(p)}^{\circledast}N \to T_p^{\circledast}M$ is injective). The locus of points $p \in M$ where f is a submersion is evidently an open set.

Geometrically speaking, a map $f : X \to Y$ is a submersion when it is a submersion on non-degenerate loci $f^{\square} : X^{\square} \to Y^{\square}$ and the inverse to $Tf^{\square}$ is 'uniformly bounded' in log coordinates as one approaches the ideal locus. No condition, however, is imposed on how f interacts with the compactifications $X^{\square} \subseteq X$ and $Y^{\square} \subseteq Y$. Any injective map of polyhedral cones $Q \to P$ gives a submersion $X_P \to X_Q$, and such maps are in general very far from being topologically locally trivial (2.1.9).

2.7.60 Exercise. Use the log cotangent short exact sequence (2.7.44) to show that f is a submersion at p iff its restriction $f : M_p \to N_{f(p)}$ to the strata containing p and $f(p)$, respectively, is a submersion of smooth manifolds and the snake map $\ker (f_p^{\flat\flat})^{\mathrm{gp}} \to \operatorname{coker} T_p^* f|_{M_p}$ is injective.

2.7.61 Exercise. Show that the normalization (2.7.43) of a submersion is a submersion (show that for $f : M \to N$, a point $p \in M$, and a face $F \subseteq T_p^{\circledast}M$, the restriction of

$T_p^\circledast f : T_{f(p)}^\circledast N \to T_p^\circledast M$ to the inverse image of F coincides with the derivative of $\tilde{f} : \tilde{M} \to \tilde{N}$ at the lift of p corresponding to F).

2.7.62 Lemma (Local normal form of a submersion)**.** *A map of log smooth manifolds is a submersion iff it is locally (on the source) modelled on monomial maps associated to injective maps of polyhedral cones.*

Proof. This will be a relative version of (2.7.58), similar to (2.7.23).

Let $f : M \to N$ be a submersion at $p \in M$. We seek to construct a diagram

$$\begin{tikzcd} (M,p) \arrow[r] \arrow[d] & X_{T_p^\circledast M} \arrow[d, "T_p^\circledast f"] \\ (N,f(p)) \arrow[r] & X_{T_{f(p)}^\circledast N} \end{tikzcd} \tag{2.7.62.1}$$

in which the horizontal maps induce the identity on cotangent cones, hence are local diffeomorphisms (2.7.57). Such a diagram is equivalent to the data of compatible sections in the following diagram.

$$\begin{tikzcd} \mathcal{A}_{M,p}^{\geq 0} \arrow[r] & T_p^\circledast M \arrow[l, bend right, dashed] \\ \mathcal{A}_{N,f(p)}^{\geq 0} \arrow[u] \arrow[r] & T_{f(p)}^\circledast N \arrow[u, "T_p^\circledast f"'] \arrow[l, bend right, dashed] \end{tikzcd} \tag{2.7.62.2}$$

The bottom map has a section by (**??**) as in (2.7.58), and it extends to a section of the top map by (**??**) again since $T_p^\circledast f : T_{f(p)}^\circledast N \to T_p^\circledast M$ is injective. ☐

Exact submersions

We now introduce exact submersions of log smooth manifolds, which have better local behavior than submersions. Exact submersions are also preserved under pullback, which is crucial for a great number of applications. It is not surprising that the relevant notion of 'exactness' (introduced by Kato [59, Definition (4.6)] and since recognized as a key notion in log geometry) is a condition on the derivative on cotangent cones.

2.7.63 Definition (Local)**.** Let $f : Q \to P$ be a map of polyhedral cones, and let $P_0 \subseteq P$ and $Q_0 \subseteq Q$ denote the minimal strata (equivalently, the subgroups of invertible elements). It is always the case that $Q_0 \subseteq f^{-1}(P_0)$ (the image of an invertible element is always invertible). The map f is called *local* when this inclusion is an equality, that is $Q_0 = f^{-1}(P_0)$.

2.7.64 Exercise. Show that $Q \to P$ is local iff the associated monomial map $X_P \to X_Q$ sends the minimal stratum of X_P to the minimal stratum of X_Q. Show that the monomial map $X_P \to X_Q$ associated to $f : Q \to P$ is locally modelled near $x \in X_P$ on the monomial map associated to $Q + f^{-1}(F_x)^{\mathrm{gp}} \to P + F_x^{\mathrm{gp}}$ (compare (2.7.20)). Show that every map $Q + f^{-1}(F)^{\mathrm{gp}} \to P + F^{\mathrm{gp}}$ is local.

2.7.65 Exercise. Show that $Q \to P$ is local iff $Q^{\#} = Q/Q_0 \to P/P_0 = P^{\#}$ is local. Conclude that the derivative $T_x^{\circledast} f : T_{f(x)}^{\circledast} N \to T_x^{\circledast} M$ of a log smooth map $f : M \to N$ is always local (recall that $f_x^{\flat\flat} : \mathcal{Z}_{N,f(x)} \to \mathcal{Z}_{M,x}$ is always local (2.7.23)).

2.7.66 Definition (Exact; Kato [59, Definition (4.6)])**.** A map of real polyhedral cones $f : Q \to P$ is called *exact* when $(f^{\mathrm{gp}})^{-1}(P) = Q$.

2.7.67 Exercise. Show that $Q \to P$ is exact iff $Q^{\#} \to P^{\#}$ is exact. Conclude that $T_x^{\circledast} f$ is exact iff $f_x^{\flat\flat}$ is exact.

2.7.68 Exercise. Show that if $f : Q \to P$ is exact, then so is $f^{-1}(F) \to F$ for every face $F \subseteq P$.

2.7.69 Definition (Locally exact; Illusie–Kato–Nakayama [45, (A.3.2)(iii)] and Nakayama–Ogus [88, Definition 2.1(3)])**.** A map of real polyhedral cones $f : Q \to P$ is called *locally exact* when for every face $F \subseteq P$, the localized map

$$Q + f^{-1}(F)^{\mathrm{gp}} \to P + F^{\mathrm{gp}} \tag{2.7.69.1}$$

is exact.

2.7.70 Exercise. Show that $Q \to P$ is locally exact iff $Q^{\#} \to P^{\#}$ is locally exact. Conclude that $T_x^{\circledast} f$ is locally exact iff $f_x^{\flat\flat}$ is locally exact.

2.7.71 Exercise. Let $f : M \to N$ be a map of log smooth manifolds. Show that $f_x^{\flat\flat}$ is locally exact iff $f_y^{\flat\flat}$ is exact for all y in a neighborhood of x (use (2.7.23)).

2.7.72 Definition (Exact; Kato [59, Definition (4.6)])**.** Let f be a map of log smooth manifolds. We say that f is *exact* at x when the following equivalent conditions hold:
(2.7.72.1) $T_y^{\circledast} f$ (equivalently $f_y^{\flat\flat}$ (2.7.67)) is exact for all y in a neighborhood of x.
(2.7.72.2) $T_x^{\circledast} f$ (equivalently $f_x^{\flat\flat}$ (2.7.70)) is locally exact.
Exactness is evidently an open condition.

2.7.73 Exercise. Show that an exact local map of sharp polyhedral cones is injective. Conclude that if f is exact then $f_x^{\flat\flat}$ is injective for every x.

2.7.74 Exercise (Relative depth)**.** The *(relative) depth* of an *exact* map of log smooth manifolds $f : M \to N$ at a point $x \in M$ is the difference

$$\operatorname{depth}_x(f) = \dim \mathcal{Z}_{M,x} - \dim \mathcal{Z}_{N,f(x)} \tag{2.7.74.1}$$
$$= (\dim M - \dim N) - (\dim M_x - \dim N_{f(x)}). \tag{2.7.74.2}$$

Use (2.7.73) to show that the relative depth is non-negative and upper semicontinuous on M. Show that an exact map has depth zero iff it is strict (use (2.6.23)). What is the depth (as a function on the source) of the multiplication map $'\mathbb{R}^2_{\geq 0} \to {}'\mathbb{R}_{\geq 0}$?

2.7.75 Exercise. Conclude from (2.7.68) that normalization (2.7.43) preserves exactness.

The significance of the notions of exactness and local exactness, at least for us, comes from the fact that they behave well under pushout (corresponding to pullback of log smooth manifolds).

2.7.76 Lemma. *Locally exact (resp. exact and locally exact) morphisms of polyhedral cones are preserved under pushout (in the category of $\mathbb{R}_{\geq 0}$-linear monoids).*

Proof. We begin with a criterion for the existence of the pushout $P \sqcup_Q R$ of a diagram of polyhedral cones $R \leftarrow Q \to P$. Let I^{gp} denote the pushout of groupifications (a finite-dimensional real vector space).

$$\begin{array}{ccc} Q^{\mathrm{gp}} & \longrightarrow & P^{\mathrm{gp}} \\ \downarrow & & \downarrow \\ R^{\mathrm{gp}} & \longrightarrow & I^{\mathrm{gp}} \end{array} \tag{2.7.76.1}$$

Now define the polyhedral cone $I \subseteq I^{\mathrm{gp}}$ to be the image of $P \oplus R \to P^{\mathrm{gp}} \oplus R^{\mathrm{gp}} \twoheadrightarrow I^{\mathrm{gp}}$. The notation is consistent: the groupification of I is indeed I^{gp}. We now wish to formulate a condition under which the resulting diagram

$$\begin{array}{ccc} Q & \longrightarrow & P \\ \downarrow & & \downarrow \\ R & \longrightarrow & I \end{array} \tag{2.7.76.2}$$

is a pushout.

Consider the following two equivalence relations $\sim_Q$ and $\sim_{Q^{\mathrm{gp}}}$ on the set $P \oplus R$. For pairs $(p, r), (p', r') \in P \oplus R$, we declare that $(p, r) \sim_{Q^{\mathrm{gp}}} (p', r')$ iff

$$p' = p + q \tag{2.7.76.3}$$
$$r = r' + q \tag{2.7.76.4}$$

for some $q \in Q^{\mathrm{gp}}$. We set $(p, r) \sim_Q^{\mathrm{pre}} (p', r')$ when (2.7.76.3)–(2.7.76.4) hold for some $q \in Q$ (as opposed to Q^{gp}), and we take $\sim_Q$ to be the equivalence relation closure of $\sim_Q^{\mathrm{pre}}$. Now a map out of $P \oplus R$ comes from a (necessarily unique) map out of I iff Q^{gp}-equivalent pairs have the same image, while it comes from a (necessarily unique) map out of $R \leftarrow Q \to P$ iff Q-equivalent pairs have the same image. We thus conclude that if $\sim_Q$ and $\sim_{Q^{\mathrm{gp}}}$ coincide, then (2.7.76.2) is a pushout.

Now let us argue that if $f : Q \to P$ is locally exact, then this criterion is satisfied, namely Q^{gp}-equivalence implies Q-equivalence. Consider a point $(p, r) \in P \oplus R$, and let $F \subseteq P$ denote the minimal face containing p. Local exactness of $Q \to P$ means that $(f^{\mathrm{gp}})^{-1}(P + F^{\mathrm{gp}}) = Q + f^{-1}(F)^{\mathrm{gp}}$, and $Q + f^{-1}(F)^{\mathrm{gp}}$ is equivalently the set of differences $Q - f^{-1}(F)$, so

$$(f^{\mathrm{gp}})^{-1}(P + F^{\mathrm{gp}}) = Q - f^{-1}(F). \tag{2.7.76.5}$$

For any point $(p',r') \in P \oplus R$, the difference $p'-p$ lies in $P+F^{\mathrm{gp}}$. A lift of this difference to Q^{gp} is thus an element of $Q-f^{-1}(F)$. Now if (p',r') is Q^{gp}-equivalent and sufficiently close to (p,r), then the element of Q^{gp} lifting $p'-p$ realizing this equivalence can be taken arbitrarily small. It is thus an element of Q minus an arbitrarily small element of $f^{-1}(F)$. Now $p \in F^\circ$, so p minus a sufficiently small element of F lies in P, and we thus conclude that (p',r') is Q-equivalent to (p,r). To conclude that Q^{gp}-equivalence implies Q-equivalence in general, it suffices to note that Q^{gp}-equivalence classes in $P \oplus R$ are convex (being the intersection of the convex set $P \oplus R \subseteq P^{\mathrm{gp}} \oplus R^{\mathrm{gp}}$ with the inverse image of a point of $P^{\mathrm{gp}} \sqcup_{Q^{\mathrm{gp}}} R^{\mathrm{gp}}$) hence connected.

Now let us show that if $Q \to P$ is exact, then so is $R \to I$. Fix an element $r \in R^{\mathrm{gp}}$, and suppose that its image in I^{gp} is contained in I. This means $(0,r)$ is Q^{gp}-equivalent to some $(p',r') \in P \oplus R$, namely there is $q \in Q^{\mathrm{gp}}$ lifting $p' \in P$ and $r-r' \in R^{\mathrm{gp}}$. Exactness of $Q \to P$ means $q \in Q$, so $r = r'+q \in R$ as desired.

Finally, we should show that $R \to I$ is locally exact. A face $A \subseteq I$ pulls back to faces $F \subseteq P$, $G \subseteq Q$, and $H \subseteq R$, and we have a resulting localized diagram.

$$\begin{tikzcd} Q+G^{\mathrm{gp}} \arrow[r] \arrow[d] & P+F^{\mathrm{gp}} \arrow[d] \\ R+H^{\mathrm{gp}} \arrow[r] & I+A^{\mathrm{gp}} \end{tikzcd} \tag{2.7.76.6}$$

Now the pullback of A to $P \oplus R$ is the face $F \oplus H \subseteq P \oplus R$ (indeed, every face of $P \oplus R$ is a product, hence is the direct sum of its pullbacks to P and R). Thus A^{gp} is the image of $F^{\mathrm{gp}} \oplus H^{\mathrm{gp}}$, which implies that $I+A^{\mathrm{gp}}$ is the image of $(P+F^{\mathrm{gp}}) \oplus (R+H^{\mathrm{gp}})$. The top map $Q+G^{\mathrm{gp}} \to P+F^{\mathrm{gp}}$ is locally exact, so the localized diagram remains a pushout. Exactness of the top map thus implies exactness of the bottom map $R+H^{\mathrm{gp}} \to I+A^{\mathrm{gp}}$ as desired. □

2.7.77 Example. The map $'\mathbb{R}^2_{\geq 0} \to {}'\mathbb{R}_{\geq 0}$ given by $(x,y) \mapsto xy = \lambda$ (i.e. corresponding to the diagonal embedding $\mathbb{R}_{\geq 0} \to \mathbb{R}^2_{\geq 0}$) is an exact submersion. Although the fibers of this map over points $\lambda \in {}'\mathbb{R}_{\geq 0}$ develop a singularity as $\lambda \to 0$, the family is at least topologically locally trivial on the source. In fact, all exact submersions are topologically locally trivial on the source by a result of Nakayama–Ogus [88, Theorem 0.2] (though we will not appeal to this result).

⋆ **2.7.78 Proposition.** *Exact submersions are preserved under pullback.*

Proof. Since submersions are locally monomial (2.7.62), it suffices to consider monomial maps $X_P \to X_Q$ associated to injective and locally exact maps $Q \to P$.

Now let $Z \to X_Q$ be an arbitrary log smooth map, and let us show that $Z \times_{X_Q} X_P$ exists and maps exactly submersively to Z. The map $Z \to X_Q$ need not be locally monomial, but it is at least expressible in local coordinates $Z = X_R$ as the product ug of a monomial map $g : X_R \to X_Q$ and a log smooth map $u : X_R \to X_Q^\square$. Let us argue that the pullbacks of $X_P \to X_Q$ under the two maps g and ug are identified. It suffices to consider the 'universal' case of $Z = X_R = X_Q \times X_Q^\square$ where g and u are the first and second projections, respectively.

To make the desired identification, it is enough to lift the action of $X_Q^\square$ on X_Q to X_P. Now $X_P^\square$ acts on X_P, and the map $X_P \to X_Q$ is equivariant for the map $X_P^\square \to X_Q^\square$. It thus suffices to fix a section of $X_P^\square \to X_Q^\square$, which is just the map of vector spaces $(P^{\mathrm{gp}})^* \to (Q^{\mathrm{gp}})^*$, hence has a section since it is surjective (since $Q \to P$ is injective).

We have thus reduced our problem to showing existence of the pullback $X_R \times_{X_Q} X_P$ of log smooth manifolds for maps of polyhedral cones $R \leftarrow Q \to P$ in which $Q \to P$ is injective and locally exact. Now log smooth maps $Z \to X_P$ are in natural bijection with $\mathbb{R}_{\geq 0}$-linear maps $P \to \mathcal{A}_Z^{\geq 0}$. It follows that $X_R \times_{X_Q} X_P = X_{R \sqcup_Q P}$. Local exactness of $Q \to P$ implies the pushout $R \sqcup_Q P$ exists and that the map $R \to R \sqcup_Q P$ is locally exact (2.7.76), so $X_{R \sqcup_Q P} \to X_R$ is an exact submersion. $\square$

The fibers of an exact submersion over *non-degenerate* points of the base are log smooth manifolds by stability of exact submersions under pullback (2.7.78). Stability under pullback says nothing about fibers over ideal points of the base (note that a map of log smooth manifolds $* \to M$ must land inside the non-degenerate locus $M^\square$). Such fibers may be 'singular' as in (2.7.77), and may be called 'broken' log smooth manifolds (2.7.77). While not log smooth manifolds, such fibers are objects in a certain 'hybrid' category (2.10). Here is another way to make sense of the fibers of an exact submersion:

2.7.79 Definition (Normalized fiber)**.** Let $f : M \to N$ be an exact submersion of log smooth manifolds, and let $n \in N$ be a (possibly ideal) point. The induced map on normalizations $\tilde{f} : \tilde{M} \to \tilde{N}$ remains an exact submersion (2.7.61)(2.7.75). Now a point $n \in N$ has a unique inverse image in $\tilde{N}^\square$ (the non-degenerate locus of the normalization), so the fiber of $\tilde{f}$ over this point is a log smooth manifold (2.7.78) which we call the *normalized fiber* of f over n, denoting it $\tilde{M}_n$.

2.7.80 Exercise. Compute the fiber and the normalized fiber of the multiplication map $'\mathbb{R}^2_{\geq 0} \to {}'\mathbb{R}_{\geq 0}$ (which is an exact submersion) over the point $0 \in {}'\mathbb{R}_{\geq 0}$.

Here are two special classes of exact submersions of interest.

2.7.81 Example (Strict submersion)**.** A map f of log smooth manifolds is strict precisely when $f^{\flat\flat}$ is an isomorphism at every point (2.7.23). In particular, a strict submersion is exact. A strict submersion is locally on the source a pullback of $\mathbb{R}^k \to *$ (2.7.62).

2.7.82 Definition (Simply-broken submersion)**.** A *simply-broken submersion* of log smooth manifolds is a map which is locally a pullback of $\mathbb{R}^k \times {}'\mathbb{R}_{\geq 0} \to *$ or a product of the multiplication map $'\mathbb{R}^2_{\geq 0} \to {}'\mathbb{R}_{\geq 0}$ and $\mathbb{R}^k \to *$.

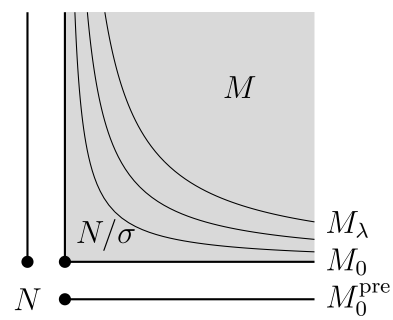
We now explore generalizations of Ehresmann's Theorem (2.4.17) (proper submersions of smooth manifolds are trivial locally on the target) to log smooth manifolds.

2.7.83 Proposition. *A proper submersion of log smooth manifolds which is trivial locally on the source is trivial locally on the target.*

Proof. We generalize the proof for smooth manifolds (2.4.17) as follows. Let $M \to B$ be a proper submersion which is trivial locally on the source, and let us show that $M \to B$ is trivial in a neighborhood of a given point $0 \in B$. Since $M \to B$ is trivial locally on the source, the fiber $M_0 = M \times_B 0$ exists.

We now seek to construct a local retraction $M \to M_0$. Such a retraction may be defined locally near any point of M_0 using the fact that $M \to B$ is trivial locally on the source (2.7.81). To patch together these local retractions, we appeal to the averaging operation for measures on log smooth manifolds (2.7.54). The resulting map $M \to M_0 \times B$ is an isomorphism on cotangent cones (inspection) so the log inverse function theorem (2.7.57) applies to show that it is a local isomorphism. ☐

⋆ **2.7.84 Definition** (Gluing coordinates)**.** Let M_0^{pre} be a log smooth manifold, let $i : N \times {}'\mathbb{R}_{\geq 0} \hookrightarrow M_0^{\mathrm{pre}}$ be an open embedding covering all points of positive depth (thus M_0^{pre} has depth one), and let $\sigma : N \to N$ be a free involution. Associated to this data $(M_0^{\mathrm{pre}}, N, i, \sigma)$ is a standard 'gluing coordinates' family $M \to {}'\mathbb{R}_{\geq 0}$ as we now recall.



(2.7.84.1)

The fiber M_0 over $0 \in {}'\mathbb{R}_{\geq 0}$ is the quotient of M_0^{pre} by the involution σ acting on $N \times 0 \subseteq N \times {}'\mathbb{R}_{\geq 0} \subseteq M_0^{\mathrm{pre}}$. The fiber M_λ for $\lambda > 0$ is the quotient of $(M_0^{\mathrm{pre}})^\square$ by the relation $i(n, x) = i(\sigma(n), y)$ whenever $xy = \lambda$. Thus M_λ is a 'gluing' of M_0 with 'gluing parameter' $\lambda \in {}'\mathbb{R}_{\geq 0}$.

The map $M \to {}'\mathbb{R}_{\geq 0}$ is defined as follows. Consider the two maps

$$\begin{tikzcd} (M_0^{\mathrm{pre}})^\square \times {}'\mathbb{R}_{\geq 0} \arrow[dr] & & (N \times {}'\mathbb{R}^2_{\geq 0})/(\sigma \times s) \arrow[dl] \\ & {}'\mathbb{R}_{\geq 0} & \end{tikzcd} \tag{2.7.84.2}$$

given by projection to ${}'\mathbb{R}_{\geq 0}$ and projection to ${}'\mathbb{R}^2_{\geq 0}$ followed by multiplication, respectively, where s denotes the involution $(x, y) \mapsto (y, x)$ of ${}'\mathbb{R}^2_{\geq 0}$. We glue these total spaces together by identifying $(n, x, y) = (\sigma(n), y, x)$ with $(i(n, x), xy)$ and $(i(\sigma(n), y), xy)$ to obtain M.

$$\begin{tikzcd} M \arrow[d] = \Big((M_0^{\mathrm{pre}})^\square \times {}'\mathbb{R}_{\geq 0}\Big) \displaystyle\bigcup_{(N \times ({}'\mathbb{R}^2_{\geq 0} \setminus (0,0)))/(\sigma \times s)} \Big((N \times {}'\mathbb{R}^2_{\geq 0})/(\sigma \times s)\Big) \arrow[dl] \\ {}'\mathbb{R}^2_{\geq 0} \end{tikzcd} \tag{2.7.84.3}$$

When M_0^{pre} is compact Hausdorff, the resulting family $M \to {}'\mathbb{R}_{\geq 0}$ is proper.

Finally, let us note that we may also take a separate gluing parameter for every connected component of N/σ to produce a family $M \to {}'\mathbb{R}_{\geq 0}^{\pi_0 N/\sigma}$.

2.7.85 Proposition. *Every proper simply-broken submersion is, locally on the base, a pullback of a standard gluing family* (2.7.84)*.*

Proof. This is a generalization of (2.4.17).

Let $Q \to B$ be a proper simply-broken submersion. Fix a basepoint $b \in B$, and let Q_b denote the fiber over b. The map $Q \to B$ is covered by local pullback diagrams.

$$\begin{tikzcd} Q \arrow[r] \arrow[d] & {}'\mathbb{R}_{\geq 0}^2 \times \mathbb{R}^k \arrow[d, "(x,y,t)\mapsto xy"] \\ B \arrow[r, "b\mapsto 0"] & {}'\mathbb{R}_{\geq 0} \end{tikzcd} \tag{2.7.85.1}$$

Our task is to patch together these local charts into gluing coordinates (2.7.84) near the basepoint $b \in B$. In fact, we will not do exactly this, rather we will show how to recover such charts intrinsically from the map $Q \to B$, and we will then globalize this intrinsic construction.

Let us call a point $q \in Q$ *non-singular* when the map $\pi : Q \to B$ is strict in a neighborhood of q, equivalently when $\mathcal{Z}_{B,\pi(q)} \to \mathcal{Z}_{Q,q}$ is an isomorphism (2.7.24). At a non-singular point $q \in Q$, the map $Q \to B$ is a strict submersion, hence is locally of the form $Q = B \times \mathbb{R}^k \to B$ (2.7.81). Thus the stratum $Q_q \subseteq Q$ of q is the unique local stratum lying over the stratum $B_b \subseteq B$ of b, and the restriction $Q_q \to B_b$ is a submersion. Thus the (open) non-singular locus of Q_b is canonically a smooth manifold. In a neighborhood of any non-singular point of Q_b, a local trivialization $Q = Q_b \times B$ may be constructed intrinsically as follows: construct a retraction $Q \to Q_b$ simply by extension of smooth functions from strata, and note that the induced map $Q \to Q_b \times B$ is a local isomorphism by the inverse function theorem (2.7.57).

Let us now investigate what happens near the singular points of Q. Singular points of Q are precisely those lying over $0 \times 0 \times \mathbb{R}^k$ in the local charts (2.7.85.1) (points not lying over $0 \times 0 \times \mathbb{R}^k$ are non-singular since the map ${}'\mathbb{R}_{\geq 0}^2 \to {}'\mathbb{R}_{\geq 0}$ is strict away from $(0,0)$, and it will be clear from the present discussion that conversely all points lying over $0 \times 0 \times \mathbb{R}^k$ are in fact singular). At a singular point $q \in Q_b$, a choice of local chart (2.7.85.1) induces a pushout square of cotangent cones (2.7.78).

$$\begin{tikzcd} T_q^{\circledast} Q & \mathbb{R}_{\geq 0}^2 \times \mathbb{R}^k \arrow[l] \\ T_b^{\circledast} B \arrow[u] & \mathbb{R}_{\geq 0} \arrow[l] \arrow[u, "a\mapsto(a,a,0)"'] \end{tikzcd} \tag{2.7.85.2}$$

It is somewhat more convenient to quotient by the invertible elements to obtain the following pushout square.

$$\begin{tikzcd} \mathcal{Z}_{Q,q} & \mathbb{R}^2_{\geq 0} \arrow[l] \\ \mathcal{Z}_{B,b} \arrow[u] & \mathbb{R}_{\geq 0} \arrow[l] \arrow[u, "a\mapsto(a,a)"'] \end{tikzcd} \tag{2.7.85.3}$$

By the construction of such pushouts (2.7.76), this means that $\mathcal{Z}_{Q,q}$ is the image of $\mathcal{Z}_{B,b}\oplus\mathbb{R}^2_{\geq 0}$ inside the pushout of vector spaces $\mathcal{Z}^{\mathrm{gp}}_{B,b}\sqcup_{\mathbb{R}}\mathbb{R}^2$. We now make some deductions by inspecting this description of $\mathcal{Z}_{Q,q}$. There exist precisely two non-zero faces of $\mathcal{Z}_{Q,q}$ whose intersection with $\mathcal{Z}_{B,b}$ is zero. Both are rays $\mathbb{R}_{\geq 0}\subseteq\mathcal{Z}_{Q,q}$, and the quadrant $\mathbb{R}^2_{\geq 0}\subseteq\mathcal{Z}_{Q,q}$ they span (namely the upper map in (2.7.85.3)) intersects $\mathcal{Z}_{B,b}$ in a ray $\mathbb{R}_{\geq 0}\subseteq\mathcal{Z}_{B,b}$ (namely the lower map in (2.7.85.3), not necessarily a face). We conclude that the pushout diagram (2.7.85.3) is actually determined uniquely by the map $\mathcal{Z}_{B,b}\to\mathcal{Z}_{Q,q}$, up to simultaneous scaling of $\mathbb{R}_{\geq 0}\to\mathbb{R}^2_{\geq 0}$. Recall that the strata of B (resp. Q) near b (resp. q) are indexed by the faces of $\mathcal{Z}_{B,b}$ (resp. $\mathcal{Z}_{Q,q}$) and that the stratum of Q corresponding to a face $F\subseteq\mathcal{Z}_{Q,q}$ maps to the stratum of B corresponding to $F\cap\mathcal{Z}_{B,b}\subseteq\mathcal{Z}_{B,b}$. There are thus precisely three strata of Q lying over the stratum of b, namely those corresponding to zero and to the two distinguished rays inside $\mathcal{Z}_{Q,q}$. The stratum Q_q of Q corresponding to the zero face of $\mathcal{Z}_{Q,q}$ is precisely the singular locus near q. The stalks of $\mathcal{Z}_Q$ at nearby singular points are identified canonically, so they have 'the same' distinguished rays. In particular, every component of the singular locus of Q_b determines a ray in $\mathcal{Z}_{B,b}$.

Given this knowledge of the structure of the map $\mathcal{Z}_{B,b}\to\mathcal{Z}_{Q,q}$ for singular points $q\in Q_b$, we can now give an 'intrinsic' construction of local charts (2.7.85.1) near such q. Suppose $x,y:(Q,q)\to({}'\mathbb{R}_{\geq 0},0)$ and $\lambda:(B,b)\to({}'\mathbb{R}_{\geq 0},0)$ are maps whose classes in $\mathcal{Z}_{Q,q}$ and $\mathcal{Z}_{B,b}$ generate the distinguished rays in these polyhedral cones. We therefore have $\lambda=e^fx^ay^b$ for some smooth $f:M\to\mathbb{R}$ and some real numbers $a,b>0$. By replacing (x,y) with $(e^{f/2}x^a,e^{f/2}y^b)$, we may achieve that $\lambda=xy$ on M, and hence that we have a diagram of the following form.

$$\begin{tikzcd} Q \arrow[r] \arrow[d] & {}'\mathbb{R}^2_{\geq 0} \arrow[d] \\ B \arrow[r] & {}'\mathbb{R}_{\geq 0} \end{tikzcd} \tag{2.7.85.4}$$

Now fix in addition a function on the singular stratum of Q to $\mathbb{R}^k$ which is a local diffeomorphism, and extend it to a smooth function on a neighborhood of q. The resulting diagram (with ${}'\mathbb{R}^2_{\geq 0}\times\mathbb{R}^k$ in the upper right corner) is a pullback square: the pullback $B\times_{{}'\mathbb{R}_{\geq 0}}{}'\mathbb{R}^2_{\geq 0}\times\mathbb{R}^k$ exists since the map being pulled back is an exact submersion, and the inverse function theorem (2.7.57) guarantees that the map from Q to this pullback is an isomorphism (its derivative at q is an isomorphism by construction).

Now let us extract from the fiber Q_b the data necessary to define a standard gluing chart (2.7.84). The map $b:*\to B$ is a map of topological spaces, but not of log topological spaces unless b is non-degenerate, so the fiber $Q_b=Q\times_B *$ is merely a topological space. We can, however, refine the topological fiber Q_b to a log smooth manifold $\tilde{Q}_b$ (the 'normalized fiber') mapping to Q_b (2.7.79). To do this, we appeal to normalization (2.7.43). There is a unique point $\tilde{b} \in \tilde{B}^\square$ lying over $b \in B$ (namely, it is the inverse image of b corresponding to the local stratum of B containing b). The normalized fiber $\tilde{Q}_b$ is the fiber of $\tilde{Q} \to \tilde{B}$ over $\tilde{b}$, which is a log smooth manifold since it is a pullback of the exact submersion $\tilde{Q} \to \tilde{B}$ (2.7.75)(2.7.61).

There is an evident map of topological spaces $\tilde{Q}_b \to Q_b$ (induced by $\tilde{Q} \to Q$ and $\tilde{B} \to B$), and we can describe it concretely as follows (by inspection). Near a non-singular point of Q_b, the map $\tilde{Q}_b \to Q_b$ is a homeomorphism and $\tilde{Q}_b$ is a smooth manifold (and this coincides with the smooth manifold structure on Q_b defined above). A singular point of Q_b has three inverse images in $\tilde{Q}_b$, corresponding to the three local strata of Q lying over the stratum of $b \in B$. This decomposes $\tilde{Q}_b$ into the union of a smooth manifold N/σ (in bijection with the singular points of Q_b) and a log smooth manifold M_0^{pre} of depth one whose ideal locus $N = (M_0^{\mathrm{pre}})^\infty$ has a free involution σ with quotient N/σ.

Now let $M \to {}'\mathbb{R}_{\geq 0}^{\pi_0 N/\sigma}$ denote the gluing family (2.7.84) associated to the data $(M_0^{\mathrm{pre}}, \sigma)$ defined above, and let us construct a pullback diagram of the desired shape.

$$\begin{tikzcd} Q \arrow[r] \arrow[d] & M \arrow[d] \\ B \arrow[r, "\lambda"] & {}'\mathbb{R}_{\geq 0}^{\pi_0 N/\sigma} \end{tikzcd} \tag{2.7.85.5}$$

We have already seen how to construct such pullback diagrams locally on Q, so we just need to globalize. Each component of $\pi_0 N/\sigma$ (i.e. the singular locus of Q_b) gives a distinguished ray in $\mathcal{Z}_{B,b}$, and we fix any bottom map λ inducing the same rays. To define a lift $Q \to M$ in a neighborhood of the singular locus of Q_b, we should first construct functions $x, y : Q \to {}'\mathbb{R}_{\geq 0}$ satisfying $\lambda = xy$ (of course, really (x, y) is an *unordered* pair of functions indexed by the two local branches of $M_0^{\mathrm{pre}} \subseteq \tilde{Q}_b$, but we will stick with the abuse of notation for simplicity). We can certainly fix functions $x, y : Q \to {}'\mathbb{R}_{\geq 0}$ in a neighborhood of the singular locus of Q_b whose classes lie in the two relevant distinguished rays in $\mathcal{Z}_{Q,q}$ for singular points q, and as before we have $\lambda = e^f x^a y^b$ for locally constant $a, b : Q \to \mathbb{R}_{>0}$ and smooth $f : Q \to \mathbb{R}$, so replacing (x, y) with $(e^{f/2}x^a, e^{f/2}y^b)$ we may achieve $\lambda = xy$ on Q. A choice of local retraction $Q \to N/\sigma$ (construct it locally and then average (2.4.15)) completes the data of a lift $Q \to M$ near the singular locus of Q_b. Finally, to extend the lift $Q \to M$ to a neighborhood the rest of Q_b, we patch together local retractions $Q \to M_0^{\mathrm{pre}}$ as in (**??**). As we already saw above, the resulting diagram is a pullback square by the inverse function theorem. □

2.7.86 Exercise. Let $Q \to B$ be a proper simply-broken submersion, and let $B \to X_P$ be strict. Show that $Q \to B$ is, locally on the target, the pullback of a standard gluing family along a monomial map $X_P \to {}'\mathbb{R}_{\geq 0}^n$ (note that in the above construction of such pullbacks (2.7.85.5), the map $\lambda : B \to {}'\mathbb{R}_{\geq 0}^{\pi_0 N/\sigma}$ just needs to induce the correct rays in $\mathcal{Z}_{B,b}$).

2.7.87 Definition (Gluing coordinates with vector bundles)**.** The gluing construction (2.7.84) may be enhanced to carry along vector bundles. Recall that the input to the gluing construction is a log smooth manifold M_0^{pre} of depth one (let $N = (M_0^{\mathrm{pre}})^\infty$ denote its ideal locus, in this case a smooth manifold), a collar $N \times {}'\mathbb{R}_{\geq 0} \hookrightarrow M_0^{\mathrm{pre}}$, and a free involution $\sigma : N \to N$. The output is a family $M \to {}'\mathbb{R}_{\geq 0}^{\pi_0 N/\sigma}$. Now enhance everything with a vector bundle: fix a vector bundle $V_0^{\mathrm{pre}} \to M_0^{\mathrm{pre}}$ (its restriction to the ideal locus denoted $W \to N$), an isomorphism $W = V_0^{\mathrm{pre}}$ over the collar, and a lift of the involution σ to W. Such data evidently gives rise to a vector bundle $V \to M$.

2.7.88 Exercise. Enhance the proof of 'simply-broken Ehresmann' (2.7.85) to show that a proper simply-broken submersion with a vector bundle on its total space is locally on the base a pullback of a standard gluing family (2.7.87).

Local structure of log mapping stacks

We analyzed the local structure of the mapping stacks $\underline{\mathrm{Sec}}_B(M, Q)$ for submersions of smooth manifolds $Q \to M \to B$ in (2.4.18)(2.4.19)(2.4.20)(2.4.21). Specifically, we showed that they were 'locally linear', namely locally isomorphic to stacks of the same flavor but with $Q \to M$ a vector bundle.

We now generalize these results to the setting of log smooth manifolds. The main new feature in this setting, discussed already in (2.6.29), is the presence of 'ideal points' of $\underline{\mathrm{Sec}}_B(M, Q)$, namely those not corresponding to pairs $(b \in B^\square, u : M_b \to Q_b)$, reflecting non-triviality of the log structure on $\underline{\mathrm{Sec}}_B(M, Q)$. The log structure on $\underline{\mathrm{Sec}}_B(M, Q)$ has two sources: the log structure on the base B, and the log structure on Q relative M.

* *

Let us first dispense with the case that $Q \to M$ is a strict submersion, where the statements and proofs of the results for smooth manifolds (2.4.18)(2.4.19)(2.4.20)(2.4.21) apply without much change.

2.7.89 Lemma (Local structure of $\underline{\mathrm{Sec}}(M, Q)$; compare (2.4.18))**.** *Let $Q \to M$ be a strict submersion of log smooth manifolds. If M is paracompact Hausdorff, then any section $s : M \to Q$ extends to an open embedding $(s^* T_{Q/M}, 0) \to (Q, s)$ over M.*

Proof. The proof in (2.4.18) of the smooth case applies without change. Indeed, we just the source-local normal form for strict submersions (2.7.62), log bump functions (2.7.50), and the log inverse function theorem (2.7.57). □

⋆ **2.7.90 Corollary** (Local structure of $\underline{\mathrm{Sec}}(M, Q)$; compare (2.4.19))**.** *Let $Q \to M$ be a strict submersion of log smooth manifolds. If M is compact Hausdorff, then the moduli stack $\underline{\mathrm{Sec}}(M, Q)$ is covered by the open substacks $\underline{\mathrm{Sec}}(M, Q^\circ) \subseteq \underline{\mathrm{Sec}}(M, Q)$ associated to open subsets $Q^\circ \subseteq Q$ for which $Q^\circ \to M$ can be equipped with the structure of a vector bundle.*

Proof. The proof in (2.4.19) applies without change (using (2.7.89) in place of (2.4.18)). □

2.7.91 Lemma (Local structure of $\underline{\mathrm{Sec}}_B(M,Q)$; compare (2.4.20))**.** *Let $Q \xrightarrow{\text{strict}} M \xrightarrow{\text{simply-broken}} B$ be submersions of log smooth manifolds. If $M \to B$ is proper, then for any $b \in B$ and any section s of $\tilde{Q}_b \to \tilde{M}_b$ (normalized fibers (2.7.79)) which descends to a section of $Q_b \to M_b$ (topological fibers), there are open embeddings $B^\circ \subseteq B$, $M^\circ = M \times_B B^\circ$, and $Q^\circ \subseteq Q \times_B B^\circ$, where $Q^\circ \to M^\circ \to B^\circ$ is a pullback of a standard gluing family* (2.7.84) *carrying a standard gluing family vector bundle* (2.7.87)*, with $b \in B^\circ$ and $s(M_b) \subseteq Q_b^\circ$. Moreover, we may assume that the map $B^\circ \to {}'\mathbb{R}^n_{\geq 0}$ (along which the standard gluing family is pulled back) factors through any given strict map $B \to B'$ defined near b.*

Proof. This is similar to (2.4.20). Since $M \to B$ is proper, it may be identified locally on the target with the pullback of a standard gluing family (2.7.85), and the refinement (2.7.86) ensures the pullback map may be taken to factor through any given strict map $B \to B'$ defined near b. Now the pullback $s^*T_{Q/M}$ is a vector bundle on the normalized fiber $\tilde{M}_b$ with matching data over the nodal locus. Choosing (arbitrarily) collar trivializations for it, we obtain a standard gluing family vector bundle $Q^\circ \to M^\circ$ (2.7.87). Finally, to construct the desired open embedding $Q^\circ \hookrightarrow Q \times_B B^\circ$, it suffices to construct a map $Q \to Q^\circ$ over M sending s to the zero section with vertical derivative along s being the identity map. □

⋆ **2.7.92 Corollary** (Local structure of $\underline{\mathrm{Sec}}_B(M,Q)$; compare (2.4.21))**.** *Let $Q \xrightarrow{\text{strict}} M \xrightarrow{\text{simply-broken}} B$ be submersions of log smooth manifolds. If $M \to B$ is proper, then the moduli stack $\underline{\mathrm{Sec}}_B(M,Q)$ is covered by the open substacks $\underline{\mathrm{Sec}}_{B^\circ}(M^\circ,Q^\circ) \subseteq \underline{\mathrm{Sec}}(M,Q)$ associated to open subsets $B^\circ \subseteq B$, $M^\circ = M \times_B B^\circ$, and $Q^\circ \subseteq Q \times_B B^\circ$, for which $Q^\circ \to M^\circ \to B^\circ$ is a pullback of a standard gluing family* (2.7.84) *carrying a standard gluing family vector bundle* (2.7.87)*. Moreover, every point $(b,s) \in \underline{\mathrm{Sec}}_B(M,Q)$ is covered by such an open substack $\underline{\mathrm{Sec}}_{B^\circ}(M^\circ,Q^\circ)$ in which the classifying map $B^\circ \to {}'\mathbb{R}^n_{\geq 0}$ (along which the standard gluing family is pulled back) factors through any given strict map $B \to B'$ defined near b.*

Proof. The desired $Q^\circ \to M^\circ \to B^\circ$ covering a given point of $\underline{\mathrm{Sec}}_B(M,Q)$ is produced by (2.7.91). The map $\underline{\mathrm{Sec}}_{B^\circ}(M^\circ,Q^\circ) \to \underline{\mathrm{Sec}}_B(M,Q)$ is an open embedding since $M \to B$ is universally closed (2.3.59). □

* *

Now we move on to the case of sections of general (not necessarily strict) submersions.

⋆ **2.7.93 Definition** (Artin cone)**.** Let P be a polyhedral cone. Its associated *Artin cone* $\mathcal{A}_P$ is the stack quotient $X_P/X_P^\square$.

2.7.94 Lemma. *The Artin cone $\mathcal{A}_P$ represents the functor sending any quasi-integral log topological space Z to the set of monoid homomorphisms $P \to \mathcal{Z}_Z$.*

Proof. Recall that X_P represents the functor $Z \mapsto \mathrm{Hom}(P, \mathcal{O}_Z^{\geq 0})$ (2.7.12) and that $X_P^\square$ represents the functor $Z \mapsto \mathrm{Hom}(P, \mathcal{O}_Z^{>0})$ (note that $X_P^\square = X_{P^{\mathrm{gp}}}$). □

Log smooth stacks

We now explore the theory of log smooth stacks and generalize the basic results about smooth stacks (2.5) to the log context.

⋆ **2.7.95 Definition** (Log smooth (Lie) orbifold). A *log smooth (resp. Lie) orbifold* is a log smooth stack X which around every $p \in X$ has a germ of open embedding from the quotient $(X_P/G, 0)$ of a real affine toric variety X_P (2.7.7) by a linear action of a compact discrete (resp. Lie) group G on P.

2.7.96 Lemma. *The quotient of a log smooth manifold by a proper action of a finite group is a log smooth orbifold.*

Proof. We follow the proof of the corresponding result for smooth manifolds (2.5.7).

Given a local (topological) retraction $r : M \to Gp$ to an orbit $Gp \subseteq M$, we now choose a local map s from M to $X_{T^{\circledast}_{Gp}M} = \coprod_{q \in Gp} X_{T^{\circledast}_q M}$ inducing the identity map on cotangent polyhedral cones at all $q \in Gp$. Such a map s may be averaged over the action of G as before to make it G-equivariant. Now applying the log inverse function theorem (2.7.57) to s, we may conclude as before. □

2.7.97 Lemma. *The quotient of a log smooth manifold by a proper action of a compact Lie group is a log smooth Lie orbifold.*

Proof. We follow the prof of the corresponding result for smooth manifolds (2.5.8).

The orbit $Gp = G/H$ remains a smooth manifold (it is locally a smooth submanifold of some stratum of M), so a germ of equivariant retraction $r : M \to Gp$ may be constructed as before using equivariant averaging (2.4.16).

Now $\ker(T^{\circledast}M|_{Gp} \to T^*(Gp))$ is an extension of the local system of polyhedral cones $\mathcal{Z}_M|_{Gp}$ by the vector bundle $\ker(T^*M_{Gp}|_{Gp} \to T^*(Gp))$. We now lift r to a germ of map s to the associated bundle of real affine toric varieties over Gp, whose action on cotangent polyhedral cones is the canonical inclusion $\ker(T^{\circledast}M|_{Gp} \to T^*(Gp)) \to T^{\circledast}M|_{Gp}$. We may now average s to make it G-equivariant and apply the log inverse function (2.7.57) to it to conclude as before. □

2.8 Topological sites

In (2.3), we studied sheaves on the category of topological spaces. Later, we will want to consider sheaves on other similar categories (e.g. smooth manifolds) which share a common fundamental structure: their objects are topological spaces X equipped with some extra structure $\mathcal{S}_X$ of a local nature, and their morphisms are maps of underlying topological spaces $f : X \to Y$ together with some correspondence between $\mathcal{S}_X$ and $\mathcal{S}_Y$ over f, also of a local nature. We formulate the notion of a *perfect topological site* (2.8.2)(2.8.9)(2.8.44) which is a precise axiomatization of this idea. It makes sense to consider sheaves on any such category, and we show how to carry over elements of the theory of topological stacks (2.3) to this setting. The reader who desires even more abstraction is referred to the notion of a Grothendieck site from [1, Exposé II] and the notion of a 'geometry' from Lurie [73] (neither of which we will need here).

Topological sites

Any functor $|\cdot| : \mathsf{C} \to \mathsf{Top}$ may be regarded as specifying an 'underlying topological space' for each object of C (and an 'underlying continuous map' for every morphism). Such data is not particularly useful without additional axioms. We introduce the relevant axioms one by one.

⋆ **2.8.1 Definition** (Open embedding)**.** Let C be an ∞-category equipped with a functor $|\cdot| : \mathsf{C} \to \mathsf{Top}$. An *open embedding* in C is a morphism which is cartesian (**??**) over an open embedding in Top. An *open covering* of $X \in \mathsf{C}$ is a collection of open embeddings into X which after applying $|\cdot|$ becomes an open covering of $|X|$.

In other words, a morphism $U \to X$ in C is an open embedding when $|U| \to |X|$ is an open embedding and U represents the functor of maps to X which upon applying $|\cdot|$ factor through $|U| \subseteq |X|$. Open embeddings are closed under composition (since cartesian morphisms are closed under composition (**??**) and open embeddings in Top are closed under composition).

⋆ **2.8.2 Definition** (Topological site)**.** A *topological site* is a pair $(\mathsf{C}, |\cdot| : \mathsf{C} \to \mathsf{Top})$ which has *all open embeddings*, meaning that for every $X \in \mathsf{C}$, every open subset of $|X|$ is realized by an open embedding $U \to X$ in C (in other words, every map $(\Delta^1, 1) \to (\mathsf{Top}, \mathsf{C})$ whose underlying morphism in Top is an open embedding has a cartesian lift).

In any topological ∞-site C, the functor

$$(\mathsf{C} \downarrow^{\mathsf{opemb}} X) \to (\mathsf{Top} \downarrow^{\mathsf{opemb}} |X|) = \mathsf{Open}(|X|) \tag{2.8.2.1}$$

is an equivalence (**??**) since $\mathsf{C}^{\mathsf{opemb}} \to \mathsf{Top}^{\mathsf{opemb}}$ is cartesian.

2.8.3 Exercise. Show that the following are topological sites.

(2.8.3.1) The category Top of topological spaces with $|\cdot|$ the identity functor.

(2.8.3.2) The category of open subsets of any fixed topological space.

(2.8.3.3) The category Sm of smooth manifolds with the underlying topological space functor.

(2.8.3.4) The category of pairs (X,ω_X) where $X \in \mathsf{Sm}$ and ω_X a closed 3-form on X and morphisms $(X,\omega_X) \to (Y,\omega_Y)$ given by smooth maps $f : X \to Y$ satisfying $f^*\omega_Y = \omega_X$.

(2.8.3.5) The category $\mathsf{Vect} \rtimes \mathsf{Top}$ whose objects are pairs (X,V) where X is a topological space and $V \to X$ is a vector bundle, and in which a morphism $(X,V) \to (Y,W)$ is a continuous map $X \to Y$ covered by a map of vector bundles $V \to W$ (with underlying topological space functor $(X,V) \mapsto X$).

(2.8.3.6) The arrow category $\mathsf{Fun}(\Delta^1,\mathsf{Top})$ with the functor $\mathsf{Fun}(\Delta^1,\mathsf{Top}) \to \mathsf{Top}$ sending an arrow to its target (that is, $(X \to Y) \mapsto Y$).

(2.8.3.7) The category $\mathsf{Top}_{/X}$ of topological spaces over a fixed topological space X.

(2.8.3.8) The category of sets.

(2.8.3.9) The category whose objects are pairs $(I,\{X_i\}_{i\in I})$ where I is a set and X_i is a pointed topological space for every $i \in I$ and whose morphisms $(I,\{X_i\}_{i\in I}) \to (J,\{Y_j\}_{j\in J})$ are maps $f : I \to J$ with finite fibers together with pointed maps $\prod_{f(i)=j} X_i \to Y_j$ for every $j \in J$ (with underlying topological space functor $(I,\{X_i\}_{i\in I}) \mapsto I$).

(2.8.3.10) The category of schemes.

(2.8.3.11) The ∞-category $\mathsf{C}^{\mathsf{opemb}}$ for any topological ∞-site C.

(2.8.3.12) The ∞-category $\mathsf{C}^{\triangleleft}$ for any ∞-category C with underlying topological space functor sending the cone point to $\varnothing$ and sending all objects of C to $*$.

(2.8.3.13) Any full subcategory $\mathsf{C}^- \subseteq \mathsf{C}$ of a topological ∞-site C with the property that if $X \in \mathsf{C}^-$ and $U \to X$ is an open embedding in C, then $U \in \mathsf{C}^-$.

(2.8.3.14) An ∞-category E with a cartesian fibration $\mathsf{E} \to \mathsf{C}$ where C is a topological ∞-site (more generally, it is enough to assume that every map $(\Delta^1,1) \to (\mathsf{C},\mathsf{E})$ whose underlying morphism in C is an open embedding has a cartesian lift). For example, this applies to $\mathsf{P}(-)^{\mathsf{op}} \rtimes \mathsf{Top}$ and $\mathsf{Shv}(-)^{\mathsf{op}} \rtimes \mathsf{Top}$ (2.2.22). Which of the above examples are special cases of this?

2.8.4 Lemma. *Let C be an ∞-category with a functor $|\cdot| : \mathsf{C} \to \mathsf{Top}$. Consider a square*

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & Y' \arrow[d] \\ X \arrow[r] & Y \end{tikzcd} \tag{2.8.4.1}$$

in C whose bottom arrow is an open embedding and whose image in Top is a pullback. In this case, the diagram (2.8.4.1) is a pullback iff $X' \to Y'$ is an open embedding.

Proof. This is simply a special case of (1.4.151). □

2.8.5 Lemma. *Open embeddings in a topological ∞-site are preserved under pullback, and $|\cdot|$ sends pullbacks of open embeddings in C to pullbacks of open embeddings in Top.*

Proof. While (1.4.152) does not apply directly since the functor $|\cdot| : \mathsf{C} \to \mathsf{Top}$ is not cartesian, its proof applies without change (the only cartesian lifting problems encountered are over open embeddings in Top). □

2.8.6 Exercise. Conclude from the description of pullbacks of open embeddings (2.8.5) that open coverings are preserved under pullback in any topological ∞-site.

2.8.7 Exercise. Conclude from (2.8.5) that a fiber product of open embeddings $U, V \to X$ in a topological ∞-site is the open embedding corresponding to the intersection $|U| \cap |V| \subseteq |X|$.

2.8.8 Exercise. Consider the cartesian fibration $\mathsf{Open} \rtimes \mathsf{Top} \to \mathsf{Top}$ where $\mathsf{Open} \rtimes \mathsf{Top}$ is the full subcategory of $\mathsf{Fun}(\Delta^1, \mathsf{Top})$ spanned by open embeddings and the map to Top is evaluation at $1 \in \Delta^1$. This cartesian fibration encodes the functor $\mathsf{Open} : \mathsf{Top}^{\mathsf{op}} \to \mathsf{Po} \subseteq \mathsf{Cat}$ (where Po is partially ordered sets (1.1.31)).

Now let C be a topological ∞-site, and consider the pullback $\mathsf{Open}(|-|) \rtimes \mathsf{C} = (\mathsf{Open} \rtimes \mathsf{Top}) \times_{\mathsf{Top}} \mathsf{C}$. There is an evident forgetful functor from the full subcategory of $\mathsf{Fun}(\Delta^1, \mathsf{C})$ spanned by open embeddings to this pullback. Show that this forgetful functor is a trivial Kan fibration by lifting $(\Delta^1, 1)^\# \wedge (\Delta^k, \partial\Delta^k)$ against $(\mathsf{C}, \mathsf{opemb}) \to (\mathsf{Top}, \mathsf{opemb})$. Conclude that $\mathsf{Fun}_{\mathsf{opemb}}(\Delta^1, \mathsf{C}) \to \mathsf{C}$ (evaluate at $1 \in \Delta^1$) is a cartesian fibration encoding the functor $\mathsf{Open}(|\cdot|) : \mathsf{C}^{\mathsf{op}} \to \mathsf{Po}$.

Many notions and constructions in the context of topological spaces depend only on the notions of open embeddings and open coverings, hence make sense in any topological ∞-site. For example, a morphism $X \to Y$ in a topological ∞-site is called a local isomorphism (2.1.10) when there exists an open cover $X = \bigcup_i U_i$ such that each composition $U_i \to X \to Y$ is an open embedding. Of central importance is the notion of a sheaf: a presheaf on a topological ∞-site C is called a sheaf when it sends open coverings to limits (equivalently, when its pullback to $\mathsf{Open}(|X|) = (\mathsf{C} \downarrow^{\mathsf{opemb}} X)$ (2.8.2.1) is a sheaf on $|X|$ for every $X \in \mathsf{C}$).

It should be noted that the axioms of a topological ∞-site *do not* guarantee that morphisms are of a local nature. Rather, this is an additional (very important) property called being 'subcanonical'. While most topological ∞-sites of interest are subcanonical, various key foundational constructions will involve non-subcanonical topological ∞-sites in an important way.

⋆ **2.8.9 Definition** (Subcanonical)**.** A topological site C is called *subcanonical* when every Yoneda presheaf $\mathsf{C}(-, X) \in \mathsf{P}(\mathsf{C})$ is a sheaf (equivalently, when open coverings are colimits (**??**)).

2.8.10 Exercise. Which of the topological sites in (2.8.3) are subcanonical?

2.8.11 Exercise. Show that a morphism $X \to Y$ in a subcanonical topological ∞-site is an isomorphism iff it is an isomorphism locally on the target.

2.8.12 Exercise (Coproducts in a subcanonical topological ∞-site)**.** Let X be an object of a subcanonical topological ∞-site C. Let $X = \bigcup_i U_i$ be a cover by open embeddings. Show that if the $|U_i|$ are disjoint, then $X = \bigsqcup_i U_i$ is their coproduct in C. In particular, if $|X| = \varnothing$, then X is an initial object of C.

2.8.13 Exercise. Show that every subcanonical topological site for which the essential image of $|\cdot|$ is $\{\varnothing, *\} \subseteq \mathsf{Top}$ is of the form (2.8.3.12).

2.8.14 Exercise. Let $D : K \to \mathsf{C}$ be a diagram in a subcanonical topological ∞-site C and suppose that $\lim_K |D| = \varnothing$. Show that the limit $\lim_K D = \varnothing$ (the initial object).

2.8.15 Lemma. *If C is subcanonical, then isomorphism is a local property of morphisms in C.*

Proof. Suppose $f : X \to Y$ is a morphism in C which is a local isomorphism, meaning there exists an open cover $Y = \bigcup_i U_i$ for which each restriction $X \times_Y U_i \to U_i$ is an isomorphism. Denote by $S \in J(Y)$ the covering sieve on Y consisting of those open $U \subseteq Y$ for which $X \times_Y U \to U$ is an isomorphism. Now consider the colimit over S of these isomorphisms.

$$\begin{tikzcd} \operatorname*{colim}_{U \in S} X \times_Y U \arrow[r, "\sim"] \arrow[d, "\sim"] & X \arrow[d] \\ \operatorname*{colim}_{U \in S} U \arrow[r, "\sim"] & Y \end{tikzcd} \tag{2.8.15.1}$$

The result is isomorphic to $X \to Y$ since coverings are colimits in a subcanonical topological ∞-site (2.8.9). □

Topological functors

⋆ **2.8.16 Definition** (Topological functor)**.** Let $(\mathsf{C}, |\cdot|_{\mathsf{C}})$ and $(\mathsf{D}, |\cdot|_{\mathsf{D}})$ be topological sites. A *topological functor* $(\mathsf{C}, |\cdot|_{\mathsf{C}}) \to (\mathsf{D}, |\cdot|_{\mathsf{D}})$ is a functor $f : \mathsf{C} \to \mathsf{D}$ preserving open embeddings, together with a natural transformation $\pi : |f(\cdot)|_{\mathsf{D}} \to |\cdot|_{\mathsf{C}}$ which sends open embeddings to pullbacks.

$$\begin{tikzcd} \mathsf{C} \arrow[r, "\times 0"] \arrow[d, "f"'] & \mathsf{C} \times \Delta^1 \arrow[d, "\pi"'] & \mathsf{C} \arrow[l, "\times 1"'] \arrow[dl, "|\cdot|_{\mathsf{C}}"] \\ \mathsf{D} \arrow[r, "|\cdot|_{\mathsf{D}}"] & \mathsf{Top} & \end{tikzcd} \tag{2.8.16.1}$$

A topological functor (f, π) is called *strict* when π is a natural isomorphism. Topological functors from C to D form an ∞-category denoted $\mathsf{Top}(\mathsf{C}, \mathsf{D})$, namely the full subcategory $\mathsf{Fun}(\mathsf{C}, \mathsf{D})_{(|\cdot|_{\mathsf{D}} \circ -)/|\cdot|_{\mathsf{C}}}$ spanned by those pairs (f, π) for which f preserves open embeddings and π sends open embeddings to pullbacks.

2.8.17 Exercise. Show that the following are topological functors.

(2.8.17.1) The forgetful functor $\mathsf{Sm} \to \mathsf{Top}$.

(2.8.17.2) The functor $|\cdot| : \mathsf{C} \to \mathsf{Top}$ for any topological site C.

(2.8.17.3) The functor $\mathsf{Vect} \rtimes \mathsf{Top} \to \mathsf{Top}$ sending (X, V) to the total space of V.

(2.8.17.4) The forgetful functor $\mathsf{Sm}^n \to \mathsf{Sm}^m$ for $n \geq m$, where Sm^k denotes the category of C^k-manifolds (i.e. with transition maps of class C^k rather than smooth).

(2.8.17.5) The inverse image functor $f^{-1} = \mathsf{Open}(f) : \mathsf{Open}(Y) \to \mathsf{Open}(X)$ associated to a continuous map of topological spaces $f : X \to Y$.

(2.8.17.6) The functor $\mathsf{Top} \to \mathsf{Top}$ given by sending a topological space to its underlying set equipped with the discrete topology.

(2.8.17.7) The functor $\mathsf{Top} \to \mathsf{Top}$ given by $X \mapsto X \times A$ (any fixed topological space A).

(2.8.17.8) The functor $\mathsf{Sm} \to \mathsf{Sm}$ given by $X \mapsto TX$ (the tangent bundle).

(2.8.17.9) The functor $\mathsf{Sm}^{\mathsf{lociso}} \to \mathsf{Sm}^{\mathsf{lociso}}$ given by sending a smooth manifold to its frame bundle.

2.8.18 Exercise. Show that a topological functor preserves pullbacks of open embeddings.

2.8.19 Exercise. Show that a natural transformation of topological functors $f \to g$ sends open embeddings to pullbacks.

2.8.20 Exercise. Show that a topological functor preserves open coverings, and hence that presheaf pullback along a topological functor sends sheaves to sheaves.

2.8.21 Exercise. Let $f : \mathsf{C} \to \mathsf{D}$ be a strict topological functor. Show that the essential image of f is a topological ∞-site and that the functors $\mathsf{C} \to \operatorname{im}(f) \to \mathsf{D}$ are both strict topological.

2.8.22 Exercise. Let $(f, \pi) : (\mathsf{C}, |\cdot|_{\mathsf{C}}) \to (\mathsf{D}, |\cdot|_{\mathsf{D}})$ be a topological functor. Since f preserves open embeddings, it restricts to a functor $(\mathsf{C} \downarrow^{\mathsf{opemb}} X) \to (\mathsf{D} \downarrow^{\mathsf{opemb}} f(X))$. Show that under the identifications $(\mathsf{C} \downarrow^{\mathsf{opemb}} X) = \mathsf{Open}(|X|)$ and $(\mathsf{D} \downarrow^{\mathsf{opemb}} f(X)) = \mathsf{Open}(|f(X)|)$, this functor is canonically identifed with $\mathsf{Open}(\pi_X)$.

2.8.23 Lemma (Converting a category-theoretic universal property into a site-theoretic universal property)**.** *Let $i : \mathsf{C} \to \overline{\mathsf{C}}$ be a strict topological functor, and let E be a topological ∞-site. Let α and $\overline{\alpha}$ be conditions on functors to E from C and $\overline{\mathsf{C}}$ (respectively), and suppose the following conditions hold:*

(2.8.23.1) *The pullback functor $i^* : \mathsf{Fun}_{\overline{\alpha}}(\overline{\mathsf{C}}, \mathsf{E}) \xrightarrow{\sim} \mathsf{Fun}_{\alpha}(\mathsf{C}, \mathsf{E})$ is an equivalence.*

(2.8.23.2) *The unit map $|\cdot|_{\overline{\mathsf{C}}} \to i_* i^* |\cdot|_{\overline{\mathsf{C}}} = i_* |\cdot|_{\mathsf{C}}$ (associated to the adjunction (i^*, i_*) of functors $i^* : \mathsf{Fun}(\overline{\mathsf{C}}, \mathsf{Top}) \rightleftarrows \mathsf{Fun}(\mathsf{C}, \mathsf{Top}) : i_*$, which exists since Top is complete) is an isomorphism.*

(2.8.23.3) *For all $f : \overline{\mathsf{C}} \to \mathsf{E}$ satisfying $\overline{\alpha}$ and all $\pi : |\cdot|_{\mathsf{E}} \circ f \to |\cdot|_{\overline{\mathsf{C}}}$, if f and π send open embeddings in C to open embeddings and pullbacks (respectively), then they do the same for open embeddings in $\overline{\mathsf{C}}$.*

In this case, the pullback map $i^* : \mathsf{Top}_{\overline{\alpha}}(\overline{\mathsf{C}}, \mathsf{E}) \xrightarrow{\sim} \mathsf{Top}_{\alpha}(\mathsf{C}, \mathsf{E})$ is an equivalence. In place of (2.8.23.2)*, the following condition may also be substituted:*

(2.8.23.4) *There exist conditions β and $\overline{\beta}$ on functors to Top from C and $\overline{\mathsf{C}}$ (respectively) satisfied by $|\cdot|_{\overline{\mathsf{C}}}$ and $|\cdot|_{\mathsf{C}}$ such that composition with $|\cdot|_{\mathsf{E}}$ sends α (resp. $\overline{\alpha}$) functors to β (resp. $\overline{\beta}$) functors and $i^* : \mathsf{Fun}_{\overline{\beta}}(\overline{\mathsf{C}}, \mathsf{Top}) \xrightarrow{\sim} \mathsf{Fun}_{\beta}(\mathsf{C}, \mathsf{Top})$ is an equivalence.*

Proof. The first step is to upgrade the equivalence $i^* : \mathsf{Fun}_{\overline{\alpha}}(\overline{\mathsf{C}}, \mathsf{E}) \xrightarrow{\sim} \mathsf{Fun}_{\alpha}(\mathsf{C}, \mathsf{E})$ (2.8.23.1) to an equivalence of slice categories

$$i^* : \mathsf{Fun}_{\overline{\alpha}}(\overline{\mathsf{C}}, \mathsf{E})_{(|\cdot|_{\mathsf{E}} \circ -)/|\cdot|_{\overline{\mathsf{C}}}} \xrightarrow{\sim} \mathsf{Fun}_{\alpha}(\mathsf{C}, \mathsf{E})_{(|\cdot|_{\mathsf{E}} \circ -)/|\cdot|_{\mathsf{C}}}. \tag{2.8.23.5}$$

To do this, it suffices to show that for $f : \overline{\mathsf{C}} \to \mathsf{E}$ satisfying $\overline{\alpha}$, the tautological map

$$\mathrm{Hom}_{\mathsf{Fun}(\overline{\mathsf{C}},\mathsf{Top})}(|f(\cdot)|_{\mathsf{E}}, |\cdot|_{\overline{\mathsf{C}}}) \to \mathrm{Hom}_{\mathsf{Fun}(\mathsf{C},\mathsf{Top})}(|f(\cdot)|_{\mathsf{E}}, |\cdot|_{\mathsf{C}}) \tag{2.8.23.6}$$

is a homotopy equivalence. By the adjunction (i^*, i_*), the right side may also be written as $\mathrm{Hom}_{\mathsf{Fun}(\overline{\mathsf{C}},\mathsf{Top})}(|f(\cdot)|_{\mathsf{E}}, i_*|\cdot|_{\mathsf{C}})$, and the map is then identified with composition with the unit map $|\cdot|_{\overline{\mathsf{C}}} \to i_*i^*|\cdot|_{\overline{\mathsf{C}}} = i_*|\cdot|_{\mathsf{C}}$, which is an isomorphism by hypothesis (2.8.23.2). Alternatively, it is evident that (2.8.23.4) also implies that (2.8.23.6) is an isomorphism.

The second step is to upgrade the equivalence of slice categories (2.8.23.5) to the desired equivalence $i^* : \mathsf{Top}_{\overline{\alpha}}(\overline{\mathsf{C}}, \mathsf{E}) \xrightarrow{\sim} \mathsf{Top}_{\alpha}(\mathsf{C}, \mathsf{E})$ of ∞-categories of topological functors. These are full subcategories of the domain and target of (2.8.23.5), and the assertion that they coincide under i^* is precisely the condition (2.8.23.3). □

2.8.24 Lemma. *Let $i : \mathsf{C} \to \overline{\mathsf{C}}$ be a strict topological functor. Fix a class $\mathcal{Q}$ of pullbacks in $\overline{\mathsf{C}}$, and suppose that every open embedding in $\overline{\mathsf{C}}$ is a $\mathcal{Q}$-pullback of an open embedding in C. Let $f : \overline{\mathsf{C}} \to \mathsf{E}$ be a functor sending $\mathcal{Q}$-pullbacks to pullbacks, and let $\pi : |\cdot|_{\mathsf{E}} \circ f \to |\cdot|_{\overline{\mathsf{C}}}$ be a natural transformation. If f and π send open embeddings in C to open embeddings and pullbacks (respectively), then they do the same for open embeddings in $\overline{\mathsf{C}}$.*

Proof. Suppose that f and π send open embeddings in C to open embeddings and pullbacks (respectively), and let us show they do the same for open embeddings in $\overline{\mathsf{C}}$. Fix an open embedding $X \to Y$ in $\overline{\mathsf{C}}$, and express it as a $\mathcal{Q}$-pullback of an open embedding $U \to M$ in C.

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ U \arrow[r] & M \end{tikzcd} \tag{2.8.24.1}$$

Since f send $\mathcal{Q}$-pullbacks to pullbacks, we see that $f(X) \to f(Y)$ is a pullback of $f(U) \to f(M)$. The latter is an open embedding by hypothesis on f, hence so is the former since open embeddings are preserved under pullback in any topological ∞-site E (2.8.5). Applying π to the pullback square (2.8.24.1) yields a cubical diagram.

$$\begin{tikzcd} {|f(X)|_{\mathsf{E}}} \arrow[rr] \arrow[dr] \arrow[dd] & & {|f(Y)|_{\mathsf{E}}} \arrow[dr] \arrow[dd] & \\ & {|X|_{\mathsf{C}}} \arrow[rr] \arrow[dd] & & {|Y|_{\mathsf{C}}} \arrow[dd] \\ {|f(U)|_{\mathsf{E}}} \arrow[rr] \arrow[dr] & & {|f(M)|_{\mathsf{E}}} \arrow[dr] & \\ & {|U|_{\mathsf{C}}} \arrow[rr] & & {|M|_{\mathsf{C}}} \end{tikzcd} \tag{2.8.24.2}$$

By hypothesis on π, the lower square $\pi(U \to M)$ is a pullback. The squares $|\cdot|_{\mathsf{C}}(2.8.24.1)$ and $|f(\cdot)|_{\mathsf{E}}(2.8.24.1)$ are pullbacks since $|\cdot|_{\mathsf{C}}$ and $|\cdot|_{\mathsf{E}}$ preserve pullbacks of open embeddings (2.8.5). It follows from cancellation (1.1.57) that the top square $\pi(X \to Y)$ is a pullback. □

Presheaves on a topological site

We have already seen how the category of presheaves $\mathsf{P}(\mathsf{C})$ is a useful enlargement of a category C. We now explore this construction in the case that C is a topological site.

2.8.25 Definition. Let C be a topological ∞-site. We equip the ∞-category of presheaves $\mathsf{P}(\mathsf{C})$ with the functor $|\cdot|_{\mathsf{P}(\mathsf{C})}$ defined as the composition

$$\mathsf{P}(\mathsf{C}) \xrightarrow{|\cdot|_{\mathsf{C}!}} \mathsf{P}(\mathsf{Top}) \xrightarrow{\mathrm{colim}} \mathsf{Top} \tag{2.8.25.1}$$

of left Kan extension along $|\cdot|_{\mathsf{C}}$ and the colimit functor. Thus $|\cdot|_{\mathsf{P}(\mathsf{C})} : \mathsf{P}(\mathsf{C}) \to \mathsf{Top}$ is left adjoint to the composition $|\cdot|^*_{\mathsf{C}} \circ \text{よ}_{\mathsf{Top}} : \mathsf{Top} \to \mathsf{P}(\mathsf{C})$ which we will usually abbreviate as $|\cdot|^*_{\mathsf{C}}$.

The following technical characterization of open embeddings in $\mathsf{P}(\mathsf{C})$ will be very useful.

2.8.26 Lemma. *Let C be a topological ∞-site. For a morphism $F \to G$ in $\mathsf{P}(\mathsf{C})$, the following are equivalent:*

(2.8.26.1) *$F \to G$ is an open embedding with respect to $|\cdot|_{\mathsf{P}(\mathsf{C})}$ in the sense of* (2.8.1)*.*

(2.8.26.2) *$|F|_{\mathsf{P}(\mathsf{C})} \to |G|_{\mathsf{P}(\mathsf{C})}$ is an open embedding and the unit map from $F \to G$ to $|\cdot|^*_{\mathsf{C}}(|F|_{\mathsf{P}(\mathsf{C})} \to |G|_{\mathsf{P}(\mathsf{C})})$ is a pullback.*

(2.8.26.3) *$F \to G$ is a pullback of $|\cdot|^*_{\mathsf{C}}$ of an open embedding in Top.*

(2.8.26.4) *$F \to G$ is an open embedding in the sense of* (1.1.124) *(the pullback $F \times_G c \to c$ is an open embedding in C for every $c \in \mathsf{C}$ and every map $c \to G$).*

(2.8.26.5) *$F \to G$ is a colimit in $\mathsf{Fun}(\Delta^1, \mathsf{P}(\mathsf{C}))$ of a diagram $K \to \mathsf{Fun}(\Delta^1, \mathsf{C})$ which sends vertices to open embeddings and sends edges to pullbacks.*

Proof. The equivalence (2.8.26.1)$\Leftrightarrow$(2.8.26.2) is a general categorical fact (1.4.150) (using the fact that $|\cdot|^*_{\mathsf{C}} : \mathsf{Top} \to \mathsf{P}(\mathsf{C})$ is right adjoint to $|\cdot|_{\mathsf{P}(\mathsf{C})}$). Certainly (2.8.26.2)$\Rightarrow$(2.8.26.3).

Let us show (2.8.26.3) $\Longrightarrow$ (2.8.26.4). Property (2.8.26.3) is certainly preserved under pullback, so it suffices to show that if G is representable and $F \to G$ is a pullback of $|\cdot|^*_{\mathsf{C}}$ of an open embedding in Top, then $F \to G$ is an open embedding in C. This holds since C has all open embeddings.

Let us show (2.8.26.4) $\Longrightarrow$ (2.8.26.5). Write G as the colimit $G = \mathrm{colim}_K\, p$ in $\mathsf{P}(\mathsf{C})$ of a diagram $p : K \to \mathsf{C}$. Since presheaf fiber product is cocontinuous (**??**), we have $F = \mathrm{colim}_K\,(p \times_G F)$. Thus $(F \to G)$ is the colimit in $\mathsf{Fun}(\Delta^1, \mathsf{P}(\mathsf{C}))$ of the diagram $p \times_G (F \to G) : K \to \mathsf{Fun}(\Delta^1, \mathsf{C})$. This diagram sends vertices in K to open embeddings in C by hypothesis, and it sends edges in K to pullbacks by construction.

Let us show (2.8.26.5) $\Longrightarrow$ (2.8.26.2). Suppose $F \to G$ is a colimit in $\mathsf{Fun}(\Delta^1, \mathsf{P}(\mathsf{C}))$ of a diagram $p : K \to \mathsf{Fun}(\Delta^1, \mathsf{C})$ which sends vertices to open embeddings and sends edges to pullbacks. The functor $|\cdot|_{\mathsf{C}}$ preserves open embeddings and pullbacks of open embeddings (2.8.5), so the diagram $|p| : K \to \mathsf{Fun}(\Delta^1, \mathsf{Top})$ has the same property. It follows from the explicit description of colimits of topological spaces that $(|F| \to |G|) = \mathrm{colim}_K\, |p|_{\mathsf{C}}$ is an open embedding of topological spaces and that the map from $|p(k)|$ to this open embedding is a pullback for every vertex $k \in K$. It follows that the map from $p(k)$ to $|\cdot|^*_{\mathsf{C}}|F| \to |\cdot|^*_{\mathsf{C}}|G|$ is a pullback for every $k \in K$. Since presheaf fiber product is cocontinuous (**??**), we conclude that the map from $F \to G$ to $|\cdot|^*_{\mathsf{C}}|F| \to |\cdot|^*_{\mathsf{C}}|G|$ is a pullback. $\square$

2.8.27 Corollary. *The ∞-category of presheaves $\mathsf{P(C)}$ with the functor $|\cdot|_{\mathsf{P(C)}}$ is a topological ∞-site.*

Proof. We must show that $\mathsf{P(C)}$ has all open embeddings. Let $G \in \mathsf{P(C)}$, and write G as the colimit $G = \operatorname{colim}_K p$ in $\mathsf{P(C)}$ of a diagram $p : K \to \mathsf{C}$. By the explicit description of colimits of topological spaces, an open subset of $|G| = \operatorname{colim}_K |p|$ is the same as a choice of open subset of $|p(k)|$ for every $k \in K$, compatible with pullback along every edge of K. Since C has all open embeddings, we can promote such a collection to a diagram $p : K \to \mathsf{Fun}(\Delta^1, \mathsf{C})$ satisfying (2.8.26.5). The resulting open subset of $|G|$ is the one we started with by construction. □

2.8.28 Exercise (Coarse local isomorphism)**.** The notion of a local isomorphism in C induces a notion of a local isomorphism in $\mathsf{P(C)}$ via pullback (1.1.124). The topological ∞-site structure on $\mathsf{P(C)}$ also gives rise to a notion when a morphism in $\mathsf{P(C)}$ is to be called a local isomorphism (2.1.10), which to distinguish from the former notion we will a *coarse local isomorphism*. Show that a coarse local isomorphism is a local isomorphism, but that the converse need not hold.

2.8.29 Lemma. *The Yoneda functor $\mathsf{C} \to \mathsf{P(C)}$ of any topological ∞-site C is a strict topological functor.*

Proof. There is a tautological isomorphism $|\text{よ}_{\mathsf{C}}(\cdot)|_{\mathsf{P(C)}} = |\cdot|_{\mathsf{C}}$. An open embedding in C remains an open embedding in $\mathsf{P(C)}$ by (2.8.26.5). □

Recall that for any functor $f : \mathsf{C} \to \mathsf{D}$, the presheaf pullback $f^* : \mathsf{P(D)} \to \mathsf{P(D)}$ has a left adjoint called *left Kan extension* $f_! : \mathsf{P(C)} \to \mathsf{P(D)}$ (**??**).

2.8.30 Lemma. *Let $f : \mathsf{C} \to \mathsf{D}$ be a topological functor. The left Kan extension functor $f_! : \mathsf{P(C)} \to \mathsf{P(D)}$ is topological when equipped with the unique transformation $|f_!(\cdot)|_{\mathsf{P(D)}} \to |\cdot|_{\mathsf{P(C)}}$ restricting to the given transformation $|f(\cdot)|_{\mathsf{D}} \to |\cdot|_{\mathsf{C}}$. If f is strict then so is $f_!$.*

Proof. The functors $|\cdot|_{\mathsf{P(C)}}$, $|\cdot|_{\mathsf{P(D)}}$, and $f_!$ are cocontinuous, so by the universal property of presheaf categories (1.1.118), natural transformations $|f_!(\cdot)|_{\mathsf{P(D)}} \to |\cdot|_{\mathsf{P(C)}}$ are the same as natural transformations $|f(\cdot)|_{\mathsf{D}} \to |\cdot|_{\mathsf{C}}$ (and moreover this correspondence respects isomorphisms).

Now let us show that $f_!$ sends open embeddings to open embeddings and $|f_!(\cdot)|_{\mathsf{P(D)}} \to |\cdot|_{\mathsf{P(C)}}$ sends open embeddings to pullbacks. Write an open embedding $F \to G$ in $\mathsf{P(C)}$ as a colimit of a diagram $p : K \to \mathsf{Fun}(\Delta^1, \mathsf{C})$ of open embeddings in C as in (2.8.26.5). The pushforward $f_!F \to f_!G$ is the colimit of $f(p)$, from which the result follows by inspection. □

2.8.31 Lemma. *Let $f : \mathsf{C} \to \mathsf{D}$ be a strict topological functor. The pullback functor $f^* : \mathsf{P(D)} \to \mathsf{P(C)}$ is topological when equipped with the natural transformation $|f^*(\cdot)|_{\mathsf{P(C)}} = |f_!f^*(\cdot)|_{\mathsf{P(D)}} \to |\cdot|_{\mathsf{P(D)}}$ induced by the adjunction $(f^*, f_!)$.*

Proof. Let $F \to G$ be an open embedding in $\mathsf{P(D)}$. Every such open embedding is a pullback of $|\cdot|^*_{\mathsf{D}}(U \to X)$ for some open embedding of topological spaces $U \to X$ (2.8.26.3). We have $f^*|\cdot|^*_{\mathsf{D}} = |\cdot|^*_{\mathsf{C}}$ since f is strict, so we may apply f^* (which is continuous) to see that $f^*F \to f^*G$

is a pullback of $|\cdot|^*_{\mathsf{C}}(U \to X)$, hence is an open embedding. Now we saw in (2.8.26) that moreover $|F|_{\mathsf{P}(\mathsf{D})} \to |G|_{\mathsf{P}(\mathsf{D})}$ and $|f^*F|_{\mathsf{P}(\mathsf{C})} \to |f^*G|_{\mathsf{P}(\mathsf{D})}$ are both pullbacks of $U \to X$. By cancellation (1.1.57), this implies $|f^*(\cdot)|_{\mathsf{P}(\mathsf{C})} \to |\cdot|_{\mathsf{P}(\mathsf{D})}$ sends $F \to G$ to a pullback. □

The ∞-category of presheaves $\mathsf{P}(\mathsf{C})$ on an ∞-category C satisfies a purely categorical universal property (1.1.118). When C is a topological ∞-site, so is $\mathsf{P}(\mathsf{C})$, and it is natural to ask whether the topological functor $\mathsf{C} \hookrightarrow \mathsf{P}(\mathsf{C})$ satisfies a universal property which characterizes $\mathsf{P}(\mathsf{C})$ uniquely as a topological ∞-site. Let us now deduce such a universal property.

2.8.32 Proposition (Universal property of presheaves on a topological ∞-site)**.** *Let C be a topological ∞-site. Let E be a cocomplete topological ∞-site for which $|\cdot|_{\mathsf{E}}$ is cocontinuous and for which the colimit of any diagram $K \to \mathsf{Fun}(\Delta^1, \mathsf{E})$ sending vertices to open embeddings and edges to pullbacks is an open embedding. Pullback along the strict topological functor $\mathsf{C} \to \mathsf{P}(\mathsf{C})$ induces an equivalence between the following ∞-categories of functors:*

(2.8.32.1) *Cocontinuous topological functors $\mathsf{P}(\mathsf{C}) \to \mathsf{E}$.*

(2.8.32.2) *Topological functors $\mathsf{C} \to \mathsf{E}$.*

Proof. We apply (2.8.23).

Take properties α and β to be vacuous and properties $\overline{\alpha}$ and $\overline{\beta}$ to be 'cocontinuous'. Pullback i^* is an equivalence by the category theoretic universal property of $\mathsf{C} \to \mathsf{P}(\mathsf{C})$ (1.1.118). The functor $|\cdot|_{\mathsf{P}(\mathsf{C})}$ is cocontinuous by definition, and composition with $|\cdot|_{\mathsf{E}}$ preserves cocontinuity since $|\cdot|_{\mathsf{E}}$ is assumed cocontinuous. This gives (2.8.23.1) and (2.8.23.4)

It remains to verify (2.8.23.3). Fix $f : \mathsf{P}(\mathsf{C}) \to \mathsf{E}$ cocontinuous and $\pi : |\cdot|_{\mathsf{E}} \circ f \to |\cdot|_{\overline{\mathsf{C}}}$, and suppose f and π send open embeddings in C to open embeddings and pullbacks (respectively). Express an open embedding $F \to G$ in $\mathsf{P}(\mathsf{C})$ as the colimit of a diagram $K \to \mathsf{Fun}(\Delta^1, \mathsf{C})$ sending vertices to open embeddings and edges to pullbacks (2.8.26.5). Since f is cocontinuous, the map $f(F \to G)$ is the colimit of the diagram $K \to \mathsf{Fun}(\Delta^1, \mathsf{E})$ obtained by composing with f. Since $(f, \pi)|_{\mathsf{C}}$ is a topological functor, this composed diagram also sends vertices to open embeddings and edges to pullbacks, hence its colimit is an open embedding by hypothesis on E. The square $\pi(F \to G)$ is a pullback by inspection (using the fact that π sends open embeddings in C to pullbacks and the explicit description of colimits of topological spaces). □

Sheaves on a topological site

We now establish some basic properties of the ∞-category of sheaves $\mathsf{Shv}(\mathsf{C}) \subseteq \mathsf{P}(\mathsf{C})$ on a topological ∞-site C.

⋆ **2.8.33 Proposition** (Universal property of sheaves on a topological ∞-site)**.** *Let C be a topological ∞-site. For any cocomplete ∞-category E, pullback along the functors $\mathsf{C} \xrightarrow{y_{\mathsf{C}}} \mathsf{P}(\mathsf{C}) \xrightarrow{\#} \mathsf{Shv}(\mathsf{C})$ defines equivalences between the following ∞-categories of functors:*

(2.8.33.1) *Cocontinuous functors $\mathsf{Shv}(\mathsf{C}) \to \mathsf{E}$.*

(2.8.33.2) *Cocontinuous functors $\mathsf{P}(\mathsf{C}) \to \mathsf{E}$ sending sheafifications to isomorphisms.*

(2.8.33.3) *Cocontinuous functors* $\mathsf{P}(\mathsf{C}) \to \mathsf{E}$ *sending nerves of open coverings in* C *to isomorphisms.*

(2.8.33.4) *Cosheaves* $\mathsf{C} \to \mathsf{E}$.

Proof. The reasoning given for the case $\mathsf{C} = \mathsf{Top}$ (2.3.3) applies without change. $\square$

* *

2.8.34 Definition. Let C be a topological ∞-site. We equip the ∞-category of sheaves $\mathsf{Shv}(\mathsf{C})$ with the functor $|\cdot|_{\mathsf{Shv}(\mathsf{C})}$ defined as the restriction of $|\cdot|_{\mathsf{P}(\mathsf{C})}$ to the full subcategory $\mathsf{Shv}(\mathsf{C}) \subseteq \mathsf{P}(\mathsf{C})$. Note that $|\cdot|_{\mathsf{Shv}(\mathsf{C})}$ is cocontinuous since $|\cdot|_{\mathsf{P}(\mathsf{C})}$ is cocontinuous and $|\cdot|_{\mathsf{C}}$ sends open coverings to colimits (2.8.33).

⋆ **2.8.35 Lemma.** *Let* C *be a topological* ∞*-site. The* ∞*-category of sheaves* $\mathsf{Shv}(\mathsf{C})$ *with the functor* $|\cdot|_{\mathsf{Shv}(\mathsf{C})}$ *is a topological* ∞*-site, and the adjoint pair* $i : \mathsf{Shv}(\mathsf{C}) \rightleftarrows \mathsf{P}(\mathsf{C}) : \#$ *are strict topological functors.*

Proof. If $G \in \mathsf{Shv}(\mathsf{C})$ and $F \to G$ is an open embedding in $\mathsf{P}(\mathsf{C})$, then F is a sheaf. Indeed, we saw in (2.8.27) that such an open embedding is a pullback of $|\cdot|_{\mathsf{C}}^*(U \to X)$ for some open embedding of topological spaces $U \to X$, and this expresses F as a fiber product of sheaves $|\cdot|_{\mathsf{C}}^*U \times_{|\cdot|_{\mathsf{C}}^*X} G$ which is thus itself a sheaf (**??**). It follows that $\mathsf{Shv}(\mathsf{C}) \subseteq \mathsf{P}(\mathsf{C})$ is a topological ∞-site and that its inclusion functor i is a strict topological functor.

Now let us show that sheafification $\#$ is topological. If $F \to G$ is an open embedding in $\mathsf{P}(\mathsf{C})$, then it is a pullback of $|\cdot|_{\mathsf{C}}^*(U \to X)$ for some open embedding of topological spaces $U \to X$. Sheafification preserves finite limits (**??**), so $F^\# \to G^\#$ is also a pullback of $|\cdot|_{\mathsf{C}}^*(U \to X)$, hence is also an open embedding. Thus sheafification preserves open embeddings. Finally, we should show that the canonical map $|\cdot|_{\mathsf{P}(\mathsf{C})} \to |\#(\cdot)|_{\mathsf{Shv}(\mathsf{C})}$ arising from the sheafification adjunction $(\#, i)$ is an isomorphism. That is, we should show that $|\cdot|_{\mathsf{P}(\mathsf{C})}$ sends sheafifications to isomorphisms. By (2.8.33), this is equivalent to $|\cdot|_{\mathsf{C}} : \mathsf{C} \to \mathsf{Top}$ being a cosheaf. Now $|\cdot|_{\mathsf{C}}$ sends open coverings to open coverings, and open coverings in Top are colimits. $\square$

⋆ **2.8.36 Definition** (Sheaf left Kan extension)**.** Let $f : \mathsf{C} \to \mathsf{D}$ be a topological functor. Since presheaf pullback f^* sends sheaves to sheaves (2.8.20) and sheaves are a reflective subcategory of presheaves (2.2.15), it follows that the adjunction $(f_!, f^*)$ of functors $f_! : \mathsf{P}(\mathsf{C}) \rightleftarrows \mathsf{P}(\mathsf{D}) : f^*$ descends to the reflective subcategories of sheaves (1.1.101), producing an adjunction $(f_!, f^*)$ of functors $f_! : \mathsf{Shv}(\mathsf{C}) \rightleftarrows \mathsf{Shv}(\mathsf{D}) : f^*$. Explicitly, sheaf pullback f^* is simply presheaf pullback restricted to sheaves, and sheaf left Kan extension $f_!$ is presheaf left Kan extension followed by sheafification (note that this notation is somewhat hazardous, as sheaf pushforward $f_!$ does not coincide with the restriction of presheaf pushforward $f_!$ to sheaves). Sheaf pushforward and presheaf pushforward are related by the following commuting diagram.

$$\begin{tikzcd} \mathsf{C} \arrow[r, "\mathcal{Y}_{\mathsf{C}}"] \arrow[d, "f"'] & \mathsf{P}(\mathsf{C}) \arrow[r, "\#"] \arrow[d, "f_!"] & \mathsf{Shv}(\mathsf{C}) \arrow[d, "f_!"] \\ \mathsf{C} \arrow[r, "\mathcal{Y}_{\mathsf{D}}"] & \mathsf{P}(\mathsf{D}) \arrow[r, "\#"] & \mathsf{Shv}(\mathsf{D}) \end{tikzcd} \tag{2.8.36.1}$$

Sheaf pushforward $f_!$ is a topological functor by (2.8.30)(2.8.35). When f is strict, sheaf pushforward $f_!$ is strict and sheaf pullback f^* is also a topological functor (2.8.31).

2.8.37 Exercise. Explain why f^* sending sheaves to sheaves implies that the right square in (2.8.36.1) commutes.

2.8.38 Lemma. *If $f : \mathsf{C} \to \mathsf{D}$ is strict, then presheaf pullback f^* commutes with sheafification (i.e. sends sheafifications to sheafifications).*

Proof. For any $X \in \mathsf{C}$, consider the functor $\mathsf{Open}(|X|) \to \mathsf{C}$ and its composition with $\mathsf{C} \to \mathsf{D}$. Pullback under $\mathsf{Open}(|X|) \to \mathsf{C}$ and the composition $\mathsf{Open}(|X|) \to \mathsf{D}$ both commute with sheafification by (**??**) (using, in the latter case, the fact that f is strict). Since the joint pullback under the functors $\mathsf{Open}(|X|) \to \mathsf{C}$ for all $X \in \mathsf{C}$ together reflect isomorphisms, this implies pullback under $\mathsf{C} \to \mathsf{D}$ also commutes with sheafification. □

2.8.39 Exercise. Conclude from (2.8.38) that if $f : \mathsf{C} \to \mathsf{D}$ is strict then sheaf pullback $f^* : \mathsf{Shv}(\mathsf{D}) \to \mathsf{Shv}(\mathsf{C})$ is cocontinuous.

2.8.40 Lemma. *If $f : \mathsf{C} \to \mathsf{D}$ is fully faithful and strict, then $f_! : \mathsf{Shv}(\mathsf{C}) \to \mathsf{Shv}(\mathsf{D})$ is fully faithful (and strict).*

Proof. This is a special case of (1.1.102), recalling that f strict implies f^* commutes with sheafification (2.8.38). □

2.8.41 Lemma. *If a topological functor $f : \mathsf{C} \to \mathsf{D}$ preserves finite products, then $f_! : \mathsf{Shv}(\mathsf{C}) \to \mathsf{Shv}(\mathsf{D})$ does as well.*

Proof. Recall that if $f : \mathsf{C} \to \mathsf{D}$ preserves finite products then $f_! : \mathsf{P}(\mathsf{C}) \to \mathsf{P}(\mathsf{D})$ preserves finite products (1.1.112). Now write sheaf left Kan extension $f_!$ as the composition $\# \circ f_! \circ i$ (where $f_!$ is presheaf left Kan extension), and note that i preserves all limits (**??**) and $\#$ preserves finite limits (**??**). □

2.8.42 Lemma. *The topological ∞-site $\mathsf{Shv}(\mathsf{C})$ is subcanonical.*

Proof. We should show that for every open cover $X = \bigcup_i U_i$ in $\mathsf{Shv}(\mathsf{C})$, the map

$$\operatorname*{colim}^{\mathsf{Shv}(\mathsf{C})}_{\boldsymbol{\Delta}^{\mathsf{op}}} N(X, \{U_i\}_i) \to X \tag{2.8.42.1}$$

is an isomorphism, where we have used the notation

$$N(X, \{U_i\}_i) = \Big(\cdots \substack{\to\\\to\\\to\\\to} \coprod_{i,j,k} U_i \times_X U_j \times_X U_k \substack{\to\\\to\\\to} \coprod_{i,j} U_i \times_X U_j \rightrightarrows \coprod_i U_i \Big) \tag{2.8.42.2}$$

for the Čech simplicial object (2.2.18). Recall that $U_i = X \times_{|\cdot|^*|X|} |\cdot|^*|U_i|$, and observe that

$$N(X, X \times_{|\cdot|^*|X|} |\cdot|^*|U_i|\}_i) = X \times_{|\cdot|^*|X|} |\cdot|^* N(|X|, \{|U_i|\}_i). \tag{2.8.42.3}$$

Now the operation $X\times_{|\cdot|^*|X|}$ commutes with colimits of spaces (**??**), hence with colimits of presheaves since these are computed pointwise (**??**), hence with colimits of sheaves since sheafification preserves finite limits (**??**). We are thus reduced to showing that the map

$$\operatorname*{colim}^{\mathsf{Shv}(\mathsf{C})}_{\mathbf{\Delta}^{\mathsf{op}}} |\cdot|^* N(|X|, \{|U_i|\}_i) \to |\cdot|^*|X| \tag{2.8.42.4}$$

is an isomorphism. The sheaf pullback functor $|\cdot|^* : \mathsf{Shv}(\mathsf{Top}) \to \mathsf{Shv}(\mathsf{C})$ is cocontinuous (2.8.39), so it suffices to show that the map

$$\operatorname*{colim}^{\mathsf{Shv}(\mathsf{Top})}_{\mathbf{\Delta}^{\mathsf{op}}} N(|X|, \{|U_i|\}_i) \to |X| \tag{2.8.42.5}$$

is an isomorphism, which is a special case of (1.4.202)(2.2.19). □

We saw just above that the ∞-category of sheaves $\mathsf{Shv}(\mathsf{C})$ on a topological ∞-site C satisfies a purely categorical universal property (2.8.33). It is natural to ask whether the topological functor $\mathsf{C} \hookrightarrow \mathsf{Shv}(\mathsf{C})$ satisfies a universal property which characterizes $\mathsf{Shv}(\mathsf{C})$ uniquely as a topological ∞-site (like we proved just above for presheaves (2.8.32)). Let us now deduce such a universal property.

2.8.43 Proposition (Universal property of sheaves on a topological ∞-site)**.** *Let C be a topological ∞-site. Let E be a cocomplete topological ∞-site with $|\cdot|_{\mathsf{E}}$ cocontinuous. Pullback along the strict topological functor $\# : \mathsf{P}(\mathsf{C}) \to \mathsf{Shv}(\mathsf{C})$ induces an equivalence between the following ∞-categories of functors:*

(2.8.43.1) *Cocontinuous topological functors $\mathsf{Shv}(\mathsf{C}) \to \mathsf{E}$.*

(2.8.43.2) *Cocontinuous topological functors $\mathsf{P}(\mathsf{C}) \to \mathsf{E}$ sending sheafifications to isomorphisms.*

(2.8.43.3) *Cocontinuous topological functors $\mathsf{P}(\mathsf{C}) \to \mathsf{E}$ sending nerves of open coverings in C to isomorphisms.*

If, in addition, the colimit of any diagram $K \to \mathsf{Fun}(\Delta^1, \mathsf{E})$ sending vertices to open embeddings and edges to pullbacks is an open embedding, then pullback along the strict topological functors $\mathsf{C} \to \mathsf{P}(\mathsf{C}) \xrightarrow{\#} \mathsf{Shv}(\mathsf{C})$ induces equivalences between the above ∞-categories of functors and:

(2.8.43.4) *Topological functors $\mathsf{C} \to \mathsf{E}$ sending coverings to colimits.*

Proof. We first address the equivalence between the first three classes of functors. The equivalence holds without the adjective 'topological' by (2.8.33). To pass from functors to topological functors, we appeal to (2.8.23). Take $\alpha_{\mathsf{Shv}(\mathsf{C})} = \beta_{\mathsf{Shv}(\mathsf{C})}$ to be 'cocontinuous' (2.8.33.1), and take $\alpha_{\mathsf{P}(\mathsf{C})} = \beta_{\mathsf{P}(\mathsf{C})}$ to be 'cocontinuous and sends sheafifications to isomorphisms' (2.8.33.2) (equivalently, 'cocontinuous and sends nerves of open coverings in C to isomorphisms' (2.8.33.3). By (2.8.33), pullback along $\#$ identifies these classes of functors from $\mathsf{P}(\mathsf{C})$ and $\mathsf{Shv}(\mathsf{C})$ to both E and Top. Since $|\cdot|_{\mathsf{E}}$ is cocontinuous, composing with it sends α functors to β functors on both $\mathsf{P}(\mathsf{C})$ and $\mathsf{Shv}(\mathsf{C})$. The condition (2.8.23.3) is satisfied since every open embedding in $\mathsf{Shv}(\mathsf{C})$ is the image of an open embedding in $\mathsf{P}(\mathsf{C})$ (namely, itself, since

$\# : \mathsf{P}(\mathsf{C}) \rightleftarrows \mathsf{Shv}(\mathsf{C}) : i$ are both topological functors (2.8.35)). Thus (2.8.23) gives the desired result.

For the equivalence with the fourth class of functors, recall that the universal property of $\mathsf{C} \hookrightarrow \mathsf{P}(\mathsf{C})$ among topological ∞-sites (2.8.32) identifies cocontinuous topological functors $\mathsf{P}(\mathsf{C}) \to \mathsf{E}$ with topological functors $\mathsf{C} \to \mathsf{E}$. It thus suffices to note that a cocontinuous functor $\mathsf{P}(\mathsf{C}) \to \mathsf{E}$ sends nerves of open coverings in C to isomorphisms iff its restriction to C sends coverings to colimits. □

Perfectness

We call a subcanonical topological ∞-site *perfect* when 'objects can be glued together along open sets' (as formalized in (2.8.44)). We will eventually see that every subcanonical topological ∞-site has a 'perfection' (left adjoint to the inclusion of perfect sites into subcanonical sites) obtained by formally adjoining such gluings (2.8.60). The significance of perfection is that it allows certain properties (e.g. representability (2.8.47) and, in particular, the existence of limits (2.8.48)) to be checked locally.

⋆ **2.8.44 Definition** (Perfect)**.** Let C be a subcanonical topological ∞-site. Consider lifting problems of the shape

$$\begin{tikzcd} S \arrow[r] \arrow[d] & \mathsf{C}^{\mathsf{opemb}} \arrow[d] \\ S^{\triangleright} \arrow[r] \arrow[ur, dashed] & \mathsf{Top}^{\mathsf{opemb}} \end{tikzcd} \tag{2.8.44.1}$$

where the bottom arrow exhibits S as a covering sieve on a topological space. When every such lifting problem has a solution, we say that C is *perfect*.

2.8.45 Lemma. *Let $f : \mathsf{C} \to \mathsf{D}$ be a fully faithful strict topological functor of subcanonical topological ∞-sites. If C is perfect, then being in the image of f is a local property of objects of D. If D is perfect, then the converse holds as well.*

Proof. Perfectness of C and perfectness of D are lifting properties (2.8.44.1). There is also a lifting property to express the statement that being in the image of f is a local property of objects of D, namely we may ask for the lifting property

$$\begin{tikzcd} S \arrow[r] \arrow[d] & \mathsf{C}^{\mathsf{opemb}} \arrow[d] \\ S^{\triangleright} \arrow[r] \arrow[ur, dashed] & \mathsf{D}^{\mathsf{opemb}} \end{tikzcd} \tag{2.8.45.1}$$

for arrows $S^{\triangleright} \to \mathsf{D}^{\mathsf{opemb}}$ arising from an object $d \in \mathsf{D}$ and a covering sieve on $|d|$ (this equivalence follows immediately upon noting that since f is fully faithful, a diagram in D lifts to C iff each object lifts to C and that in this case the lift is unique up to contractible choice).

Now the desired results can be shown as follows. If D is perfect and the lifting problem for $\mathsf{C} \to \mathsf{D}$ (2.8.45.1) is always solvable, then we can solve a lifting problem for $\mathsf{C} \to \mathsf{Top}$

(2.8.44.1) by lifting along $\mathsf{C} \to \mathsf{D} \to \mathsf{Top}$ (note that f is strict) in two steps. If C is perfect, then we can solve a lifting problem for $\mathsf{C} \to \mathsf{D}$ (2.8.45.1) by composing with $|\cdot|_{\mathsf{D}}$ to obtain a lifting problem for $\mathsf{C} \to \mathsf{Top}$ (2.8.44.1) and using the fact that C is perfect (this solves the original lifting problem since the space of solutions to a lifting problem for $\mathsf{D} \to \mathsf{Top}$ (2.8.44.1) is either empty or contractible since D is subcanonical). □

2.8.46 Lemma. *The subcanonical topological ∞-site $\mathsf{Shv}(\mathsf{C})$ is perfect.*

Proof. In fact, to show that $\mathsf{Shv}(\mathsf{C})$ is perfect, all we will use is that it is subcanonical (2.8.42), cocomplete with $|\cdot|$ cocontinuous, and satisfies the 'colimit of open embeddings property' (2.8.26.5) (which is implied for $\mathsf{Shv}(\mathsf{C})$ from the fact that it holds for $\mathsf{P}(\mathsf{C})$ since inclusion and sheafification are strict topological functors (2.8.35)).

Fix a lifting problem (2.8.44.1). The colimit of the top arrow $U : S \to \mathsf{Shv}(\mathsf{C})$ is a lift of the bottom arrow since $|\cdot|_{\mathsf{Shv}(\mathsf{C})}$ is cocontinuous and the bottom arrow is a covering, hence a colimit, in Top. It remains to check that this lift $S^{\triangleright} \to \mathsf{Shv}(\mathsf{C})$ is cartesian, i.e. that the map from U_s (any vertex $s \in S$) to the colimit $X = \operatorname{colim}_S U$ is an open embedding. This map can be expressed as the isomorphism $U_s \to \operatorname{colim}_S(U \times_X U_s)$ followed by the map $\operatorname{colim}_S(U \times_X U_s) \to \operatorname{colim}_S U$, which is an open embedding by the colimit of open embeddings property. □

⋆ **2.8.47 Exercise.** Conclude from (2.8.45) and (2.8.46) that a subcanonical topological ∞-site C is perfect iff representability is a local property for objects of $\mathsf{Shv}(\mathsf{C})$.

Given a perfect topological ∞-site C, the ∞-category $\mathsf{Shv}(\mathsf{C})$ provides a useful context in which to make constructions which may not *a priori* work in C itself. For example, while C need not be complete, the ∞-category $\mathsf{Shv}(\mathsf{C})$ is always complete, and the inclusion $\mathsf{C} \hookrightarrow \mathsf{Shv}(\mathsf{C})$ reflects and lifts limits (these assertions hold for presheaves, hence also for the reflective subcategory of sheaves (1.1.91)). Thus when studying limits in C, it is often useful to enlarge our focus to $\mathsf{Shv}(\mathsf{C})$. Note that we cannot use this strategy for colimits since the opposite of a topological ∞-site is not a topological ∞-site.

⋆ **2.8.48 Example** (Locality of limits)**.** Let C be a perfect topological site, and let us consider the question of whether a given limit $\lim_\alpha X_\alpha$ exists in C or not. This limit certainly exists in $\mathsf{Shv}(\mathsf{C})$, so it is a question of whether this limit in $\mathsf{Shv}(\mathsf{C})$ is representable. Since C is perfect, it is enough to show that $\lim_\alpha^{\mathsf{Shv}(\mathsf{C})} X_\alpha$ is locally representable (2.8.47). Given a map of diagrams $U_\alpha \to X_\alpha$ (over the same indexing shape) where all but finitely many of the constituent maps $U_\alpha \to X_\alpha$ are isomorphisms and all are open embeddings, the resulting map $\lim_\alpha U_\alpha \to \lim_\alpha X_\alpha$ of limits in $\mathsf{Shv}(\mathsf{C})$ is an open embedding (it is a finite iterated pullback of the open embeddings $U_\alpha \to X_\alpha$). In view of the canonical map $|\lim_\alpha^{\mathsf{Shv}(\mathsf{C})} X_\alpha| \to \lim_\alpha |X_\alpha|$, a collection of such 'open subdiagrams' will cover $\lim_\alpha X_\alpha$ provided the open embeddings $\lim_\alpha |U_\alpha| \to \lim_\alpha |X_\alpha|$ cover. At this point, one is naturally led to ask whether the ∞-category of formal limits $\mathsf{Lim}(\mathsf{C}) = \mathsf{Fun}(\mathsf{C}, \mathsf{Spc})^{\mathsf{op}}$ (**??**) is itself a topological ∞-site; we give an answer of sorts to this question in (2.8.65) and the discussion which follows.

2.8.49 Lemma. *Let $f : \mathsf{C} \to \mathsf{D}$ be a topological functor, and let $\mathcal{P}$ and $\mathcal{Q}$ be properties of morphisms in C and D (respectively) preserved under pullback. Suppose that D is perfect and $\mathcal{Q}$ is local on the target* (2.1.11)*. If f sends pullbacks of $\mathcal{P}$-morphisms to pullbacks of $\mathcal{Q}$-morphisms, then so does the left Kan extension functor $f_! : \mathsf{Shv}(\mathsf{C}) \to \mathsf{Shv}(\mathsf{D})$.*

Proof. It was shown in (1.4.244) that the presheaf left Kan extension functor $f_! : \mathsf{P}(\mathsf{C}) \to \mathsf{P}(\mathsf{D})$ sends pullbacks of $\mathcal{P}$-morphisms to pullbacks of $\mathcal{Q}$-morphisms. Now the inclusion of sheaves into presheaves is continuous, and sheafification preserves all finite limits (**??**). It follows that the sheaf left Kan extension $f_! : \mathsf{Shv}(\mathsf{C}) \to \mathsf{Shv}(\mathsf{D})$ sends pullbacks of $\mathcal{P}$-morphisms to pullbacks. To show that sheaf left Kan extension $f_!$ sends $\mathcal{P}$-morphisms to $\mathcal{Q}$-morphisms, it suffices to show that sheafification $\mathsf{P}(\mathsf{D}) \to \mathsf{Shv}(\mathsf{D})$ preserves $\mathcal{Q}$-morphisms. Consider a $\mathcal{Q}$-morphism $F \to G$ in $\mathsf{P}(\mathsf{D})$, and let us show that $F^\# \to G^\#$ is also a $\mathcal{Q}$-morphism. Fix a map $d \to G^\#$ from some $d \in \mathsf{D}$, and let us show $F^\# \times_{G^\#} d \to d$ has $\mathcal{Q}$. Since D is perfect and $\mathcal{Q}$ is local on the target, we may wlog replace d with the elements of an open cover. In particular, we may assume wlog that the morphism $d \to G^\#$ lifts to G (**??**). Since sheafification preserves pullbacks, we have $F^\# \times_{G^\#} d = (F \times_G d)^\# \to d^\# = d$. The morphism $F \times_G d \to d$ in $\mathsf{P}(\mathsf{D})$ lies in the full subcategory $\mathsf{D} \subseteq \mathsf{P}(\mathsf{D})$ and has $\mathcal{Q}$, so sheafification does nothing since D is subcanonical. □

Topological Morita equivalences

⋆ **2.8.50 Definition** (Topological Morita equivalence)**.** A topological functor $f : \mathsf{C} \to \mathsf{D}$ is called a *topological Morita equivalence* when $f_! : \mathsf{Shv}(\mathsf{C}) \to \mathsf{Shv}(\mathsf{D})$ (equivalently, its right adjoint $f^* : \mathsf{Shv}(\mathsf{D}) \to \mathsf{Shv}(\mathsf{C})$) is an equivalence of ∞-categories (when f is strict, it follows that $f_! : \mathsf{Shv}(\mathsf{C}) \rightleftarrows \mathsf{Shv}(\mathsf{D}) : f^*$ are equivalences of topological ∞-sites (2.8.30)).

2.8.51 Example. Let $f : X \to Y$ be a continuous map of topological spaces, and consider the inverse image map $f^{-1} = \mathsf{Open}(f) : \mathsf{Open}(Y) \to \mathsf{Open}(X)$ as a morphism of topological sites. If $\mathsf{Open}(f)$ is a bijection of posets (hence an equivalence of categories), then it is a topological Morita equivalence. It is only strict when f itself is an isomorphism (which need not be the case even if $\mathsf{Open}(f)$ is an isomorphism).

2.8.52 Exercise. Deduce from (2.8.33) that pullback along a topological Morita equivalence $f : \mathsf{C} \to \mathsf{D}$ defines an equivalence between categories of sheaves valued in any complete ∞-category E.

⋆ **2.8.53 Definition** (Topologically fully faithful)**.** Let $f : \mathsf{C} \to \mathsf{D}$ be a strict topological functor. For given $c \in \mathsf{C}$, we may consider the map of presheaves $\mathrm{Hom}_{\mathsf{C}}(-, c) \to \mathrm{Hom}_{\mathsf{D}}(f(-), f(c))$ on C. When this map induces an isomorphism on sheafifications, we say that f is *topologically fully faithful.*

2.8.54 Lemma. *Let $f : \mathsf{C} \to \mathsf{D}$ be a strict topological functor. The sheaf pushforward $f_! : \mathsf{Shv}(\mathsf{C}) \to \mathsf{Shv}(\mathsf{D})$ is fully faithful iff f is topologically fully faithful.*

Proof. The left adjoint $f_!$ is fully faithful iff the unit map $1 \to f^*f_!$ is an isomorphism (1.1.83). Since f is strict, the sheaf pullback f^* is cocontinuous (2.8.39), so the unit map $1 \to f^*f_!$ is a natural transformation between cocontinuous functors. Every object of $\mathsf{Shv}(\mathsf{C})$ is a colimit of objects in the image of sheafified Yoneda $\#よ : \mathsf{C} \to \mathsf{Shv}(\mathsf{C})$, so the unit map is an isomorphism iff its pullback under $\#よ$ is an isomorphism. The pullback of the unit map under $\#よ$ being an isomorphism is exactly what it means for f to be topologically fully faithful (2.8.53). □

2.8.55 Lemma. *The sheafified Yoneda functor* $\mathsf{C} \xrightarrow{よ_\mathsf{C}} \mathsf{P}(\mathsf{C}) \xrightarrow{\#} \mathsf{Shv}(\mathsf{C})$ *is topologically fully faithful.*

Proof. We are to show that the composition

$$\mathrm{Hom}_\mathsf{C}(-,c) \xrightarrow{よ_\mathsf{C}} \mathrm{Hom}_{\mathsf{P}(\mathsf{C})}(よ_\mathsf{C}(-),よ_\mathsf{C}(c)) \xrightarrow{\#} \mathrm{Hom}_{\mathsf{Shv}(\mathsf{C})}(\#よ_\mathsf{C}(-),\#よ_\mathsf{C}(c)) \qquad (2.8.55.1)$$

induces an isomorphism on sheafifications. The first map is an isomorphism since $よ_\mathsf{C}$ is fully faithful (1.1.108). By virtue of the adjunction $(\#, i)$, the second map is the same as

$$\mathrm{Hom}_{\mathsf{P}(\mathsf{C})}(よ_\mathsf{C}(-),よ_\mathsf{C}(c)) \xrightarrow{よ_\mathsf{C}(c)\to\#よ_\mathsf{C}(c)} \mathrm{Hom}_{\mathsf{P}(\mathsf{C})}(よ_\mathsf{C}(-),\#よ_\mathsf{C}(c)). \qquad (2.8.55.2)$$

By Yoneda (1.1.106), this map is just $よ_\mathsf{C}(c)) \to \#よ_\mathsf{C}(c)$, which is a sheafification, hence in particular becomes an isomorphism after applying sheafification. □

2.8.56 Definition (Topologically dominant)**.** A topological functor $f : \mathsf{C} \to \mathsf{D}$ is called *topologically dominant* when every object of D is a locally a retract of objects in the image of f.

⋆ **2.8.57 Lemma.** *A strict topological functor* $f : \mathsf{C} \to \mathsf{D}$ *is a topological Morita equivalence iff it is topologically fully faithful and topologically dominant.*

Proof. This is similar to (1.1.116).

We already saw that for a strict topological functor $f : \mathsf{C} \to \mathsf{D}$, the pushforward $f_! : \mathsf{Shv}(\mathsf{C}) \to \mathsf{Shv}(\mathsf{D})$ is fully faithful iff f is topologically fully faithful (2.8.54). It thus suffices to show that under these equivalent conditions, the pushforward $f_!$ is essentially surjective iff every object of D is locally a retract of an object of C. Note that under these conditions, $f_!$ is a fully faithful cocontinuous functor between the cocomplete ∞-categories, so its image is closed under colimits.

We will show, in fact, that $\#よ_\mathsf{D}(d) \in \mathsf{Shv}(\mathsf{D})$ is in the image of $f_!$ iff d is locally a retract of objects in the image of f (under the assumption that $f_!$ is fully faithful). This implies the desired result since $\mathsf{Shv}(\mathsf{D})$ is generated under colimits by the image of sheafified Yoneda $\#よ_\mathsf{D}$ (1.4.193)(**??**), so $f_!$ is essentially surjective iff its image contains $\#よ_\mathsf{D}(d)$ for all $d \in \mathsf{D}$.

Suppose $d \in \mathsf{D}$ is a retract of $f(c)$ in D. This remains the case after pushing forward under $\#よ_\mathsf{D}$ (indeed, under any functor), so we conclude that $\#よ_\mathsf{D}(d)$ is a retract of $\#よ_\mathsf{D}(f(c)) = f_!(\#よ_\mathsf{C}(c))$. Retracts are colimits (**??**), so $\#よ_\mathsf{D}(d)$ is a colimit of objects in the image of $f_!$, hence is in the image of $f_!$. Now if $d \in \mathsf{D}$ is locally a retract of objects in the image of f, there is an open cover $d = \bigcup_i d_i$ in D with each $\#よ_\mathsf{D}(d_i)$ in the image of $f_!$. The functors $よ_\mathsf{D}$

and $\#$ are topological (2.8.29)(2.8.35), so $\#よ_{\mathsf{D}}(d) = \bigcup_i \#よ_{\mathsf{D}}(d_i)$ is an open cover by objects in the image of $f_!$. Coverings in $\mathsf{Shv}(\mathsf{D})$ are colimits (2.8.42), so we conclude that $\#よ_{\mathsf{D}}(d)$ lies in the image of $f_!$.

Conversely, suppose $\#よ_{\mathsf{D}}(d)$ lies in the image of $f_!$, and let us show that d is locally a retract of objects in the image of f. Every object of $\mathsf{Shv}(\mathsf{C})$ is the colimit of the pushforward under $\#よ_{\mathsf{C}}$ of a diagram $c : K \to \mathsf{C}$ (1.4.193)(**??**) and $f_!$ is cocontinuous, so we have an isomorphism $d^\# = (\operatorname{colim}_K^{\mathsf{P}(\mathsf{D})} f(c))^\#$ in $\mathsf{Shv}(\mathsf{D})$. A morphism $d^\# \to (\operatorname{colim}_K^{\mathsf{P}(\mathsf{D})} f(c))^\#$ in $\mathsf{Shv}(\mathsf{D})$ is, by Yoneda, a section of $(\operatorname{colim}_K^{\mathsf{P}(\mathsf{D})} f(c))^\#$ over d. Every such section lifts to $\operatorname{colim}_K^{\mathsf{P}(\mathsf{D})} f(c)$ after pulling back to an open cover of d (**??**), and every section of $\operatorname{colim}_K^{\mathsf{P}(\mathsf{D})} f(c)$ lifts to $f(c_i)$ for some $i \in K$ since colimits in presheaf categories are computed pointwise (**??**). We therefore have (after replacing d with an open cover thereof) a factorization of the sheafification map $d \to d^\#$ of the form

$$d \to f(c_i) \to \operatorname*{colim}_K^{\mathsf{P}(\mathsf{D})} f(c) \to (\operatorname*{colim}_K^{\mathsf{P}(\mathsf{D})} f(c))^\# = d^\#. \tag{2.8.57.1}$$

Note that since $よ_{\mathsf{D}}$ (2.8.29), $\#$ (2.8.35), and f are strict topological functors, this gives a factorization $|d| \to |c| \to |d|$ of the identity on $|d|$ (i.e. exhibits $|d|$ as a retract of $|c|$). Now the map $f(c_i) \to d^\#$ locally lifts to d for the same reason as above. Replacing c_i with the relevant open cover and d with its pullback, we obtain a pair of maps $d \to f(c_i) \to d$ whose composition is sent to the identity of $d^\#$ by sheafification. It follows that the composition $d \to d$ coincides locally with the identity of d (**??**), so after replacing d with yet another open cover (and c_i by its pullback), we obtain the desired factorization $d \to f(c_i) \to d$ of the identity map of d. □

2.8.58 Definition (Topologically essentially surjective)**.** A topological functor $f : \mathsf{C} \to \mathsf{D}$ is called *topologically essentially surjective* when every object of D admits an open cover by objects in the image of f.

2.8.59 Corollary. *Let $f : \mathsf{C} \to \mathsf{D}$ be a strict topological Morita equivalence, and let E be subcanonical. Pullback along f induces an equivalence between topological functors $\mathsf{D} \to \mathsf{E}$ and $\mathsf{C} \to \mathsf{E}$, provided at least one of the following conditions holds:*

(2.8.59.1) *f is essentially surjective.*

(2.8.59.2) *E is perfect and f is topologically essentially surjective.*

(2.8.59.3) *E is perfect and idempotent-complete.*

Proof. Since $\mathsf{E} \hookrightarrow \mathsf{Shv}(\mathsf{E})$ is a fully faithful (since E is subcanonical) strict topological functor, there is a tautological equivalence between the following ∞-categories of functors:

(2.8.59.4) Topological functors $\mathsf{C} \to \mathsf{E}$.

(2.8.59.5) Topological functors $\mathsf{C} \to \mathsf{Shv}(\mathsf{E})$ with essential image contained in $\mathsf{E} \subseteq \mathsf{Shv}(\mathsf{E})$.

Since $\mathsf{Shv}(\mathsf{E})$ is subcanonical (2.8.42), functors of the latter sort send coverings to colimits. Thus the universal property of $\mathsf{C} \hookrightarrow \mathsf{Shv}(\mathsf{C})$ in the context of topological ∞-sites (2.8.43) implies that the latter is equivalent to:

(2.8.59.6) Cocontinuous topological functors $\mathsf{Shv}(\mathsf{C}) \to \mathsf{Shv}(\mathsf{E})$ whose pre-composition with $\mathsf{C} \to \mathsf{Shv}(\mathsf{C})$ has essential image contained in $\mathsf{E} \subseteq \mathsf{Shv}(\mathsf{E})$.

Now $f_! : \mathsf{Shv}(\mathsf{C}) \to \mathsf{Shv}(\mathsf{D})$ is an equivalence of topological ∞-sites by hypothesis, so pullback under $f_!$ defines an equivalence between:

(2.8.59.7) Cocontinuous topological functors $\mathsf{Shv}(\mathsf{C}) \to \mathsf{Shv}(\mathsf{E})$.

(2.8.59.8) Cocontinuous topological functors $\mathsf{Shv}(\mathsf{D}) \to \mathsf{Shv}(\mathsf{E})$.

To conclude the desired result, it thus suffices to show that if a cocontinuous topological functor $\mathsf{Shv}(\mathsf{D}) \to \mathsf{Shv}(\mathsf{E})$ sends all objects of C to $\mathsf{E} \subseteq \mathsf{Shv}(\mathsf{E})$, then it sends all objects of D to $\mathsf{E} \subseteq \mathsf{Shv}(\mathsf{E})$ as well. This is ensured by any of the conditions (2.8.59.1)(2.8.59.2)(2.8.59.3) (in the last case, note that f is topologically dominant since f is a strict topological Morita equivalence (2.8.57)). □

Subcanonization and perfection

Given a topological ∞-site C, we may wish to 'sheafify its morphism spaces' to obtain a subcanonical topological ∞-site $\mathsf{C}^\#$. We may also wish to 'formally adjoin gluings of objects along open embeddings' to any subcanonical topological ∞-site. Let us now argue that such operations exist and are unique up to contractible choice.

⋆ **2.8.60 Theorem** (Subcanonization, perfection, and idempotent-complete perfection)**.** *Let* C *be a topological* ∞*-site. There exist topologically fully faithful strict topological functors:*

(2.8.60.1) $\mathsf{C} \to \mathsf{C}^\#$ *essentially surjective with* $\mathsf{C}^\#$ *subcanonical.*

(2.8.60.2) $\mathsf{C} \to \mathsf{C}^\#$ *topologically essentially surjective with* $\mathsf{C}^\#$ *perfect.*

(2.8.60.3) $\mathsf{C} \to \mathsf{C}^{\#\pi}$ *topologically dominant with* $\mathsf{C}^{\#\pi}$ *perfect and idempotent-complete.*

Moreover, such functors satisfy the following universal properties, and hence are unique up to contractible choice:

(2.8.60.4) *Pullback along* $\mathsf{C} \to \mathsf{C}^\#$ *is an equivalence between the* ∞*-categories of topological functors to any subcanonical topological* ∞*-site* E.

(2.8.60.5) *Pullback along* $\mathsf{C} \to \mathsf{C}^\#$ *is an equivalence between the* ∞*-categories of topological functors to any perfect topological* ∞*-site* E.

(2.8.60.6) *Pullback along* $\mathsf{C} \to \mathsf{C}^{\#\pi}$ *is an equivalence between the* ∞*-categories of topological functors to any idempotent-complete perfect topological* ∞*-site* E.

Proof. The sheafified Yoneda functor $\mathsf{C} \xrightarrow{\mathcal{Y}_\mathsf{C}} \mathsf{P}(\mathsf{C}) \xrightarrow{\#} \mathsf{Shv}(\mathsf{C})$ is a strict topological functor (2.8.29)(2.8.35) from C to an idempotent-complete perfect topological ∞-site $\mathsf{Shv}(\mathsf{C})$ (2.8.42) (2.8.46). It is topologically fully faithful by (2.8.55).

Take $\mathsf{C}^\# \subseteq \mathsf{Shv}(\mathsf{C})$ to be the essential image of C. Take $\mathsf{C}^\# \subseteq \mathsf{Shv}(\mathsf{C})$ to consist of those objects which are locally in the essential image of C. Take $\mathsf{C}^{\#\pi} \subseteq \mathsf{Shv}(\mathsf{C})$ to consist of those objects which are locally retracts of objects in the essential image of C. It is straightforward to check that these satisfy the desired properties (2.8.60.1)(2.8.60.2)(2.8.60.3). Each is a strict topological functor which is topologically fully faithful and dominant, hence a strict topological Morita equivalence (2.8.57). The desired universal properties thus follow from (2.8.59). □

2.8.61 Exercise. Show that the functors $\mathsf{C} \to \mathsf{C}^{\#} \to \mathsf{C}^{\#} \to \mathsf{C}^{\#\pi}$ preserve all finite limits which exist in C (show the same for $\mathsf{C} \hookrightarrow \mathsf{P}(\mathsf{C}) \xrightarrow{\#} \mathsf{Shv}(\mathsf{C})$ and note that $\mathsf{C}^{\#} \subseteq \mathsf{C}^{\#} \subseteq \mathsf{C}^{\#\pi} \subseteq \mathsf{Shv}(\mathsf{C})$).

2.8.62 Lemma. *If C has finite products, then $\mathsf{C}^{\#}$ has finite products and a topological functor $\mathsf{C}^{\#} \to \mathsf{E}$ preserves finite products iff its pre-composition with $\mathsf{C} \to \mathsf{C}^{\#}$ does.*

Proof. Consider a finite collection of objects $X_i \in \mathsf{C}^{\#}$. By locality of limits (2.8.48), to show that the product $\prod_i X_i$ exists in $\mathsf{C}^{\#}$ or is preserved by a given topological functor out of $\mathsf{C}^{\#}$, it suffices to show the same for any collection of products $\prod_i U_i$ for open subsets $U_i \subseteq X_i$ which jointly cover $\prod_i |X_i|$. Now every object of X_i admits an open cover by objects in the image of $\mathsf{C} \to \mathsf{C}^{\#}$, so it suffices to show existence/preservation for products in $\mathsf{C}^{\#}$ of objects in the image of C. This is immedate from the fact that C has finite products and that they are preserved by $\mathsf{C} \to \mathsf{C}^{\#}$ (2.8.61). $\square$

2.8.63 Proposition. *If C has finite limits, then $\mathsf{C}^{\#}$ has finite limits and a topological functor $\mathsf{C}^{\#} \to \mathsf{E}$ preserves finite (resp. finite cosifted) limits iff its pre-composition with $\mathsf{C} \to \mathsf{C}^{\#}$ does.*

Proof. Given (2.8.61)(2.8.62)(1.4.234), it suffices to show that $\mathsf{C}^{\#}$ has finite cosifted limits and that they are preserved by $\mathsf{C}^{\#} \to \mathsf{E}$ if the composition $\mathsf{C} \to \mathsf{C}^{\#} \to \mathsf{E}$ preserves finite cosifted limits. By locality of limits (2.8.48), it suffices to show that every formal finite cosifted limit in $\mathsf{C}^{\#}$ admits an 'open cover' by formal finite cosifted limits in C, namely the following precise assertion:

(2.8.63.1) Every formal finite cosifted limit in $\mathsf{C}^{\#}$ may be realized by a cosimplicial object in $\mathsf{C}^{\#}$ which admits a collection of levelwise open embeddings from cosimplicial objects in C representing finite cosifted formal limits in C which together cover its topological limit (limit of underlying topological spaces).

The argument to prove (2.8.63.1) will be somewhat delicate.

Consider the cosiftedization X_{cosif} of a finite diagram $X : K \to \mathsf{C}^{\#}$. Fix a point $x \in \lim |X_{\mathrm{cosif}}|$, and consider its image under $\lim |X_{\mathrm{cosif}}| \to \lim |X|$ (not necessarily an isomorphism since $|\cdot| : \mathsf{C} \to \mathsf{Top}$ is not assumed to preserve finite products), which is, concretely, a collection of points $x_k \in |X_k|$ for all vertices $k \in K$ such that for every edge $e : k \to k'$ in K we have $|X_e|(x_k) = x_{k'}$. Our goal is to construct a levelwise open embedding into X_{cosif} which covers this chosen point x and is itself a formal finite cosifted limit.

Recall that X_{cosif} is given explicitly by the Bousfield–Kan transform $X_{\mathrm{cosif}} = X_{\triangle} = \pi_* \ell^* X$ (1.4.238)(1.4.239). In a word, our strategy is to use finiteness of K to find a *germ* of lift of X to C in a neighborhood of the point $x \in \lim |X|$; such a germ does *not* entail an open subdiagram of X, but is sufficient to feed into the Bousfield–Kan transform to produce an open subdiagram of $X_{\triangle}$ containing $x \in \lim |X_{\mathrm{cosif}}|$. It will take some time to make this precise.

To begin, let us specify precisely what we mean by a 'germ of lift' of X from $\mathsf{C}^{\#}$ to C and give the construction of it. Much of this discussion could be clarified by introducing a 'category of germs' associated to any topological ∞-site, however we have no other use for this notion, so we will instead manipulate germs in a somewhat *ad hoc* manner. By a 'germ of lift of X from $\mathsf{C}^{\#}$ to C near x', we mean the following:

(2.8.63.2) For every simplex $f : \Delta^n \to K$, we fix an open neighborhood $x_{f(0)} \in V_f \subseteq X_{f(0)}$, with the property that for every composition $\Delta^n \xrightarrow{a} \Delta^m \xrightarrow{f} K$, we have

$$V_f \subseteq (X_{f(0)} \xrightarrow{X_{f(0)\to f(a(0))}} X_{f(a(0))})^{-1} V_{fa}$$

(note that this implies V_f depends only on the underlying non-degenerate simplex associated to f).

(2.8.63.3) Recall (2.2.22) the cartesian fibration $\mathsf{Open} \rtimes_X K \to K$ classifying open subsets of X; that is, a map $Z \to \mathsf{Open} \rtimes_X K$ is a map $g : Z \to K$ together with a choice of open subset $U_z \subseteq X_{g(z)}$ for every vertex $z \in Z$, such that for every edge $e : z \to z'$ in Z we have $U_z \subseteq (X_{g(z)} \xrightarrow{X_{g(e)}} X_{g(z')})^{-1} U_{z'}$. We denote by $(\mathsf{Open} \rtimes_X K)_V \subseteq \mathsf{Open} \rtimes_X K$ the subcomplex obtained by imposing the additional requirement that for every simplex $f : \Delta^n \to Z$, we have $U_{f(0)} \subseteq V_{gf}$.

(2.8.63.4) There is a tautological functor $U : \mathsf{Open} \rtimes_X K \to \mathsf{C}^\#$ (2.8.8) with an open embedding into the composition $\mathsf{Open} \rtimes_X K \to K \xrightarrow{X} \mathsf{C}^\#$.

$$\begin{tikzcd} \mathsf{Open} \rtimes_X K \arrow[d] \arrow[dr, "U"] \arrow[dr, phantom, shift right=3, "\Downarrow"] & \\ K \arrow[r, "X"'] & \mathsf{C}^\# \end{tikzcd}$$

This functor $U : \mathsf{Open} \rtimes_X K \to \mathsf{C}^\#$ evidently sends *vertical* edges in $\mathsf{Open} \rtimes_X K$ (those whose image in K is degenerate) to open embeddings in $\mathsf{C}^\#$. We fix a lift

$$\begin{tikzcd} (\mathsf{Open} \rtimes_X K)_V \arrow[r, dashed, "\overline{U}"] \arrow[d] & \mathsf{C} \arrow[d] \\ \mathsf{Open} \rtimes_X K \arrow[r, "U"] & \mathsf{C}^\# \end{tikzcd}$$

with the same property (sending vertical edges in $(\mathsf{Open} \rtimes_X K)_V$ to open embeddings, this time in C).

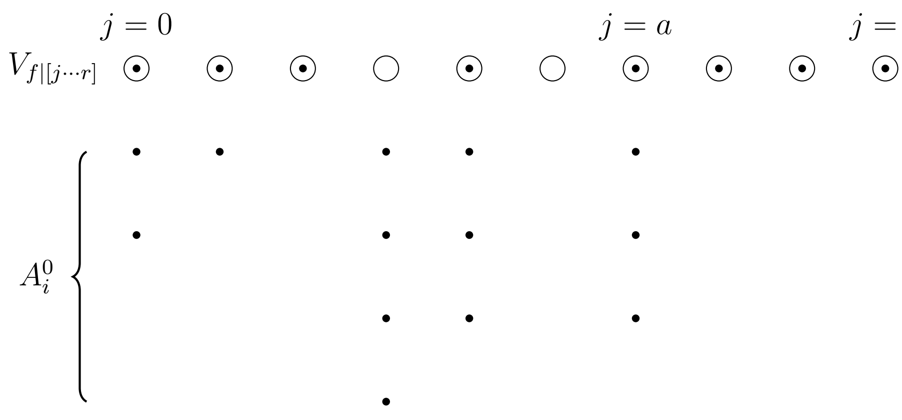
Such data may be constructed by induction on a filtration of K by pushouts of simplices $(\Delta^r, \partial\Delta^r)$. The inductive step for a particular non-degenerate simplex $f : \Delta^r \to K$ proceeds as follows. An extension of the choice of neighborhoods V (2.8.63.2) over f amounts to a choice of open neighborhood $x_{f(0)} \in V_f \subseteq X_{f(0)}$ contained (for $r > 0$) in $(X_{f(0)} \xrightarrow{X_{f(01)}} X_{f(1)})^{-1}(V_{f|[1\cdots r]})$. We claim that extending the lift $\overline{U}$ (2.8.63.4) over $(\mathsf{Open} \rtimes_{Xf} (\Delta^r, \partial\Delta^r))_V$ is equivalent (up to contractible choice) to extending it over the single simplex

$$(V_f, V_{f|[1\cdots r]}, \ldots, V_{f|[r]}) : (\Delta^r, \partial\Delta^r) \to (\mathsf{Open} \rtimes_{Xf} (\Delta^r, \partial\Delta^r))_V \tag{2.8.63.5}$$

In a word, this is true because of the requirement that $\overline{U}$ send vertical edges to open embeddings (we will postpone the precise argument until the next paragraph). To extend the lift $\overline{U}$ to this simplex (2.8.63.5) amounts to a certain lifting problem for $(\Delta^r, \partial\Delta^r)$ against

$\mathsf{C} \to \mathsf{C}^\#$ (up to natural isomorphism). When $r = 0$, this lifting problem has a solution (for a sufficiently small choice of V_f) since every object of $\mathsf{C}^\#$ admits an open cover by objects in the essential image of C. When $r = 1$, recall that lifting a simplex $(\Delta^r, \partial\Delta^r)$ against an isofibration $\mathsf{C} \to \mathsf{D}$ is equivalent to lifting $(\Delta^{r-1}, \partial\Delta^{r-1})$ against the Kan fibration $\mathrm{Hom}_\mathsf{C} \to \mathrm{Hom}_\mathsf{D}$ (1.4.87). Thus to extend $\overline{U}$ across (2.8.63.5), it is equivalent to lift (up to homotopy) $(D^{r-1}, \partial D^{r-1})$ against $\mathrm{Hom}_\mathsf{C}(\overline{V_f}, \overline{V_{f|[r]}}) \to \mathrm{Hom}_{\mathsf{C}^\#}(V_f, V_{f|[r]})$. Now this map is (the specialization to global sections of) a sheafification on $|V_f|$ (since $\mathsf{C} \to \mathsf{C}^\#$ is topologically fully faithful and $\mathsf{C}^\#$ is subcanonical), so such lifting is always possible upon replacing V_f with a sufficiently small neighborhood of $x_{f(0)}$ (**??**).

Now let us tie up the loose end from above: we show that extending $\overline{U}$ over $(\mathsf{Open} \rtimes_{Xf} (\Delta^r, \partial\Delta^r))_V$ is equivalent to extending it over the simplex (2.8.63.5). Define a filtration of the pair $(\mathsf{Open} \rtimes_{Xf} (\Delta^r, \partial\Delta^r))_V$ by counting, for any non-degenerate simplex $\Delta^n \to (\mathsf{Open} \rtimes_{Xf} \Delta^r)_V$ not lying over $\partial\Delta^r$ (which consists of a surjection $g : \Delta^n \twoheadrightarrow \Delta^r$ and distinct nested open subsets $A_i \subseteq V_{f|[g(i)\cdots r]}$ for $0 \le i \le r$) the number N of indices i for which the inclusion $A_i \subseteq V_{f|[g(i)\cdots r]}$ is proper. The case $N = 0$ is the simplex (2.8.63.5), so it suffices to show that the Nth extension problem is contractible for $N > 0$. We may further decompose the Nth extension problem according to the particular set of N distinct nested proper open subsets $A_i^0 \subsetneqq V_{f|[g(i)\cdots r]}$. Given a particular collection of N distinct nested open subsets A_i^0, the associated pair is a pushout of the pair (Δ^M, S), where $\Delta^M \to (\mathsf{Open} \rtimes_{Xf} \Delta^r)_V$ is obtained by adding to the list of A_i^0 as many of the $V_{f|[j\cdots r]}$ as we can while still preserving nesting. The subcomplex $S \subseteq \Delta^M$ consists of those simplices of Δ^M which omit some index $0 \le j \le r$ entirely or omit one of the A_i^0. This discussion is represented pictorially as follows: the circles indicate the open sets $V_{f|[j\cdots r]}$, and the dots (the vertices of Δ^M) indicate the open sets A_i^0 along with those $V_{f|[j\cdots r]}$ which preserve nesting when added to the list of A_i^0.



(2.8.63.6)

The lifting problem for (Δ^M, S) against $\mathsf{C} \to \mathsf{C}^\#$ is now solved as follows. Let a be the maximum value of j for which there is some $A_i^0 \subsetneqq V_{f|[g(i)\cdots r]}$ with $g(i) = j$. Note that every $V_{f|[b\cdots r]}$ with $a \le b \le r$ is a vertex of Δ^M (adding these preserves nesting by definition of a), so these form the final string of vertices $M - r + a, \ldots, M$ of Δ^M. Observe that S is coned at the vertex $M - r + a \in \Delta^M$ (corresponding to $V_{f|[a\cdots r]}$), meaning that adding this vertex to a non-degenerate simplex of Δ^M preserves being contained in S: indeed, adding $V_{f|[a\cdots r]}$ never destroys the property of omitting some A_i^0, and if it destroys the property of omitting an index $0 \leq j \leq r$, that index is necessarily a, which means that there is a corresponding A_i^0 omitted (namely one with $g(i) = j$). Since S is coned at $M - r + a \in \Delta^M$, a filtration of $(\Delta^{[M]-\{M-r+a\}}, S \cap \Delta^{[M]-\{M-r+a\}})$ by simplices $(\Delta^b, \partial\Delta^b)$ determines a filtration of (Δ^M, S) by pushouts of horns (which are left/right/inner according to whether the corresponding simplex $\Delta^b \subseteq \Delta^{[M]-\{M-r+a\}}$ consists of only vertices $> M - r + a$, only vertices $< M - r + a$, or both). Every simplex involving only vertices $\geq M - r + a$ omits all the A_i^0, hence (since $N > 0$) is contained in S. Every simplex involving only vertices $< M - r + a$ omits the vertex $r \in \Delta^r$, hence is contained in S, unless $a = r$. Thus if $a < r$, we conclude that (Δ^M, S) is filtered by pushouts of inner horns, which lift along $\mathsf{C} \to \mathsf{C}^\#$ since wlog it is an isofibration (1.4.86). If $a = r$, we obtain a filtration of (Δ^M, S) by pushouts of right horns, which now show how to lift along $\mathsf{C} \to \mathsf{C}^\#$. Every simplex of Δ^M not in S contains all the A_i^0, in particular the largest one inside $V_{f|[r]}$ (exists since $a = r$), which means every right horn in our filtration of (Δ^M, S) has rightmost edge $(M-1, M)$. This edge is vertical (projects to a degenerate edge in Δ^r), hence is sent to an open embedding in both C and $\mathsf{C}^\#$. Now lifting a right horn (Δ^b, Λ_b^b) against an isofibration $\mathsf{C} \to \mathsf{D}$ is equivalent to lifting $(D^b, \partial D^{b-1})$ against $\operatorname{Hom}_{\mathsf{C}}(0, b-1) \to \operatorname{Hom}_{\mathsf{D}}(0, b-1) \times_{\operatorname{Hom}_{\mathsf{D}}(-,b)} \operatorname{Hom}_{\mathsf{C}}(0, b)$ (this follows from reasoning similar to that used in the proof of (1.4.87)). Since the extremal edge $(b-1) \to b$ is, in our case, an open embedding in C and $\mathsf{C}^\#$, both maps $\operatorname{Hom}_{\mathsf{C}}(0, b-1) \to \operatorname{Hom}_{\mathsf{C}}(0, b)$ and $\operatorname{Hom}_{\mathsf{C}^\#}(0, b-1) \to \operatorname{Hom}_{\mathsf{C}^\#}(0, b)$ are pullbacks of the map of sets $\operatorname{Hom}(|-|, |A_{\max}^0|) \hookrightarrow \operatorname{Hom}(|-|, |V_{f|[r]}|)$, so we are lifting $(D^b, \partial D^{b-1})$ against a homotopy equivalence.

We have now constructed a germ of lift of X from $\mathsf{C}^\#$ to C near x (2.8.63.2)–(2.8.63.4). Now let us feed it into the Bousfield–Kan transform (1.4.238) to produce an open subdiagram of $X_\triangle$ lifting to C. For vertices $k \in K$, let

$$W_k = \bigcap_{\substack{f:\Delta^n \to K \\ f(0)=k}} V_f \subseteq p(k) \tag{2.8.63.7}$$

be the intersection of the open sets V_f associated to simplices $f : \Delta^n \to K$ with intial vertex $f(0) = k$ (this is a finite intersection since K is finite and V_f depends only on the non-degenerate simplex underlying f). We denote by $(\mathsf{Open} \rtimes_X K)_W \subseteq \mathsf{Open} \rtimes_X K$ the full subcategory spanned by the open subsets of these W_k (evidently $(\mathsf{Open} \rtimes_X K)_W \subseteq (\mathsf{Open} \rtimes_X K)_V$), and we denote by $(\mathsf{Open} \rtimes_X K)_{W,x} \subseteq (\mathsf{Open} \rtimes_X K)_W$ the full subcategory spanned by those open subsets containing x_k.

Now let us argue that there exists a lift:

$$\begin{tikzcd} & (\mathsf{Open} \rtimes_X K)_{W,x} \arrow[d] \\ \mathbf{\Delta}_{/K} \arrow[ur, dashed, "{(A,\ell)}"] \arrow[r, "\ell"] & K \end{tikzcd} \tag{2.8.63.8}$$

Choosing such a lift amounts to a choosing for every $f : \Delta^n \to K$ an open neighborhood $x_{f(n)} \in A_f \subseteq W_{f(n)}$ with the property that $A_{fa} \subseteq (X_{f(a(m))} \xrightarrow{X_{f(a(m) \to n)}} X_{f(n)})^{-1} A_f$ for every map $a : \Delta^m \to \Delta^n$ (note this implies that A_f depends only on the underlying non-degenerate simplex of f). Such open neighborhoods A_f exist by downward induction on the non-degenerate simplices of K, using the fact that K is finite.

Now a lift (2.8.63.8) determines an open embedding $(A, \ell)^*U \to (A, \ell)^*X = \ell^*X$ and a lift $(A, \ell)^*\overline{U}$ to C. This open embedding $(A, \ell)^*U \hookrightarrow \ell^*X$ pushes forward under $\pi : \mathbf{\Delta}_{/K} \to \mathbf{\Delta}$ to an open embedding $\pi_*(A, \ell)^*U \hookrightarrow \pi_*\ell^*X$ since π_* takes finite products (1.4.238) and open embeddings are closed under finite products (2.8.5)(1.1.62). The diagrams $(A, \ell)^*U$ and $(A, \ell)^*\overline{U}$ is flat (1.4.240) (as noted above, A_f depends only on the non-degenerate simplex underlying f) and K is finite, hence their pushforwards under π_* are finite cosifted limits (1.4.242). Note that $\pi_*(A, \ell)^*\overline{U}$ remains a lift of $\pi_*(A, \ell)^*U$ since $\mathsf{C} \to \mathsf{C}^{\#}$ preserves finite products (2.8.61). By construction, the open subdiagram $\pi_*(A, \ell)^*U \hookrightarrow \pi_*\ell^*X = X_\triangle$ contains the chosen point $x \in \lim |p_\triangle|$. □

Adjoining limits

We now show how to freely adjoin *finite cosifted limits* (equivalently, freely adjoin *finite limits modulo preserving finite products*) to any perfect topological ∞-site C admitting finite products (this is an adaptation of the purely categorical construction (1.4.236)); we denote this construction by $\mathsf{C} \hookrightarrow \mathcal{D}(\mathsf{C})_{\mathsf{fin}}$ and we call it the *(finite) derived ∞-site* of C (2.8.86). The derived site satisfies a universal property (2.8.88) which makes precise the slogans of formally adjoining finite cosifted limits and formally adjoining finite limits modulo finite products. Most important is probably its axiomatic characterization (**??**) which, among other things, is how one can make concrete computations with derived sites.

Note that while the constructions $\mathsf{C} \hookrightarrow \mathsf{P}(\mathsf{C})$ and $\mathsf{C} \hookrightarrow \mathsf{Shv}(\mathsf{C})$ for topological ∞-sites adjoin certain *colimits* (2.8.32)(2.8.43), the present discussion of adjoining *limits* need not be related, as the notion of a topological ∞-site is not invariant under passing to opposites.

2.8.64 Definition (Extension of $|\cdot|$ to formal limits)**.** Let C be a topological ∞-site. We equip the ∞-category $\mathsf{Lim}(\mathsf{C}) = \mathsf{Fun}(\mathsf{C}, \mathsf{Spc})^{\mathsf{op}}$ of 'formal limits in C' (**??**) with the unique continuous functor $|\cdot|_{\mathsf{Lim}(\mathsf{C})} : \mathsf{Lim}(\mathsf{C}) \to \mathsf{Top}$ (1.1.118) extending $|\cdot|_{\mathsf{C}}$.

$$\begin{tikzcd} \mathsf{C} \arrow[r] \arrow[dr, "|\cdot|_{\mathsf{C}}"'] & \mathsf{Lim}(\mathsf{C}) \arrow[d, dashed, "|\cdot|_{\mathsf{Lim}(\mathsf{C})}"] \\ & \mathsf{Top} \end{tikzcd} \tag{2.8.64.1}$$

Concretely, if $p : K \to \mathsf{C}$ is a diagram, then $|p|_{\mathsf{Lim}(\mathsf{C})} = \lim_K |p|_{\mathsf{C}}$. We may also drop the subscript $_{\mathsf{Lim}(\mathsf{C})}$ and simply write $|p|$ for $p \in \mathsf{Lim}(\mathsf{C})$ (while $|p|_! \in \mathsf{Lim}(\mathsf{Top})$ denotes the image of p under the left Kan extension functor $|\cdot|_! : \mathsf{Lim}(\mathsf{C}) \to \mathsf{Lim}(\mathsf{Top})$, so we have $|p| = \operatorname{colim} |p|_!$).

A formal limit $p \in \mathsf{Lim}(\mathsf{C})$ often contains data which is not 'local' around the topological space $|p|$. For this reason, $\mathsf{Lim}(\mathsf{C})$ is quite far from being a topological ∞-site. The first step in constructing a topological ∞-site out of $\mathsf{Lim}(\mathsf{C})$ is to introduce the notion of 'corporeality' of formal limits, which is a precise expression of the idea of $p \in \mathsf{Lim}(\mathsf{C})$ being 'local' around $|p|$.

⋆ **2.8.65 Definition** (Corporeal). Let $p \in \mathsf{Lim}(\mathsf{C})$ be a formal limit in a topological ∞-site C. For any open embedding $U \hookrightarrow M$ in C, we may consider the following diagram.

$$\begin{array}{ccc} \mathrm{Hom}(p, U) & \longrightarrow & \mathrm{Hom}(p, M) \\ \downarrow & & \downarrow \\ \mathrm{Hom}(|p|, |U|) & \longrightarrow & \mathrm{Hom}(|p|, |M|) \end{array} \tag{2.8.65.1}$$

When this diagram is a pullback for every open embedding $U \hookrightarrow M$ in C, we say that the formal limit $p \in \mathsf{Lim}(\mathsf{C})$ is *corporeal*. We denote the full subcategory spanned by corporeal formal limits by $\mathsf{Lim}_{\mathsf{cp}}(\mathsf{C}) \subseteq \mathsf{Lim}(\mathsf{C})$.

2.8.66 Exercise. Show that all objects of $\mathsf{C} \subseteq \mathsf{Lim}(\mathsf{C})$ are corporeal (use the definition of open embeddings).

2.8.67 Exercise. Consider two embeddings $f, g : [0,1] \hookrightarrow \mathbb{R}^2$ intersecting only at $f(0) = g(0)$ and $f(1) = g(1)$. Show that the formal fiber product $\lim^{\mathsf{Lim}(\mathsf{Top})}([0,1] \xrightarrow{f} \mathbb{R}^2 \xleftarrow{g} [0,1])$ is not corporeal (consider mapping to a small interval U inside $M = S^1$). Show that the formal inverse limit $\varprojlim_n^{\mathsf{Lim}(\mathsf{Top})}(-\frac{1}{n}, \frac{1}{n})$ is corporeal.

2.8.68 Lemma. *The functor* $\mathsf{C} \to \mathsf{Lim}_{\mathsf{cp}}(\mathsf{C})$ *preserves open embeddings and their pullbacks.*

Proof. Let $U \hookrightarrow M$ be an open embedding in C. To say that $U \hookrightarrow M$ is an open embedding in $\mathsf{Lim}_{\mathsf{cp}}(\mathsf{C})$ is the assertion that for any corporeal $p \in \mathsf{Lim}(\mathsf{C})$, the map $\mathrm{Hom}(p, U) \to \mathrm{Hom}(p, M)$ is the pullback of $\mathrm{Hom}(|p|, |U|) \to \mathrm{Hom}(|p|, |M|)$, which is exactly what it means for p to be corporeal.

Now suppose $X' \to Y'$ is the pullback of an open embedding $X \to Y$ in C. For $q \in \mathsf{Lim}(\mathsf{C})$, applying $\mathrm{Hom}(q, -) \to \mathrm{Hom}(|q|, |-|)$ to this pullback square produces a cube. The $\mathrm{Hom}(|q|, |-|)$ face of the cube is a pullback since $|\cdot|_{\mathsf{C}}$ preserves pullbacks of open embeddings (2.8.5). If q is corporeal, two other faces are pullbacks by definition (2.8.65.1). Using cancellation (1.1.57), we deduce that the $\mathrm{Hom}(q, -)$ face is a pullback for q corporeal. □

While corporeality of a formal limit (2.8.65) is not quite a special case of locality in the sense of (1.4.198), we will see that the ∞-category of corporeal formal limits $\mathsf{Lim}_{\mathsf{cp}}(\mathsf{C}) \subseteq \mathsf{Lim}(\mathsf{C})$ satisfies many of the same properties as local presheaves do.

⋆ **2.8.69 Lemma.** *The full subcategory of corporeal diagrams* $\mathsf{Lim}_{\mathsf{cp}}(\mathsf{C}) \subseteq \mathsf{Lim}(\mathsf{C})$ *is coreflective, and for any topological functor* $f : \mathsf{C} \to \mathsf{D}$*, the functor* $f(-)_{\mathrm{cp}} : \mathsf{Lim}(\mathsf{C}) \to \mathsf{Lim}_{\mathsf{cp}}(\mathsf{D})$ *sends corporealizations to isomorphisms.*

Proof. Let $p : K \to \mathsf{C}$ be a diagram, and let us construct its corporealization p_{cp}.

Let $|p|_! : K \to \mathsf{Top}$ denote the composition of p with the forgetful functor $|\cdot| : \mathsf{C} \to \mathsf{Top}$, and let $|p|$ denote its limit. For any vertex $\alpha \in K$, let $(|p| \downarrow \mathsf{Open}(p(\alpha)))$ denote the category of open subsets of $p(\alpha)$ which contain the image of the map $|p| \to p(\alpha)$. We will show that the corporealization of p is the diagram

$$p_{\mathrm{cp}} : (|p| \downarrow \mathsf{Open}(p)) \rtimes K \to \mathsf{C} \tag{2.8.69.1}$$

where a map $Z \to (|p| \downarrow \mathsf{Open}(p)) \rtimes K$ is a map $f : Z \to K$ together with, for every vertex $z \in Z$, a choice of open set $|p| \to U_z \subseteq p(f(z))$, such that for every edge $e : z \to z'$, we have $U_z \subseteq (p(f(e)) : p(f(z)) \to p(f(z')))^{-1}(U_{z'})$, and p_{cp} sends such a map out of Z to the evident diagram $Z \to \mathsf{C}$ given by $z \mapsto U_z$ (2.8.8). There is an evident inclusion $K \subseteq (|p| \downarrow \mathsf{Open}(p)) \rtimes K$ given by taking $U_z = p(f(z))$ for all z, giving a map of formal limits $p_{\mathrm{cp}} \to p$. Note that the natural map $|p_{\mathrm{cp}}| \to |p|$ is an isomorphism.

Let us show that p_{cp} is corporeal. That is, we should show that for any open embedding $U \hookrightarrow M$ in C and any map $f : |p| \to U$, the map

$$\operatorname*{colim}_{((|p|\downarrow \mathsf{Open}(p))\rtimes K)^{\mathsf{op}}} \operatorname{Hom}(p, U)_f \to \operatorname*{colim}_{((|p|\downarrow \mathsf{Open}(p))\rtimes K)^{\mathsf{op}}} \operatorname{Hom}(p, M)_f \tag{2.8.69.2}$$

is an isomorphism, where $\operatorname{Hom}_f \subseteq \operatorname{Hom}$ denotes the maps whose pullback to $|p|$ is f. Since the map $\pi : (|p| \downarrow \mathsf{Open}(p)) \rtimes K \to K$ is a cartesian fibration (1.4.136) (inspection), these colimits may be expressed as colimits over K^{op} of the fiberwise colimit pushforwards under π (**??**). We claim that the map between fiberwise colimits (diagrams over K^{op}) is already an isomorphism. That is, we claim that for every $X \in \mathsf{C}$, every subset $A \subseteq |X|$, and every function $f : A \to |U|$, the map

$$\operatorname*{colim}_{(A\downarrow \mathsf{Open}(|X|))^{\mathsf{op}}} \operatorname{Hom}(-, U)_f \to \operatorname*{colim}_{(A\downarrow \mathsf{Open}(|X|))^{\mathsf{op}}} \operatorname{Hom}(-, M)_f \tag{2.8.69.3}$$

is an isomorphism. This is evident since both sides are the set of germs of maps near A agreeing with f.

Now we claim that $p_{\mathrm{cp}} \to p$ is the corporealization of p. That is, we claim that for any corporeal diagram $q : L \to \mathsf{C}$, the composition map

$$\operatorname{Hom}_{\mathsf{Lim}(\mathsf{C})}(q, p_{\mathrm{cp}}) \to \operatorname{Hom}_{\mathsf{Lim}(\mathsf{C})}(q, p) \tag{2.8.69.4}$$

is an isomorphism. Both sides map to (the discrete set) $\operatorname{Hom}(|q|, |p|)$, so we may restrict to the fiber $\operatorname{Hom}_f \subseteq \operatorname{Hom}$ over a particular map $f : |q| \to |p|$. This restriction may be written as

$$\lim_{(|p|\downarrow \mathsf{Open}(p))\rtimes K} \operatorname{Hom}_{\mathsf{Lim}(\mathsf{C})}(q, p_{\mathrm{cp}}(-))_f \to \lim_K \operatorname{Hom}_{\mathsf{Lim}(\mathsf{C})}(q, p(-))_f. \tag{2.8.69.5}$$

We claim that after pushing forward the diagram on the left to K (fiberwise limit), we obtain an isomorphism of diagrams over K (and hence the map above is an isomorphism). It is enough to show that for any open embedding $U \hookrightarrow M$ and any map $f : |q| \to U$, the map

$$\operatorname{Hom}_{\mathsf{Lim}(\mathsf{C})}(q, U)_f \to \operatorname{Hom}_{\mathsf{Lim}(\mathsf{C})}(q, M)_f \tag{2.8.69.6}$$

is an isomorphism. That this is an isomorphism now follows from the fact that q is corporeal (2.8.65.1). □

⋆ **2.8.70 Exercise** (A formal limit and its corporealization are 'topologically' equivalent)**.** Use the fact that $\mathsf{Lim}_{\mathsf{cp}}(\mathsf{C}) \subseteq \mathsf{Lim}(\mathsf{C})$ is coreflective (2.8.69) and contains $\mathsf{C} \subseteq \mathsf{Lim}(\mathsf{C})$ (2.8.66) to show that the map $\lim p_{\mathrm{cp}} \to \lim p$ is an isomorphism for every formal limit $p \in \mathsf{Lim}(\mathsf{C})$

(in the sense that if either exists, then so does the other, and in this case the map is an isomorphism). Use the fact that $\mathrm{cp} \circ f_!$ sends corporealizations to isomorphisms (2.8.69) to conclude that $|\cdot|_{\mathsf{Lim}_{\mathsf{cp}}(\mathsf{C})}$ is continuous and that a topological functor $f : \mathsf{C} \to \mathsf{D}$ preserves the limit of $p \in \mathsf{Lim}(\mathsf{C})$ iff if preserves the limit of p_{cp}.

2.8.71 Exercise. Note that open embeddings in $\mathsf{Lim}_{\mathsf{cp}}(\mathsf{C})$ are preserved under pullback by (2.8.4) since $|\cdot|_{\mathsf{Lim}_{\mathsf{cp}}(\mathsf{C})}$ preserves pullbacks (in fact, preserves all limits (2.8.70)).

2.8.72 Lemma. *A left fibration $p : K \to \mathsf{C}$ is corporeal iff it satisfies the right lifting property with respect to pairs $(\Delta^1, 1) \wedge (\Delta^k, \partial\Delta^k)$ mapping to C via the projection to Δ^1 followed by an open embedding $\Delta^1 \to \mathsf{C}$, say denoted $U \to M$, for which (and this only has content when $k = 0$) the induced map $|p| \to |M|$ lands inside the open set $|U| \subseteq |M|$.*

Proof. This is a direct translation of the condition that (2.8.65.1) be a pullback (compare (1.4.200)). □

2.8.73 Lemma. *The corporealization functor $\mathsf{Lim}(\mathsf{C}) \to \mathsf{Lim}_{\mathsf{cp}}(\mathsf{C})$ sends a diagram in C to the result of applying the small object argument to the lifting problems in* (2.8.72) *(as in* (1.4.200)*).*

Proof. Following the proof of (1.4.200), it suffices to check that if $K \to \mathsf{C}$ is any diagram and $\hat{K} \to \mathsf{C}$ denotes the result of forming the pushout of a lifting problem as in (2.8.72), then for any left fibration $E \to \mathsf{C}$ satisfying the lifting property (2.8.72), the simplicial mapping space from $\hat{K}$ to E over C maps via a trivial Kan fibration to the simplicial mapping space from K to E over C. To see this, it is enough to argue that the smash product $(\Delta^1, 1) \wedge (\Delta^k, \partial\Delta^k) \wedge (\Delta^r, \partial\Delta^r)$ is filtered by pushouts of $(\Delta^1, 1) \wedge (\Delta^a, \partial\Delta^a)$ (mapping to C as in (2.8.72)). □

2.8.74 Definition (Elementary corporeal equivalence)**.** Let C be a topological ∞-site. An *elementary corporeal equivalence* is a morphism in $\mathsf{Lim}(\mathsf{C})$ of the form $p \times_M U \to p$ for an open embedding $U \hookrightarrow M$ in C and any map $\mathsf{Lim}(\mathsf{C}) \ni p \to M$ whose image $|p| \to |M|$ is contained in $|U| \subseteq |M|$.

Note that for corporeal q, the functor $\mathrm{Hom}(q, -)$ sends elementary corporeal equivalences to isomorphisms (inspection), hence corporealization sends elementary corporeal equivalences to isomorphisms (1.4.202) (hence the name). It follows that if a functor $\mathsf{Lim}(\mathsf{C}) \to \mathsf{E}$ sends corporealizations to isomorphisms then it also sends elementary corporeal equivalences to isomorphisms (1.4.203). For continuous functors, we have the converse:

2.8.75 Corollary. *A continuous functor $\mathsf{Lim}(\mathsf{C}) \to \mathsf{E}$ sends corporealizations to isomorphisms iff it sends elementary corporeal equivalences to isomorphisms.*

Proof. We cannot cite (1.4.204) since we have not shown that $\mathsf{Lim}_{\mathsf{cp}}(\mathsf{C}) \subseteq \mathsf{Lim}(\mathsf{C})$ is a category of local presheaves in the sense of (1.4.198). Nevertheless, the argument of (1.4.204) applies, given the description (2.8.73) of corporealizations as cotransfinite compositions of elementary corporeal equivalences and pullbacks of kth iterated formal diagonals ($k \geq 1$)

of open embeddings. Note that a functor which sends elementary corporeal equivalences to isomorphisms also sends formal diagonals of open embeddings to isomorphisms (apply the 2-out-of-3 property to the composition of the formal diagonal $U \to U \times_M U$ and one of the projections $U \times_M U \to U$). $\square$

2.8.76 Lemma (Universal property of corporeal formal limits)**.** *For any complete ∞-category E, pullback along $\mathsf{C} \xrightarrow{よ} \mathsf{Lim}(\mathsf{C}) \xrightarrow{\mathrm{cp}} \mathsf{Lim}_{\mathsf{cp}}(\mathsf{C})$ defines equivalences between the following ∞-categories of functors:*

(2.8.76.1) *Functors $\mathsf{Lim}_{\mathsf{cp}}(\mathsf{C}) \to \mathsf{E}$ which are continuous.*

(2.8.76.2) *Functors $\mathsf{Lim}(\mathsf{C}) \to \mathsf{E}$ which are continuous and send corporealizations $X_{\mathrm{cp}} \to X$ to isomorphisms.*

(2.8.76.3) *Functors $\mathsf{Lim}(\mathsf{C}) \to \mathsf{E}$ which are continuous and send elementary corporeal equivalences to isomorphisms.*

(2.8.76.4) *Functors $f : \mathsf{C} \to \mathsf{E}$ whose unique continuous extension to $\mathsf{Lim}(\mathsf{C})$ satisfies the above two equivalent conditions.*

Proof. This is similar to (1.4.206) and has the same proof, namely combine the universal property of a reflective subcategory of presheaves (1.1.120) with the equivalence (2.8.75). $\square$

Recall the full subcategory $\mathsf{Cosif}(\mathsf{C}) \subseteq \mathsf{Lim}(\mathsf{C})$ of formal cosifted limits (1.4.224).

⋆ **2.8.77 Definition** ($\mathsf{Cosif}_{\mathsf{cp}}(\mathsf{C})$)**.** For a topological ∞-site C, we let $\mathsf{Cosif}_{\mathsf{cp}}(\mathsf{C}) = \mathsf{Lim}_{\mathsf{cp}}(\mathsf{C}) \cap \mathsf{Cosif}(\mathsf{C}) \subseteq \mathsf{Lim}(\mathsf{C})$ denote the full subcategory of *formal cosifted corporeal limits* in C.

2.8.78 Lemma. *The corporealization functor $\mathsf{Lim}(\mathsf{C}) \to \mathsf{Lim}_{\mathsf{cp}}(\mathsf{C})$ restricts to an endofunctor of the full subcategory $\mathsf{Cosif}(\mathsf{C}) \subseteq \mathsf{Lim}(\mathsf{C})$ of formal cosifted limits.*

Proof. Let $p : K \to \mathsf{C}$ be a cosifted diagram, and let us show that its corporealization p_{cp} is cosifted. The domain of the corporealization p_{cp} (2.8.69.1) is $(|p| \downarrow \mathsf{Open}(p)) \rtimes K$. The functor $(|p| \downarrow \mathsf{Open}(p)) \rtimes K \to K$ is cartesian (2.8.8) and K is cosifted, so it suffices to show its fibers are cosifted (1.4.221). The fiber over $\alpha \in K$ is the poset category $(|p| \downarrow \mathsf{Open}(p(\alpha)))$, which is a cofiltered poset (by intersection of open sets), hence cosifted (1.4.219). $\square$

The full subcategory $\mathsf{Cosif}_{\mathsf{cp}}(\mathsf{C}) \subseteq \mathsf{Cosif}(\mathsf{C})$ is coreflective (2.8.78) and contains C (2.8.65). When C has finite products, $\mathsf{Cosif}(\mathsf{C}) \subseteq \mathsf{Lim}(\mathsf{C})$ is coreflective (1.4.230), and hence $\mathsf{Cosif}_{\mathsf{cp}}(\mathsf{C}) \subseteq \mathsf{Lim}(\mathsf{C})$ is also coreflective, with coreflection given by cosiftedization (1.4.230) followed by corporealization (2.8.78). In this case, $\mathsf{Cosif}_{\mathsf{cp}}(\mathsf{C})$ has limits, and we denote by $\mathsf{Cosif}_{\mathsf{cp}}(\mathsf{C})_{\mathsf{fin}} \subseteq \mathsf{Cosif}_{\mathsf{cp}}(\mathsf{C})$ the full subcategory spanned by finite limits of objects of C.

The ∞-category $\mathsf{Cosif}_{\mathsf{cp}}(\mathsf{C})_{\mathsf{fin}}$ has all finite limits of objects of C, essentially by definition. To show that it has all finite limits is more subtle and relies on the following key observation.

2.8.79 Corollary. *For any finite diagram $p : K \to \mathsf{C}$, every morphism to the associated object $p \in \mathsf{Cosif}_{\mathsf{cp}}(\mathsf{C})_{\mathsf{fin}}$ from another object of $\mathsf{Cosif}_{\mathsf{cp}}(\mathsf{C})_{\mathsf{fin}}$ is induced from a finite diagram $q : L \to \mathsf{C}$, an inclusion $K \hookrightarrow L$, and an isomorphism $q|_K = p$.*

Proof. Given the lifting property characterization of $\mathsf{Cosif}_{\mathsf{cp}}(\mathsf{C})$ (1.4.224)(1.4.200)(2.8.72), we may proceed as in (1.4.196). □

2.8.80 Corollary. *Let* C *be a topological* ∞*-site with finite products.* $\mathsf{Cosif}_{\mathsf{cp}}(\mathsf{C})_{\mathsf{fin}}$ *has finite cosifted limits, and they remain limits in* $\mathsf{Lim}_{\mathsf{cp}}(\mathsf{C})$*.*

Proof. Recall that $\mathsf{Cosif}(\mathsf{C})_{\mathsf{fin}}$ has finite cosifted limits and that they remain limits in $\mathsf{Lim}(\mathsf{C})$ (1.4.237). We will prove the present result using the same argument.

A formal finite cosifted limit in $\mathsf{Cosif}_{\mathsf{cp}}(\mathsf{C})_{\mathsf{fin}}$ is, by definition, the cosiftedization p_{cosif} of a finite diagram $p : K \to \mathsf{Cosif}_{\mathsf{cp}}(\mathsf{C})_{\mathsf{fin}}$ (take careful note that cosiftedization of formal limits in $\mathsf{Cosif}_{\mathsf{cp}}(\mathsf{C})_{\mathsf{fin}}$ refers to the reflection $\mathsf{Lim}(\mathsf{Cosif}_{\mathsf{cp}}(\mathsf{C})_{\mathsf{fin}}) \to \mathsf{Cosif}(\mathsf{Cosif}_{\mathsf{cp}}(\mathsf{C})_{\mathsf{fin}})$). We now claim that

$$\lim^{\mathsf{Lim}_{\mathsf{cp}}(\mathsf{C})} p_{\text{cosif}} = \lim^{\mathsf{Cosif}_{\mathsf{cp}}(\mathsf{C})} p_{\text{cosif}} = \lim^{\mathsf{Cosif}_{\mathsf{cp}}(\mathsf{C})} p \in \mathsf{Cosif}_{\mathsf{cp}}(\mathsf{C})_{\mathsf{fin}}. \tag{2.8.80.1}$$

The identification $\lim^{\mathsf{Lim}_{\mathsf{cp}}(\mathsf{C})} p_{\text{cosif}} = \lim^{\mathsf{Cosif}_{\mathsf{cp}}(\mathsf{C})} p_{\text{cosif}}$ holds since $\mathsf{Cosif}(\mathsf{C}) \subseteq \mathsf{Lim}(\mathsf{C})$ is closed under cosifted limits (1.4.226) and corporealization preserves $\mathsf{Cosif}(\mathsf{C})$ (2.8.78). The identification $\lim^{\mathsf{Cosif}_{\mathsf{cp}}(\mathsf{C})} p_{\text{cosif}} = \lim^{\mathsf{Cosif}_{\mathsf{cp}}(\mathsf{C})} p$ holds since p_{cosif} is the cosiftedization of p in $\mathsf{Cosif}_{\mathsf{cp}}(\mathsf{C})$ (it is, by definition, the cosiftedization of p in $\mathsf{Cosif}_{\mathsf{cp}}(\mathsf{C})_{\mathsf{fin}}$, which is the same as the cosiftedization in $\mathsf{Cosif}_{\mathsf{cp}}(\mathsf{C})$ since $\mathsf{Cosif}_{\mathsf{cp}}(\mathsf{C})_{\mathsf{fin}} \subseteq \mathsf{Cosif}_{\mathsf{cp}}(\mathsf{C})$ is closed under finite products, in fact under all finite limits (**??**)(1.4.233)). Finally, we have $\lim^{\mathsf{Cosif}_{\mathsf{cp}}(\mathsf{C})} p \in \mathsf{Cosif}_{\mathsf{cp}}(\mathsf{C})_{\mathsf{fin}}$ since $\mathsf{Cosif}_{\mathsf{cp}}(\mathsf{C})_{\mathsf{fin}} \subseteq \mathsf{Cosif}_{\mathsf{cp}}(\mathsf{C})$ is closed under finite limits (**??**). □

2.8.81 Lemma (Universal property of $\mathsf{C} \hookrightarrow \mathsf{Cosif}_{\mathsf{cp}}(\mathsf{C})_{\mathsf{fin}}$)**.** *Let* C *be a topological* ∞*-site with finite products, and consider* $i : \mathsf{C} \hookrightarrow \mathsf{Cosif}_{\mathsf{cp}}(\mathsf{C})_{\mathsf{fin}}$*. For any complete* ∞*-category* E*, the pair of adjoint functors*

$$i^* : \mathsf{Fun}(\mathsf{C}, \mathsf{E}) \rightleftarrows \mathsf{Fun}(\mathsf{Cosif}_{\mathsf{cp}}(\mathsf{C})_{\mathsf{fin}}, \mathsf{E}) : i_* \tag{2.8.81.1}$$

restrict to an equivalence between the following full subcategories:

(2.8.81.2) *Functors* $\mathsf{C} \to \mathsf{E}$ *which are corporeal.*

(2.8.81.3) *Functors* $\mathsf{Cosif}_{\mathsf{cp}}(\mathsf{C})_{\mathsf{fin}} \to \mathsf{E}$ *which preserve finite cosifted limits and whose restriction to* C *is corporeal (equivalently, is the restriction of a continuous functor on* $\mathsf{Lim}_{\mathsf{cp}}(\mathsf{C})$*).*

Moreover, under this equivalence, functors $\mathsf{C} \to \mathsf{E}$ *preserving finite products correspond to functors* $\mathsf{Cosif}_{\mathsf{cp}}(\mathsf{C})_{\mathsf{fin}} \to \mathsf{E}$ *preserving finite products, determining an equivalence:*

(2.8.81.4) *Functors* $\mathsf{C} \to \mathsf{E}$ *which are corporeal and preserve finite products.*

(2.8.81.5) *Functors* $\mathsf{Cosif}_{\mathsf{cp}}(\mathsf{C})_{\mathsf{fin}} \to \mathsf{E}$ *which preserve finite limits and whose restriction to* C *is corporeal.*

Proof. The equivalence between corporeal functors $\mathsf{C} \to \mathsf{E}$ and functors $\mathsf{Cosif}_{\mathsf{cp}}(\mathsf{C})_{\mathsf{fin}} \to \mathsf{E}$ which are the restriction of a continuous functor on $\mathsf{Lim}_{\mathsf{cp}}(\mathsf{C})$ is the universal property of the full subcategory $\mathsf{Cosif}_{\mathsf{cp}}(\mathsf{C})_{\mathsf{fin}}$ of the reflective subcategory $\mathsf{Lim}_{\mathsf{cp}}(\mathsf{C})$ of $\mathsf{Lim}(\mathsf{C})$ (1.1.121). The claim that a functor $\mathsf{Cosif}_{\mathsf{cp}}(\mathsf{C})_{\mathsf{fin}} \to \mathsf{E}$ is the restriction of a continuous functor on $\mathsf{Lim}_{\mathsf{cp}}(\mathsf{C})$ iff it preserves finite cosifted limits and has corporeal restriction to C also follows from (1.1.121) (since finite cosifted limits in $\mathsf{Cosif}_{\mathsf{cp}}(\mathsf{C})_{\mathsf{fin}}$ remain limits in $\mathsf{Lim}_{\mathsf{cp}}(\mathsf{C})$ (2.8.80) and generate $\mathsf{Cosif}_{\mathsf{cp}}(\mathsf{C})_{\mathsf{fin}}$ from C).

Now let us show that under this equivalence, functors $\mathsf{C} \to \mathsf{E}$ preserving finite products correspond to functors $\mathsf{Cosif}_{\mathsf{cp}}(\mathsf{C})_{\mathsf{fin}} \to \mathsf{E}$ preserving finite products. One direction is immediate since $\mathsf{C} \to \mathsf{Cosif}_{\mathsf{cp}}(\mathsf{C})_{\mathsf{fin}}$ preserves finite products (indeed $\mathsf{C} \to \mathsf{Cosif}(\mathsf{C})$ preserves finite products (1.4.232)). For the other direction, first recall that if $\mathsf{C} \to \mathsf{E}$ preserves finite products then so does the induced functor $\mathsf{Cosif}(\mathsf{C}) \to \mathsf{E}$ (1.4.235). If $\mathsf{C} \to \mathsf{E}$ is in addition corporeal, we conclude that the induced functor $\mathsf{Cosif}_{\mathsf{cp}}(\mathsf{C}) \to \mathsf{E}$ also preserves finite products (1.1.91)(1.1.92) (since the coreflector $\mathsf{Cosif}(\mathsf{C}) \to \mathsf{Cosif}_{\mathsf{cp}}(\mathsf{C})$ is the restriction of the coreflector $\mathsf{Lim}(\mathsf{C}) \to \mathsf{Lim}_{\mathsf{cp}}(\mathsf{C})$ (2.8.78)). The restriction to $\mathsf{Cosif}_{\mathsf{cp}}(\mathsf{C})_{\mathsf{fin}} \subseteq \mathsf{Cosif}_{\mathsf{cp}}(\mathsf{C})$ hence also preserves finite products since $\mathsf{Cosif}_{\mathsf{cp}}(\mathsf{C})_{\mathsf{fin}} \subseteq \mathsf{Cosif}_{\mathsf{cp}}(\mathsf{C})$ is closed under finite limits (**??**).

Finally, recall that preserving finite limits is equivalent to preserving finite products and finite cosifted limits (1.4.234). □

We equip $\mathsf{Cosif}_{\mathsf{cp}}(\mathsf{C})$ with the functor $|\cdot|_{\mathsf{Cosif}_{\mathsf{cp}}(\mathsf{C})}$ given by the restriction of $|\cdot|_{\mathsf{Lim}(\mathsf{C})}$. The restriction $|\cdot|_{\mathsf{Cosif}(\mathsf{C})}$ preserves cosifted limits since $\mathsf{Cosif}(\mathsf{C}) \subseteq \mathsf{Lim}(\mathsf{C})$ is closed under cosifted limits (1.4.226). Since $|\cdot|_{\mathsf{Lim}(\mathsf{C})}$ sends corporealizations to isomorphisms (**??**) and the coreflection $\mathsf{Cosif}(\mathsf{C}) \to \mathsf{Cosif}_{\mathsf{cp}}(\mathsf{C})$ is the restriction of corporealization $\mathsf{Lim}(\mathsf{C}) \to \mathsf{Lim}_{\mathsf{cp}}(\mathsf{C})$ (2.8.78), we conclude that the restriction $|\cdot|_{\mathsf{Cosif}_{\mathsf{cp}}(\mathsf{C})}$ also preserves cosifted limits. We note that it need not preserve all limits (e.g. finite products in C remain products in $\mathsf{Cosif}_{\mathsf{cp}}(\mathsf{C})$, and $|\cdot|_{\mathsf{C}}$ need not preserve finite products). However, if $|\cdot|_{\mathsf{C}}$ does preserve finite product, then $|\cdot|_{\mathsf{Cosif}(\mathsf{C})}$ preserves all limits (1.4.235), hence so does $|\cdot|_{\mathsf{Cosif}_{\mathsf{cp}}(\mathsf{C})}$.

2.8.82 Exercise (Flat extension for formal sifted colimits)**.** Let K be an ∞-category with finite coproducts, and let $p : K \to \mathsf{C}$ be a diagram with finitely many vertices. Consider the extension $\check{p} : \check{K} = K \cup_{K_0} (K_0)^{\triangleright} \to \mathsf{C}$ whose restriction to $(K_0)^{\triangleright}$ is a colimit diagram (call this the 'flat extension' of p). Conclude from (**??**) that the natural map $\operatorname{colim}_K p \to \operatorname{colim}_{\check{K}} \check{p}$ is an isomorphism. Apply this fact after composing p and $\check{p}$ with the inclusion $\mathsf{C} \subseteq \mathsf{Sif}(\mathsf{C})$ to conclude that p and $\check{p}$ represent the same formal sifted colimit in C (and note where the argument fails if we instead try to compose with $\mathsf{C} \subseteq \mathsf{P}(\mathsf{C})$ to conclude that p and $\check{p}$ represent the same formal colimit in C).

2.8.83 Lemma. *Let C be a topological ∞-site with finite products. Every object of $\mathsf{Cosif}_{\mathsf{cp}}(\mathsf{C})_{\mathsf{fin}}$ has a map to an object of C which is an embedding* (2.1.7.3) *of underlying topological spaces.*

Proof. Write a given $X \in \mathsf{Cosif}_{\mathsf{cp}}(\mathsf{C})_{\mathsf{fin}}$ as the limit $X = \lim_K^{\mathsf{Cosif}_{\mathsf{cp}}(\mathsf{C})} p$ of a finite diagram $p : K \to \mathsf{C}$. Let $\hat{p} : \hat{K} \to \mathsf{C}$ denote the flat extension (2.8.82) of p to $\hat{K} = K_0^{\triangleleft} \cup_{K_0} K$. We have $\lim_K^{\mathsf{Cosif}(\mathsf{C})} p = \lim_{\hat{K}}^{\mathsf{Cosif}(\mathsf{C})} \hat{p}$ (2.8.82), and hence the same equality for limits in $\mathsf{Cosif}_{\mathsf{cp}}(\mathsf{C})$ as well (2.8.78).

Now let us compute $|X|$. We have $|X| = \left|\lim_K^{\mathsf{Cosif}_{\mathsf{cp}}(\mathsf{C})} p\right| = |(p_{\mathrm{cosif}})_{\mathrm{cp}}| = |p_{\mathrm{cosif}}|$, which by the Bousfield–Kan formula (1.4.239) is $\lim_{[n]\in\mathbf{\Delta}} \left|\prod_{f:[n]\to K} p(f(n))\right|$. In particular, the natural map $|X| \to \left|\prod_{k\in K} p(k)\right|$ is an embedding (**??**). Now note that this map is $|\cdot|$ applied to the map $X = \lim_{\hat{K}}^{\mathsf{Cosif}_{\mathsf{cp}}(\mathsf{C})} \hat{p} \to \hat{p}(*) \in \mathsf{C}$. □

⋆ **2.8.84 Proposition.** *Let C be a topological ∞-site with finite products. The ∞-category $\mathsf{Cosif}_{\mathsf{cp}}(\mathsf{C})_{\mathsf{fin}}$ is a topological ∞-site, and the functor $\mathsf{Cosif}_{\mathsf{cp}}(\mathsf{C})_{\mathsf{fin}} \hookrightarrow \mathsf{Lim}_{\mathsf{cp}}(\mathsf{C})$ preserves open embeddings and pullbacks thereof.*

Proof. Fix $X \in \mathsf{Cosif}_{\mathsf{cp}}(\mathsf{C})_{\mathsf{fin}}$, choose a map $X \to c \in \mathsf{C}$ which is an embedding on underlying topological spaces (this always exists (2.8.83)), and let $c' \hookrightarrow c$ be an open embedding in C. Realize the map $X \to c$ as the restriction map $X = \lim_K^{\mathsf{Cosif}_{\mathsf{cp}}(\mathsf{C})} p \to p(k) = c$ for some finite diagram $p : K \to \mathsf{C}$, some vertex $k \in K$, and some identification $p(k) = c$ (this exists by (2.8.79), or alternatively by construction of the map $X \to c$ (2.8.83)). We may assume wlog that:

(2.8.84.1) For every $n \geq 0$, there is exactly one n-simplex $\sigma : \Delta^n \to K$ with final vertex $\sigma(n) = k$ (namely the constant n-simplex over k).

To see this, either note that it holds for the construction (2.8.83), or replace (K, k) with $(\Delta^1 \cup_{1\sim k} K, 0)$ and replace p with its pullback (which has the same limit since $(\Delta^1, 1)^\#$ is a marked right horn (1.4.109)). Let $p' \to p$ be the morphism of diagrams which is an isomorphism except for $p'(k) = c' \hookrightarrow c = p(k)$ (to define $p' \to p$, argue by induction on the filtration of $(\Delta^1, 1) \wedge K$ induced by a filtration of K by simplices $(\Delta^r, \partial\Delta^r)$: for $r = 0$ we take the map $c' \hookrightarrow c$ at k and the identity map on all other vertices, and for $r > 0$ we extend $(\Delta^1, 1)^\# \wedge (\Delta^r, \partial\Delta^r) \to \mathsf{C}$ using the fact that it is filtered by pushouts of marked right horns (1.4.54) whose marked edge is sent to an isomorphism in C since our simplex $\Delta^r \to K$ does not send r to k (2.8.84.1)). Now our key claim is that

$$\begin{tikzcd} \lim\limits_K^{\mathsf{Cosif}_{\mathsf{cp}}(\mathsf{C})} p' \arrow[r] \arrow[d] & \lim\limits_K^{\mathsf{Cosif}_{\mathsf{cp}}(\mathsf{C})} p \arrow[d] \\ c' \arrow[r] & c \end{tikzcd} \tag{2.8.84.2}$$

is a pullback in $\mathsf{Lim}_{\mathsf{cp}}(\mathsf{C})$ (hence $|\cdot|$ sends it to a pullback in Top (2.8.70)). Given this claim, the fact that $c' \hookrightarrow c$ is an open embedding in $\mathsf{Lim}_{\mathsf{cp}}(\mathsf{C})$ (2.8.68) implies that $\lim_K^{\mathsf{Cosif}_{\mathsf{cp}}(\mathsf{C})} p' \to \lim_K^{\mathsf{Cosif}_{\mathsf{cp}}(\mathsf{C})} p$ is as well (2.8.4), so we may conclude that $\mathsf{Cosif}_{\mathsf{cp}}(\mathsf{C})_{\mathsf{fin}}$ is a topological ∞-site. To prove the key claim that (2.8.84.2) is a pullback in $\mathsf{Lim}_{\mathsf{cp}}(\mathsf{C})$, write $\lim_K^{\mathsf{Cosif}_{\mathsf{cp}}(\mathsf{C})} p = \lim^{\mathsf{Lim}_{\mathsf{cp}}(\mathsf{C})} p_{\mathrm{cosif}}$ (2.8.78) and recall that p_{cosif} is given by the Bousfield–Kan transform $p_{\mathrm{cosif}} = p_\triangle$ (1.4.239). Thus the diagram (2.8.84.2) is given by $\lim_{\boldsymbol{\Delta}}^{\mathsf{Lim}_{\mathsf{cp}}(\mathsf{C})}$ applied to the square of Bousfield–Kan transforms $p'_\triangle \to p_\triangle$ mapping to $c' \to c$ (constant cosimplicial objects), which is a pullback of cosimplicial objects since K has exactly one simplex of every dimension with final vertex k (2.8.84.1)(1.4.238.3), so the claim follows since limits commutes with limits.

We have shown that $\mathsf{Cosif}_{\mathsf{cp}}(\mathsf{C})_{\mathsf{fin}}$ is a topological ∞-site and that $\mathsf{Cosif}_{\mathsf{cp}}(\mathsf{C})_{\mathsf{fin}} \hookrightarrow \mathsf{Lim}_{\mathsf{cp}}(\mathsf{C})$ preserves open embeddings. Now let us show that $\mathsf{Cosif}_{\mathsf{cp}}(\mathsf{C})_{\mathsf{fin}} \hookrightarrow \mathsf{Lim}_{\mathsf{cp}}(\mathsf{C})$ preserves pullbacks of open embeddings. Fix $Y \to X \in \mathsf{Cosif}_{\mathsf{cp}}(\mathsf{C})_{\mathsf{fin}}$, choose a map $X \to c \in \mathsf{C}$ which is an embedding on underlying topological spaces, and let $c' \hookrightarrow c$ be an open embedding in C. Realize $Y \to X \to c$ as restriction maps on limits in $\mathsf{Cosif}_{\mathsf{cp}}(\mathsf{C})_{\mathsf{fin}}$ of finite diagrams $k \in K \subseteq L \xrightarrow{q} \mathsf{C}$ with $Y = \lim_L q$, $X = \lim_K p$ ($p = q|_K$), and $q(k) = c$ (2.8.79). As above, we may assume that (2.8.84.1) holds for $k \in L$ (hence also for $k \in K$), and we may define $q' \to q$ (inducing $p' \to p$ by restriction to K) to be an isomorphism except for sending k to the open embedding $c' \hookrightarrow c$. The resulting diagrams (2.8.84.2) for $q' \to q$ and $p' \to p$ are pullbacks in $\mathsf{Lim}_{\mathsf{cp}}(\mathsf{C})$ as shown above, so we conclude from cancellation (1.1.57) that

$$\begin{array}{ccc} \lim\limits_{L}^{\mathsf{Cosif}_{\mathsf{cp}}(\mathsf{C})} q' & \longrightarrow & \lim\limits_{L}^{\mathsf{Cosif}_{\mathsf{cp}}(\mathsf{C})} q \\ \downarrow & & \downarrow \\ \lim\limits_{K}^{\mathsf{Cosif}_{\mathsf{cp}}(\mathsf{C})} p' & \longrightarrow & \lim\limits_{K}^{\mathsf{Cosif}_{\mathsf{cp}}(\mathsf{C})} p \end{array} \tag{2.8.84.3}$$

is a pullback in $\mathsf{Lim}_{\mathsf{cp}}(\mathsf{C})$. This is an arbitrary open embedding pullback in $\mathsf{Cosif}_{\mathsf{cp}}(\mathsf{C})_{\mathsf{fin}}$, so we are done. □

⋆ **2.8.85 Proposition** (Universal property of $\mathsf{C} \hookrightarrow \mathsf{Cosif}_{\mathsf{cp}}(\mathsf{C})_{\mathsf{fin}}$)**.** *Let* C *be a topological* ∞*-site with finite products. For any complete topological* ∞*-site* E*, pullback under* $\mathsf{C} \to \mathsf{Cosif}_{\mathsf{cp}}(\mathsf{C})_{\mathsf{fin}}$ *defines an equivalence between the following* ∞*-categories of topological functors:*

(2.8.85.1) *Topological functors* $\mathsf{C} \to \mathsf{E}$*.*

(2.8.85.2) *Topological functors* $\mathsf{Cosif}_{\mathsf{cp}}(\mathsf{C})_{\mathsf{fin}} \to \mathsf{E}$ *which preserve finite cosifted limits.*

Moreover, this restricts to an equivalence:

(2.8.85.3) *Topological functors* $\mathsf{C} \to \mathsf{E}$ *which preserve finite products.*

(2.8.85.4) *Topological functors* $\mathsf{Cosif}_{\mathsf{cp}}(\mathsf{C})_{\mathsf{fin}} \to \mathsf{E}$ *which preserve finite limits.*

More generally, the same holds for E *not assumed complete, once we restrict to those functors* $\mathsf{C} \to \mathsf{E}$ *which send every finite cosifted diagram in* C *to a diagram in* E *whose limit exists.*

Proof. In view of the strict topological functor $\mathsf{E} \hookrightarrow \mathsf{P}(\mathsf{E})$ which preserves all limits, it suffices to treat the case that E is complete.

To compare ∞-categories of topological functors out of C and $\mathsf{Cosif}_{\mathsf{cp}}(\mathsf{C})_{\mathsf{fin}}$, we apply (2.8.23) to the strict topological functor $\mathsf{C} \to \mathsf{Cosif}_{\mathsf{cp}}(\mathsf{C})_{\mathsf{fin}}$ and conditions α and $\overline{\alpha}$ given by (2.8.81.2) and (2.8.81.3). Hypothesis (2.8.23.1) (that restriction from $\overline{\alpha}$ functors to α functors is an equivalence) holds by (2.8.81).

To verify hypothesis (2.8.23.2), note that the unit map $|\cdot|_{\mathsf{Cosif}_{\mathsf{cp}}(\mathsf{C})_{\mathsf{fin}}} \to (\mathsf{C} \to \mathsf{Cosif}_{\mathsf{cp}}(\mathsf{C})_{\mathsf{fin}})_*|\cdot|_{\mathsf{C}}$ is an isomorphism by definition (indeed, right Kan extension $(\mathsf{C} \to \mathsf{Cosif}_{\mathsf{cp}}(\mathsf{C})_{\mathsf{fin}})_*$ is the same as right Kan extension $(\mathsf{C} \to \mathsf{Lim}(\mathsf{C}))_*$ followed by restriction $(\mathsf{Cosif}_{\mathsf{cp}}(\mathsf{C})_{\mathsf{fin}} \to \mathsf{Lim}(\mathsf{C}))^*$, and $|\cdot|_{\mathsf{Cosif}_{\mathsf{cp}}(\mathsf{C})_{\mathsf{fin}}}$ is defined as the restriction of $|\cdot|_{\mathsf{Lim}(\mathsf{C})}$ (2.8.64) to $\mathsf{Cosif}_{\mathsf{cp}}(\mathsf{C})_{\mathsf{fin}} \subseteq \mathsf{Lim}(\mathsf{C})$).

To verify hypothesis (2.8.23.3), we appeal to the characterization of open embeddings in $\mathsf{Cosif}_{\mathsf{cp}}(\mathsf{C})_{\mathsf{fin}}$ (2.8.84) and (2.8.24). This works since all $\overline{\alpha}$-functors $\mathsf{Cosif}_{\mathsf{cp}}(\mathsf{C})_{\mathsf{fin}} \to \mathsf{E}$ preserve pullback diagrams in $\mathsf{Cosif}_{\mathsf{cp}}(\mathsf{C})_{\mathsf{fin}}$ which remain pullbacks in $\mathsf{Lim}_{\mathsf{cp}}(\mathsf{C})$ (by definition of $\overline{\alpha}$).

Finally, note that topological functors satisfying α and $\overline{\alpha}$ are precisely the classes of functors (2.8.88.1)(2.8.88.2), since every topological functor is corporeal (2.8.70).

The second equivalence (that with the additional condition of preservation of finite products) follows by using the second equivalence from (2.8.81) in place of the first. □

⋆ **2.8.86 Definition** (Derived site)**.** For any perfect topological ∞-site C with finite products, its *finite derived* ∞*-site* $\mathcal{D}_{\mathsf{fin}}\mathsf{C}$ is defined as $(\mathsf{Cosif}_{\mathsf{cp}}(\mathsf{C})_{\mathsf{fin}})^{\#}$, namely the perfection # (2.8.60) of formal finite corporeal cosifted limits $\mathsf{Cosif}_{\mathsf{cp}}(\mathsf{C})_{\mathsf{fin}}$ (2.8.77).

For simplicity, we will drop the adjective 'finite' and the subscript 'fin': that is we will write $\mathcal{D}\mathsf{C}$ for $\mathcal{D}_{\mathsf{fin}}\mathsf{C}$ and just call it the 'derived site' of C. We must emphasize, however, that it can be useful to consider a larger version of the derived site where we adjoin more corporeal cosifted limits (not just the finite ones), inside which $\mathcal{D}_{\mathsf{fin}}\mathsf{C}$ would be termed the full subcategory of locally finitely presented objects.

2.8.87 Lemma. *If C has finite products, then $\mathcal{D}\mathsf{C}$ has finite limits.*

Proof. If C has finite products, then $\mathsf{Cosif}_{\mathsf{cp}}(\mathsf{C})_{\mathsf{fin}}$ has finite limits (**??**), and perfection $\#$ preserves having finite limits (2.8.63). ☐

⋆ **2.8.88 Theorem** (Universal property of the derived ∞-site)**.** *Let C be a topological ∞-site with finite products. For any complete perfect topological ∞-site E, pullback under $\mathsf{C} \to \mathcal{D}\mathsf{C}$ defines an equivalence between the following ∞-categories of topological functors:*

(2.8.88.1) *Topological functors $\mathsf{C} \to \mathsf{E}$.*

(2.8.88.2) *Topological functors $\mathcal{D}\mathsf{C} \to \mathsf{E}$ which preserve finite cosifted limits.*

Moreover, this restricts to an equivalence:

(2.8.88.3) *Topological functors $\mathsf{C} \to \mathsf{E}$ which preserve finite products.*

(2.8.88.4) *Topological functors $\mathcal{D}\mathsf{C} \to \mathsf{E}$ which preserve finite limits.*

More generally, the same holds for E not assumed complete, once we restrict to those functors $\mathsf{C} \to \mathsf{E}$ which send every finite cosifted diagram in C to a diagram in E whose limit exists.

Proof. Combine the universal property of $\mathsf{C} \to \mathsf{Cosif}_{\mathsf{cp}}(\mathsf{C})_{\mathsf{fin}}$ (2.8.85) with the universal property of perfection (2.8.60.5) and the fact that the latter (for sites with finite limits, in this case $\mathsf{Cosif}_{\mathsf{cp}}(\mathsf{C})_{\mathsf{fin}}$ (**??**)) respects preservation of finite limits and finite cosifted limits (2.8.63). ☐

The universal property of the derived site (2.8.88) characterizes it uniquely, hence can be useful for comparing different explicit constructions of it. It also gives some philosophical justification that the rather complex construction we gave of it does indeed define a reasonably 'correct' object of interest. On the other hand, it does not give any way of performing computations in a derived site. The explicit definition $\mathcal{D}\mathsf{C} = (\mathsf{Cosif}_{\mathsf{cp}}(\mathsf{C})_{\mathsf{fin}})^{\#}$ does, at least theoretically speaking, allow for concrete computations, however it is not efficient for this purpose.

Our next goal is to give an 'intrinsic characterization' of the derived site of a topological ∞-site, analogous to the intrisic characterization of presheaf categories (**??**). This characterization, or axiomatization, of derived sites provides a direct method for computations.

2.8.89 Proposition. *Let C be a topological ∞-site with finite products. The essential image of the left Kan extension functor*

$$(\mathsf{C} \to \mathcal{D}\mathsf{C})_! : \mathsf{Shv}(\mathsf{C}) \to \mathsf{Shv}(\mathcal{D}\mathsf{C}) \tag{2.8.89.1}$$

consists precisely of those sheaves on $\mathcal{D}\mathsf{C}$ which topologically preserve (2.9.2) *finite cosifted limits* (1.4.236) *(equivalently, finite cosifted limits of objects of C).*

Proof. Yoneda functors $\operatorname{Hom}(-,N)$ for $N \in \mathsf{C}$ topologically preserve finite cosifted limits (**??**). The essential image of left Kan extension is the closure of such Yoneda functors under colimits. We claim that the collection of sheaves on $\mathcal{D}\mathsf{C}$ which topologically preserve finite cosifted limits is closed under colimits. It suffices to show that for any fixed diagram $K^{\triangleleft} \to \mathsf{Top}$, the collection of lifts to $\mathsf{Shv}(-)^{\mathsf{op}} \rtimes \mathsf{Top}$ which are relative limit diagrams is closed under limits inside the ∞-category of all lifts. Since relative limits are limits in fibers (functorially) (1.4.154) and the pullback functors between sheaf categories preserve colimits (being left adjoints), we are reduced to the fact (**??**) that limit diagrams inside $\mathsf{Fun}(K^{\triangleleft}, \mathsf{E})$ are closed under limits for any ∞-category E (in this case $\mathsf{E} = \mathsf{Shv}(X)$ for $X \in \mathsf{Top}$ the cone point of our fixed diagram $K^{\triangleleft} \to \mathsf{Top}$).

We have thus shown that if $F \in \mathsf{Shv}(\mathcal{D}\mathsf{C})$ is left Kan extended from C then F topologically preserves finite cosifted limits. To show the converse, it suffices (1.1.94) to check that if $F, G \in \mathsf{Shv}(\mathcal{D}\mathsf{C})$ both topologically preserve finite cosifted limits (even just of objects of C), then a morphism $F \to G$ is an isomorphism iff its restriction to C is an isomorphism. This fact follows immediately from the fact that every object of $\mathcal{D}\mathsf{C}$ is locally a finite cosifted limit of objects of C. $\square$

2.9 Derived smooth manifolds

The ∞-category of derived smooth manifolds $\mathcal{D}\mathsf{Sm}$ is an enlargement of the category of smooth manifolds Sm, obtained by *formally adjoining finite limits, modulo transverse limits in* Sm (equivalently, formally adjoining cosifted limits as in (**??**)) within the realm of topological ∞-sites. For example, every diagram of smooth manifolds $X \to Y \leftarrow Z$ has a fiber product $X \times_Y Z$ in the category of derived smooth manifolds. When the diagram is transverse, this is simply the usual fiber product; otherwise it is a more exotic sort of object. The theory of derived smooth manifolds originates in work of Spivak [101, 102] with further developments by Joyce [50, 54] and many others. It is a homotopical analogue of the theory of locally finitely presented C^∞-schemes [23, 83, 53] and falls within the general framework of derived geometry of Lurie [72] and Toën–Vezzosi [105, 106]. We introduce a new axiomatic approach to the ∞-category of derived smooth manifolds.

* *

The ∞-category of derived smooth manifolds is best understood through certain properties it satisfies. It is not hard to see that these properties characterize it uniquely, so they may be regarded as the definition of derived smooth manifolds, modulo a proof of existence. Recall the notion of a 'perfect topological ∞-site' (2.8) which is an '∞-category of topological spaces equipped with additional local structure'.

⋆ **2.9.1 Definition** (Derived smooth manifold)**.** The topological ∞-site $\mathcal{D}\mathsf{Sm}$ of derived smooth manifolds is the derived ∞-site (2.8.86) of the topological site of smooth manifolds Sm.

⋆ **2.9.2 Definition** (Topological preservation of limits)**.** Let C be a topological ∞-site, and recall that to each sheaf $F : \mathsf{C}^{\mathsf{op}} \to \mathsf{Spc}$ we may associate a strict topological functor $F : \mathsf{C} \to \mathsf{Shv}(-)^{\mathsf{op}} \rtimes \mathsf{Top}$ lifting $|\cdot| : \mathsf{C} \to \mathsf{Top}$ (**??**). Given a limit in C which is preserved by $|\cdot|$, we say that F *topologically preserves* this limit when its associated functor $\mathsf{C} \to \mathsf{Shv}(-)^{\mathsf{op}} \rtimes \mathsf{Top}$ preserves said limit (equivalently, sends it to a relative limit (1.4.153) over Top).

Concretely, a diagram $K^{\triangleleft} \to \mathsf{Shv}(-)^{\mathsf{op}} \rtimes \mathsf{Top}$ encoding a diagram of sheaved topological spaces (X_α, F_α) all receiving a map from a sheaved topological space (X, F) is a relative limit diagram when the natural map

$$\operatorname*{colim}_{\alpha} \pi_\alpha^* F_\alpha \to F \tag{2.9.2.1}$$

is an isomorphism (that is, relative limits in $\mathsf{Shv}(-)^{\mathsf{op}} \rtimes \mathsf{Top} \to \mathsf{Top}$ are limits in fibers (1.4.154)(2.2.22)).

⋆ **2.9.3 Definition** (Derived smooth manifold)**.** The ∞-category $\mathcal{D}\mathsf{Sm}$ of derived smooth manifolds together with the functor $\mathsf{Sm} \to \mathcal{D}\mathsf{Sm}$ is defined by the following properties:

(2.9.3.1) $\mathsf{Sm} \to \mathcal{D}\mathsf{Sm}$ is a strict topological functor between perfect topological ∞-sites (2.8).

(2.9.3.2) $\mathsf{Sm} \to \mathcal{D}\mathsf{Sm}$ is fully faithful and preserves finite products.

(2.9.3.3) $\mathcal{D}\mathsf{Sm}$ has finite limits, and every object of $\mathcal{D}\mathsf{Sm}$ is locally isomorphic to a finite limit of smooth manifolds.

(2.9.3.4) $|\cdot| : \mathcal{D}\mathsf{Sm} \to \mathsf{Top}$ preserves finite limits.

(2.9.3.5) For any $N \in \mathsf{Sm}$, the Yoneda sheaf $\operatorname{Hom}(-, N) \in \mathsf{Shv}(\mathcal{D}\mathsf{Sm})$ topologically preserves finite cosifted limits (**??**).

(2.9.3.6) (Universal Property) For every complete perfect topological site E, the ∞-category of topological functors $\mathcal{D}\mathsf{Sm} \to \mathsf{E}$ preserving finite cosifted limits is equivalent via restriction to the ∞-category of topological functors $\mathsf{Sm} \to \mathsf{E}$.

The universal property (**??**) evidently characterizes the functor $\mathsf{Sm} \to \mathcal{D}\mathsf{Sm}$ uniquely up to contractible choice, provided it exists. The existence of this functor (satisfying all the above properties) shown in (**??**) below.

2.9.4 Exercise. Conclude from the fact that $\mathsf{Sm} \to \mathcal{D}\mathsf{Sm}$ preserves finite products that it more generally preserves all submersive pullbacks (recall that submersions of smooth manifolds are locally products (2.4.5) and appeal to the local nature of limits of limits in topological sites (2.8.48)). This will be used later to prove that $\mathsf{Sm} \to \mathcal{D}\mathsf{Sm}$ more generally preserves all 'finite transverse limits' (2.9.8)(2.9.22).

Transverse diagrams

We now discuss the notion of transversality for diagrams of smooth manifolds.

⋆ **2.9.5 Definition** (Transverse diagram of vector spaces)**.** A diagram of vector spaces $D : K \to \mathsf{Vect}$ is called *transverse* when the canonical map

$$\lim_K^{\mathsf{Vect}} D \to \lim_K^{\mathsf{K}(\mathsf{Vect})} D \tag{2.9.5.1}$$

(from the limit of D in the category of vector spaces to the limit of D in the ∞-category of complexes of vector spaces (**??**)) is an isomorphism. Transversality of D evidently depends only on its class in $\mathsf{Lim}(\mathsf{Vect})$.

Since $\mathsf{K}^{\geq 0}(\mathsf{Vect}) \subseteq \mathsf{K}(\mathsf{Vect})$ is closed under limits, we could just as well replace the limit in $\mathsf{K}(\mathsf{Vect})$ with the limit in $\mathsf{K}^{\geq 0}(\mathsf{Vect})$ in the definition of transversality. Since $\mathsf{Vect} \subseteq \mathsf{K}^{\geq 0}(\mathsf{Vect})$ is a coreflective subcategory, the comparison map from the limit in Vect to the limit in $\mathsf{K}^{\geq 0}(\mathsf{Vect})$ is an isomorphism iff the limit in $\mathsf{K}^{\geq 0}(\mathsf{Vect})$ lies in the full subcategory $\mathsf{Vect} \subseteq \mathsf{K}^{\geq 0}(\mathsf{Vect})$. In other words, a diagram $D : K \to \mathsf{Vect}$ is transverse iff its limit in $\mathsf{K}^{\geq 0}(\mathsf{Vect})$ lies in $\mathsf{Vect} \subseteq \mathsf{K}^{\geq 0}(\mathsf{Vect})$ (equivalently, has no higher cohomology).

It may help to recall that the limit of a diagram $D : K \to \mathsf{K}^{\geq 0}(\mathsf{Vect})$ is given explicitly by the total complex

$$\prod_{\sigma:[0]\to K} D(\sigma(0)) \to \prod_{\sigma:[1]\to K} D(\sigma(1)) \otimes \mathfrak{o}(1)^\vee \to \prod_{\sigma:[2]\to K} D(\sigma(2)) \otimes \mathfrak{o}(2)^\vee \to \cdots \tag{2.9.5.2}$$

which may be regarded as 'simplicial cochains on K with coefficients in D' (**??**). Also recall that the limit of a cosimplicial vector space $p : \mathbf{\Delta} \to \mathsf{Vect}$ is the object of $\mathsf{K}^{\geq 0}(\mathsf{Vect})$ associated to p by the Dold–Kan correspondence (**??**).

2.9.6 Exercise. Show that a diagram of vector spaces $V \to W \leftarrow U$ is transverse iff the sum map $V \oplus U \to W$ is surjective.

The notion of transversality for a diagram D evidently depends only on the 'formal ∞-limit' represented by D, namely its limit in the ∞-category $\mathsf{Lim}(\mathsf{Vect}) = \mathsf{P}(\mathsf{Vect}^{\mathsf{op}})^{\mathsf{op}} = \mathsf{Fun}(\mathsf{Vect}, \mathsf{Spc})^{\mathsf{op}}$ of formal ∞-limits (**??**) in Vect. Indeed, recall that every object of $\mathsf{Lim}(\mathsf{C})$ is represented by a diagram $K \to \mathsf{C}$, and two diagrams represent the same object iff they are related by pullback under initial functors. Thus a property of objects of $\mathsf{Lim}(\mathsf{C})$ (i.e. a property of formal ∞-limits in C) is a property of diagrams in C which is invariant under pullback under initial functors.

2.9.7 Lemma. *A formal limit of vector spaces is transverse iff its cosiftedization is transverse.*

Proof. The functor $\mathsf{Vect} \to \mathsf{K}^{\geq 0}(\mathsf{Vect})$ preserves finite products, hence commutes with cosiftedization (1.4.233). □

⋆ **2.9.8 Definition** (Transverse diagram of smooth manifolds)**.** Let $D : K \to \mathsf{Sm}$ be a diagram. A point

$$p \in \lim_K^{\mathsf{Top}} D \tag{2.9.8.1}$$

of the limit of D in the category of topological spaces determines a lift of D to Sm_* (pointed smooth manifolds and basepoint preserving maps).

$$\begin{tikzcd} & \mathsf{Sm}_* \arrow[d] \\ K \arrow[ur, dashed, "D_p"] \arrow[r, "D"'] & \mathsf{Sm} \end{tikzcd} \tag{2.9.8.2}$$

We can now compose this lift D_p with the 'tangent space at the basepoint' functor $T_* : \mathsf{Sm}_* \to \mathsf{Vect}_{\mathbb{R}}$ to obtain a diagram

$$T_p D : K \to \mathsf{Vect}_{\mathbb{R}}. \tag{2.9.8.3}$$

We say that D is *transverse at p* when $T_p D$ is transverse (2.9.5), and we say that D is *transverse* when it is transverse at every point of its topological limit $\lim_K^{\mathsf{Top}} D$. Transversality of D evidently depends only on its class in $\mathsf{Lim}(\mathsf{Sm})$.

2.9.9 Exercise. Show, using the corresponding statement for vector spaces (2.9.7), that a formal limit of smooth manifolds is transverse iff its cosiftedization is transverse.

2.9.10 Exercise. Show that a diagram of smooth manifolds $D : J \to \mathsf{Sm}$ with only 0-cells and 1-cells is transverse in the sense of (2.9.8) iff it is transverse in the sense of (2.4.8).

Cosimplicial presentations

We now study presentations of derived smooth manifolds by cosimplicial smooth manifolds. The relation between cosimplicial smooth manifolds csSm and derived smooth manifolds $\mathcal{D}\mathsf{Sm}$ is analogous to the relation between the category of complexes $\mathsf{Kom}^{\geq 0}(\mathsf{Vect})$ (1.1.141) (**??**) and the ∞-category of complexes $\mathsf{K}^{\geq 0}(\mathsf{Vect})$ (**??**). Recall that a cosimplicial object is called *n-truncated* when its matching maps in degrees $> n$ are isomorphisms (1.2.13)(1.2.16) and is called *truncated* when it is n-truncated for some $n < \infty$.

$\star$ **2.9.11 Lemma** (Existence of cosimplicial presentations)**.** *Any derived smooth manifold may be expressed locally as the limit of a truncated cosimplicial smooth manifold. Any map of derived smooth manifolds may be expressed locally as a levelwise submersive map of truncated cosimplicial smooth manifolds.*

Proof. Every derived smooth manifold is locally the limit of a finite diagram of smooth manifolds. Since $\mathsf{Sm} \to \mathcal{D}\mathsf{Sm}$ preserves finite products, this limit unchanged by applying cosiftedization, which turns a finite diagram into a truncated cosimplicial diagram (**??**).

Every map of derived smooth manifolds is locally the map from the limit of a finite diagram of smooth manifolds to the limit of a subdiagram thereof (**??**), and upon applying cosimplicialization this becomes a levelwise submersion of truncated cosimplicial smooth manifolds. □

2.9.12 Exercise. Conclude from (2.9.11) that every derived smooth manifold X has local bump functions (2.1.41) (use the fact that $\lim_{\mathbf{\Delta}} X^\bullet \to X^0$ is an embedding for any cosimplicial topological space $X^\bullet$), hence, if paracompact Hausdorff, partitions of unity (2.1.51).

Recall that a cosimplicial object is called *Reedy* $\mathcal{P}$ when its matching maps have $\mathcal{P}$ (1.2.17) (any property of morphisms $\mathcal{P}$). We saw earlier that a map of cosimplicial vector spaces $V^\bullet \to W^\bullet$ is Reedy surjective iff the corresponding map of chain complexes is surjective (1.2.26). Recall that a 'point' x of a cosimplicial smooth manifold $X^\bullet$ means a point of its topological limit $\lim_{\mathbf{\Delta}}^{\mathsf{Top}} X^\bullet$ or, equivalently, a map $x : * \to X^\bullet$ from the constant cosimplicial smooth manifold $*$. Thus for a point x of a cosimplicial smooth manifold $X^\bullet$, a map $X^\bullet \to Y^\bullet$ is levelwise submersive at x iff it is Reedy submersive at x.

Let us now see how to upgrade levelwise (equivalently, Reedy) submersivity over the topological limit to (true) levelwise submersivity and Reedy submersivity over an open cosimplicial submanifold containing the topological limit.

2.9.13 Exercise. Let $X^\bullet$ be a cosimplicial smooth manifold. Given an open subset $V^k \subseteq X^k$, consider the cosimplicial smooth manifold $U^\bullet$ with a levelwise open embedding $U^\bullet \to X^\bullet$ defined by

$$U^j = \bigcap_{f:[j]\to[k]} (X^j \xrightarrow{f_*} X^k)^{-1}(V^k). \tag{2.9.13.1}$$

Show that if $M^iX^\bullet$ exists, then so does $M^iU^\bullet$ and the map $M^iU^\bullet \to M^iX^\bullet$ is an open embedding. Show that if $i > k$, then the induced square of matching maps

$$\begin{tikzcd} U^i \arrow[r] \arrow[d] & X^i \arrow[d] \\ M^iU^\bullet \arrow[r] & M^iX^\bullet \end{tikzcd} \tag{2.9.13.2}$$

is a pullback (so, in particular, if the ith matching map of $X^\bullet$ is an isomorphism, then so is that of $U^\bullet$).

2.9.14 Exercise. Let $X^\bullet \to Y^\bullet$ be a map of cosimplicial smooth manifolds. Show that if $X^\bullet \to Y^\bullet$ is Reedy submersive in degrees $< n$, then the nth matching object $M^nX^\bullet \times_{M^nY^\bullet} Y^n$ is a transverse limit and maps submersively to Y^n (use (1.2.18)). Conclude that if $X^\bullet \to Y^\bullet$ is Reedy submersive in degrees $\leq n$, then it is levelwise submersive in degrees $\leq n$.

2.9.15 Lemma. *Let $X^\bullet \to Y^\bullet$ be a map of cosimplicial smooth manifolds that is Reedy submersive (equivalently, levelwise submersive* (1.2.26)*) at every point of the topological limit of $X^\bullet$. If $X^\bullet$ is n-truncated, then there exists an n-truncated levelwise open embedding $U^\bullet \to X^\bullet$ containing the topological limit such that the restriction $U^\bullet \to Y^\bullet$ is Reedy submersive (hence levelwise submersive* (2.9.14)*).*

Proof. By induction, suppose $X^\bullet \to Y^\bullet$ is Reedy submersive in degrees $< k$. Let $V^k \subseteq X^k$ be the open locus where the kth matching map $X^k \to M^kX^\bullet \times_{M^kY^\bullet} Y^k$ is submersive. By hypothesis, V^k contains the topological limit of $X^\bullet$. Now consider the open embedding $U^\bullet \subseteq X^\bullet$ associated to $V^k \subseteq X^k$ (2.9.13). Thus $U^\bullet \to X^\bullet$ is n-truncated and $U^\bullet \to Y^\bullet$ is Reedy submersive in degrees $\leq k$ (2.9.13). □

2.9.16 Corollary. *Every map of derived smooth manifolds $X \to Y$ is, locally near any point $x \in X$, a finite composition $X = Z_N \to \cdots \to Z_0 \to Z_{-1} = Y$ in which $Z_i \to Z_{i-1}$ is locally a pullback of the ith diagonal of $\mathbb{R}^{a_i}$ for some integers $a_i \geq 0$.*

Proof. Realize our given map $X \to Y$ (locally) as a levelwise submersion of truncated cosimplicial smooth manifolds $X^\bullet \to Y^\bullet$ (2.9.11). By replacing $X^\bullet$ with an open cosimplicial submanifold thereof, we may assume $X^\bullet \to Y^\bullet$ is also Reedy submersive (2.9.15). Since $X^\bullet \to Y^\bullet$ is Reedy submersive, its relative matching maps all exist (2.9.14). Now for any map of cosimplicial objects $X^\bullet \to Y^\bullet$, the induced map on totalizations $X = \lim_{\boldsymbol{\Delta}} X^\bullet \to \lim_{\boldsymbol{\Delta}} Y^\bullet = Y$ factors canonically as a (co-transfinite) composition $X = \varprojlim_i Z_i \to \cdots \to Z_2 \to Z_1 \to Z_0 \to Z_{-1} = Y$ where each map $Z_i \to Z_{i-1}$ is a pullback of the ith diagonal of the ith matching map $X^i \to M^iX^\bullet \times_{M^iY^\bullet} Y^i$ (??). In our case, the inverse limit is achieved at some finite i (indeed, $X^\bullet$ and $Y^\bullet$ are both k-truncated for some $k < \infty$, so their ith matching maps are isomorphisms for $i > k$ (1.2.16), so the inverse limit is achieved at all $i \geq k$). The ith matching map is submersive at x since $X^\bullet \to Y^\bullet$ is Reedy submersive, and the diagonal of a pullback is a pullback of the diagonal (1.1.66). □

A given cosimplicial presentation of a derived smooth manifold (or of a morphism of derived smooth manifolds) may be much larger than necessary. Our next goal is to show (2.9.21) how to transform a given cosimplicial presentation into one which is 'minimal' in the following sense.

⋆ **2.9.17 Definition** (Minimal)**.** A cosimplicial vector space will be called minimal when the associated complex of vector spaces (1.2.20) has vanishing differential. A cosimplicial smooth manifold $X^\bullet$ will be called minimal at a point $x \in X^\bullet$ when the cosimplicial vector space $T_xX^\bullet$ is minimal. More generally, a levelwise submersion of cosimplicial smooth manifolds $X^\bullet \to Y^\bullet$ will be called minimal at $x \in X^\bullet$ when the cosimplicial vector space $T_x(X^\bullet/Y^\bullet)$ is minimal. The term 'minimal' without qualification means minimal at all points.

Recall that the chain complex $[\mathbb{Z}[k+1] \to \mathbb{Z}[k]] \in \mathsf{Kom}_{\geq 0}(\mathsf{Ab})$ corresponds under Dold–Kan (1.2.20) to the simplicial abelian group $C^k_{\mathrm{cell}}(\Delta^\bullet)$ (1.2.23.1). In what follows, we will default to real coefficients, so $C^k_{\mathrm{cell}}(\Delta^\bullet) = C^k_{\mathrm{cell}}(\Delta^\bullet;\mathbb{R})$ corresponds to $[\mathbb{R}[k+1] \to \mathbb{R}[k]]$. The dual cosimplicial vector spaces $C_k^{\mathrm{cell}}(\Delta^\bullet)$ will be of interest to us as cosimplicial smooth manifolds. The augmented cosimplicial diagram $* \to C_k^{\mathrm{cell}}(\Delta^\bullet)$ is a transverse limit diagram in Sm since the complex of vector spaces corresponding to the cosimplicial vector space $C_k^{\mathrm{cell}}(\Delta^\bullet)$ is acyclic.

2.9.18 Lemma. *For every $k \geq 0$, the augmented cosimplicial diagram $* \to C_k^{\mathrm{cell}}(\Delta^\bullet)$ is a limit diagram in $\mathcal{D}\mathsf{Sm}$.*

Proof. It suffices to show that they have the same space of maps to $\mathbb{R}$. The space of maps from $\lim_{\mathbf{\Delta}} C_k^{\mathrm{cell}}(\Delta^\bullet)$ to $\mathbb{R}$ is the colimit $\operatorname{colim}_{\mathbf{\Delta}} C^\infty(C_k^{\mathrm{cell}}(\Delta^\bullet))_0$ by (**??**) (where the subscript $_0$ indicates taking germs near zero). Thus we should show that the augmented simplicial diagram $C^\infty(C_k^{\mathrm{cell}}(\Delta^\bullet))_0 \to \mathbb{R}$ is a colimit diagram. Note that this augmented simplicial diagram can really just be denoted $C^\infty(C_k^{\mathrm{cell}}(\Delta^\bullet))_0$ if we follow the convention that $\Delta^{-1} = \varnothing$. It suffices to prove the extension property for maps from $(\Delta^r, \partial\Delta^r)$ to $C^\infty(C_k^{\mathrm{cell}}(\Delta^\bullet))_0$. That is, given a collection of smooth functions $f_I : C_k^{\mathrm{cell}}(\Delta^I) \to \mathbb{R}$ (or rather germs near zero of such) for every $I \subsetneq \{0,\ldots,r\}$, we should produce a function $C_k^{\mathrm{cell}}(\Delta^r) \to \mathbb{R}$ whose restriction to $\Delta^I \subseteq \Delta^r$ is f_I for every $I \subsetneq \{0,\ldots,r\}$ (the case $I = \varnothing$ corresponds to Δ^{-1}). We can take the function

$$\sum_{J \subsetneq \{0,\ldots,r\}} (-1)^{r-1-|J|} f_J \circ (\Delta^J \to \Delta^r)^! \tag{2.9.18.1}$$

where $(\Delta^J \to \Delta^r)^! : C_k^{\mathrm{cell}}(\Delta^r) \to C_k^{\mathrm{cell}}(\Delta^J)$ is the brutal restriction of chains (simply throw away any k-simplices not contained in $\Delta^J \subseteq \Delta^r$). We should check that evaluating this function on $C_k^{\mathrm{cell}}(\Delta^I) \subseteq C_k^{\mathrm{cell}}(\Delta^r)$ yields f_I for every $I \subsetneq \{0,\ldots,r\}$. It suffices to consider the case $I = \{0,\ldots,r\} \setminus a$ for some $0 \leq a \leq r$. The term $J = I$ gives the desired result, and the remaining terms cancel in pairs with the same intersection $J \cap I$. □

2.9.19 Definition (Elementary derived open embedding)**.** A map of truncated cosimplicial smooth manifolds $X^\bullet \to Y^\bullet$ will be called an *elementary derived open embedding* when it is a finite composition of the following sorts of maps:

(2.9.19.1) A levelwise open embedding $X^\bullet \to Y^\bullet$.
(2.9.19.2) A levelwise submersive pullback

$$\begin{tikzcd} X^\bullet \arrow[r] \arrow[d] & Y^\bullet \arrow[d] \\ 0 \arrow[r] & C_k^{\mathrm{cell}}(\Delta^\bullet) \end{tikzcd}$$

for some $k \geq 0$.

If $X^\bullet \to Y^\bullet$ is an elementary derived open embedding, then the induced map on derived limits $\lim_{\mathbf{\Delta}} X^\bullet \to \lim_{\mathbf{\Delta}} Y^\bullet$ is an open embedding since $\mathsf{Sm} \to \mathcal{D}\mathsf{Sm}$ preserves submersive pullbacks (2.9.4) and $\lim_{\mathbf{\Delta}} C_k^{\mathrm{cell}}(\Delta^\bullet) = *$ (2.9.18).

2.9.20 Exercise. Show that for any elementary derived open embedding $f : X^\bullet \to Y^\bullet$, the induced map on tangent spaces $T_x X^\bullet \to T_{f(x)} Y^\bullet$ corresponds to a quasi-isomorphism of cochain complexes under Dold–Kan (note the use of (1.2.24)).

⋆ **2.9.21 Proposition** (Existence of minimal cosimplicial presentations)**.** *Let $X^\bullet \to Y^\bullet$ be a levelwise submersion of cosimplicial smooth manifolds, and suppose $X^\bullet$ is truncated. For every point $x \in X^\bullet$, there exists an elementary derived open embedding $U^\bullet \to X^\bullet$ and a lift of x to $u \in U^\bullet$ such that the composition $U^\bullet \to Y^\bullet$ is submersive and minimal* (2.9.17) *at u.*

Proof. Suppose that f is non-minimal at x. That is, the simplicial vector space $T_x^*(X^\bullet/Y^\bullet)$ is non-minimal, meaning that the corresponding chain complex $N_\bullet T_x^*(X^\bullet/Y^\bullet)$ has non-vanishing differential. Intuitively, this means that there are some 'transverse directions' of $X^\bullet$ at x which we can 'cancel' (take transverse limit in Sm) while preserving submersivity of f.

Fix an injective map $[\mathbb{R}[k+1] \to \mathbb{R}[k]] \to N_\bullet T_x^*(X^\bullet/Y^\bullet)$ for some $k \geq 0$. Denote by $C^\infty(X^\bullet, x) \subseteq C^\infty(X^\bullet)$ the smooth functions vanishing at x, and let us try to find a lift $[\mathbb{R}[k+1] \to \mathbb{R}[k]] \to N_\bullet C^\infty(X^\bullet, x)$.

$$\begin{tikzcd} & N_\bullet C^\infty(X^\bullet, x) \arrow[d] \\ {[\mathbb{R}[k+1] \to \mathbb{R}[k]]} \arrow[r] \arrow[ur, dashed] & N_\bullet T_x^*(X^\bullet/Y^\bullet) \end{tikzcd} \tag{2.9.21.1}$$

The maps $C^\infty(X^\bullet, x) \to T_x^* X^\bullet \to T_x^*(X^\bullet/Y^\bullet)$ are levelwise surjective, so the corresponding maps of complexes $N_\bullet C^\infty(X^\bullet, x) \to N_\bullet T_x^* X^\bullet \to N_\bullet T_x^*(X^\bullet/Y^\bullet)$ are degreewise surjective (1.2.24), hence the desired lift exists. This lift corresponds to a linear map $C^k_{\mathrm{cell}}(\Delta^\bullet) \to C^\infty(X^\bullet, x)$ (1.2.23.1), which is equivalently a smooth map

$$(X^\bullet, x) \to (C_k^{\mathrm{cell}}(\Delta^\bullet), 0). \tag{2.9.21.2}$$

By construction, the derivative of this map at x is the map $C^k_{\mathrm{cell}}(\Delta^\bullet) \to T_x^*(X^\bullet/Y^\bullet)$ corresponding to our chosen injection $[\mathbb{R}[k+1] \to \mathbb{R}[k]] \to N_\bullet T_x^*(X^\bullet/Y^\bullet)$. The derivative

$C^k_{\mathrm{cell}}(\Delta^\bullet) \to T^*_x(X^\bullet/Y^\bullet)$ is thus also injective (1.2.24), so our map (2.9.21.2) is levelwise submersive at x.

We would now like to form the pullback of $0 \to C_k^{\mathrm{cell}}(\Delta^\bullet)$ under our map (2.9.21.2).

$$\begin{tikzcd} U^\bullet \arrow[r] \arrow[d] & X^\bullet \arrow[d] \\ 0 \arrow[r] & C_k^{\mathrm{cell}}(\Delta^\bullet) \end{tikzcd} \tag{2.9.21.3}$$

Since $X^\bullet \to C_k^{\mathrm{cell}}(\Delta^\bullet)$ is submersive at x, we may use (2.9.15) to replace $X^\bullet$ by an open cosimplicial submanifold thereof over which the map $X^\bullet \to C_k^{\mathrm{cell}}(\Delta^\bullet)$ is levelwise submersive, thus ensuring that the pullback $U^\bullet$ exists.

The map $U^\bullet \to Y^\bullet$ is levelwise submersive at $u = x \times_0 0$ by construction. Applying (2.9.15) again, we may find inside $U^\bullet$ an open cosimplicial submanifold over which this map is levelwise submersive. Finally, the rank of the differential of $N_\bullet T_x(U^\bullet/Y^\bullet)$ is one less than that of $N_\bullet T_x(X^\bullet/Y^\bullet)$, so by iterating this construction we eventually reach a $U^\bullet$ which is minimal over $Y^\bullet$ at u. □

⋆ **2.9.22 Corollary** (Finite products generate finite transverse limits)**.** *The category of smooth manifolds* Sm *has all finite transverse limits* (2.9.8)*, and a topological functor* $\mathsf{Sm} \to \mathsf{C}$ *preserves finite transverse limits iff it preserves finite products of copies of* $\mathbb{R}$*. In particular,* $\mathsf{Sm} \to \mathcal{D}\mathsf{Sm}$ *preserves finite transverse limits.*

Proof. Due to the local nature of limits in topological ∞-sites (2.8.48), a topological functor $\mathsf{Sm} \to \mathsf{C}$ preserves finite products iff it preserves finite products of copies of $\mathbb{R}$. By the universal property of $\mathsf{Sm} \to \mathcal{D}\mathsf{Sm}$ (**??**), a topological functor $\mathsf{Sm} \to \mathsf{C}$ preserving finite products extends uniquely to a topological functor $\mathcal{D}\mathsf{Sm} \to \mathsf{C}$ preserving finite limits. It therefore suffices to show that Sm has finite transverse limits and that they are preserved by $\mathsf{Sm} \to \mathcal{D}\mathsf{Sm}$.

Fix a finite transverse diagram of smooth manifolds $D : J \to \mathsf{Sm}$, and let us show that $\lim D$ exists in Sm and is preserved by $\mathsf{Sm} \to \mathcal{D}\mathsf{Sm}$. Since Sm has finite products and $\mathsf{Sm} \to \mathcal{D}\mathsf{Sm}$ preserves finite products, we may wlog replace D with its cosiftedization, which is represented by a truncated cosimplicial object $X^\bullet : \mathbf{\Delta} \to \mathsf{Sm}$ (**??**). We may moreover assume $X^\bullet$ is minimal by (2.9.21) (noting that an elementary derived open embedding also induces an open embedding on limits in Sm provided these exist, and that it preserves transversality (2.9.20)). Now if $X^\bullet$ is minimal at $x \in X^\bullet$ and transverse, we can construct an open embedding covering x which is constant. Indeed, just work by induction applying (2.9.13) to replace each level X^n with an open subset over which the nth matching map is an open embedding; this makes all matching maps isomorphisms, hence we win.

The functor $\mathsf{Sm} \to \mathcal{D}\mathsf{Sm}$ preserves finite products (2.9.3.2), hence preserves finite transverse limits. □

Amplitude

We saw earlier that every morphism of derived smooth manifolds is, locally on the source, a finite composition of maps which are, locally on the source, a pullback of $\mathbb{R} \to *$ or one of its iterated diagonals (2.9.16). The 'amplitude' of a morphism of derived smooth manifolds records which iterated diagonals are relevant.

$\star$ **2.9.23 Definition** (Amplitude)**.** Let $I \subseteq \mathbb{Z}_{\geq 0}$. A morphism of derived smooth manifolds is said to have *amplitude* $\subseteq I$ when it is, locally on the source, a finite composition of maps which are, locally on the source, a pullback of the ith diagonal (1.1.63) of $\mathbb{R}$ for some non-negative integer $i \in I$.

Let us get a handle on the iterated diagonals of $\mathbb{R} \to *$. The first diagonal is $\mathbb{R} \to \mathbb{R} \times \mathbb{R}$, which is a pullback of $* \to \mathbb{R}$, and conversely $* \to \mathbb{R}$ is a pullback of $\mathbb{R} \to \mathbb{R} \times \mathbb{R}$; both are transverse pullbacks in Sm, hence are pullbacks in $\mathcal{D}\mathsf{Sm}$ as well. Now the diagonal of a pullback is a pullback of the diagonal (1.1.66), so being a pullback of the kth diagonal of $\mathbb{R} \to *$ is, for $k \geq 1$, equivalent to being a pullback of the $(k-1)$st diagonal of $* \to \mathbb{R}$. The ath iterated diagonal of $* \to \mathbb{R}$ is $* \to \Omega^a\mathbb{R}$, where Ω^a is the ath based loop space in the ∞-categorical sense, namely the limit over the D^a-shaped diagram in $\mathcal{D}\mathsf{Sm}$ taking the value $*$ on the boundary and the value $\mathbb{R}$ in the interior.

2.9.24 Exercise. Show that having amplitude $\subseteq I$ is preserved under pullback and closed under composition. Show that if $X \to Y$ has amplitude $\subseteq I$, then its relative diagonal $X \to X \times_Y X$ has amplitude $\subseteq I+1$. Formulate the resulting cancellation (1.1.68) statement for amplitude. Conclude, in particular, that every morphism of smooth manifolds has amplitude ≤ 1. Unwind the reasoning to explicitly express any morphism of smooth manifolds as the composition of an immersion followed by a submersion.

2.9.25 Definition (Submersion)**.** A morphism of derived smooth manifolds is called a submersion when it has amplitude 0 (equivalently, when it is, locally on the source, a pullback of $\mathbb{R}^a \to *$).

Vector bundles and perfect complexes

We now explain the notion of a vector bundle on a derived smooth manifold. Since this is an ∞-categorical context, a certain amount of abstraction is required to describe the relevant systems of higher homotopies in a manageable way. Explicitly, a vector bundle on a derived smooth manifold X is an open cover $X = \bigcup_i U_i$, integers $n_i \geq 0$, transition functions $\varphi_{ij} : U_i \cap U_j \to \mathrm{Hom}(\mathbb{R}^{n_j}, \mathbb{R}^{n_i})$ ($\varphi_{ii} = \mathbf{1}$), homotopies $\varphi_{ijk} : \varphi_{ij}\varphi_{jk} \to \varphi_{ik}$ over $U_i \cap U_j \cap U_k$, and higher homotopies $\varphi_{i_0 \cdots i_p}$ over $U_{i_0} \cap \cdots \cap U_{i_p}$ for all $p \geq 3$; morphisms of vector bundles may be defined similarly. It becomes prohibitively complex to manipulate explicitly such systems of homotopies, so a more categorical perspective is required.

$\star$ **2.9.26 Definition** (Vector bundle on a derived smooth manifold)**.** Recall (**??**) that the ∞-category $\mathsf{Rng}_{\mathsf{C}} = \mathsf{Fun}_\times((\mathsf{Rng}^{\mathsf{finfree}})^{\mathsf{op}}, \mathsf{C})$ of *(commutative) ring objects* in an ∞-category C

with finite products is the ∞-category of functors $(\mathsf{Rng}^{\mathsf{finfree}})^{\mathsf{op}} \to \mathsf{C}$ preserving finite products, where $\mathsf{Rng}^{\mathsf{finfree}}$ is the category of integer polynomial rings $\mathbb{Z}[x_1, \ldots, x_n]$ in finitely many variables. For example, the real line $\mathbb{R}$ (with its usual addition and multiplication structure) is a ring object in the category of smooth manifolds. Given a derived smooth manifold X, the space of maps $X \to \mathbb{R}$ may be upgraded to a ring object in Spc by composing the functor $(\mathsf{Rng}^{\mathsf{finfree}})^{\mathsf{op}} \to \mathsf{Sm}$ representing the ring object $\mathbb{R} \in \mathsf{Rng}_{\mathsf{Sm}}$ with $\operatorname{Hom}(X, -) : \mathsf{Sm} \to \mathsf{Spc}$ (which preserves all limits, in particular finite products). In fact, $\operatorname{Hom}(-, \mathbb{R}) : \mathcal{D}\mathsf{Sm}^{\mathsf{op}} \to \mathsf{Spc}$ may be upgraded to a presheaf of ring objects by the following composition.

$$\mathcal{D}\mathsf{Sm}^{\mathsf{op}} \xrightarrow{\times \mathbb{R}} \mathcal{D}\mathsf{Sm}^{\mathsf{op}} \times \mathsf{Rng}_{\mathsf{Sm}} \xrightarrow{\operatorname{Hom}(-,-)} \mathsf{Rng}_{\mathsf{Spc}} \tag{2.9.26.1}$$

This presheaf is a sheaf (since the forgetful functor $\mathsf{Rng}_{\mathsf{Spc}} \to \mathsf{Spc}$ reflects limits (**??**)).

Tangent complexes

⋆ **2.9.27 Definition** (Tangent complex)**.** The tangent complex functor on derived smooth manifolds is a section $T : \mathcal{D}\mathsf{Sm} \to \mathsf{Perf}^{\geq 0} \rtimes \mathcal{D}\mathsf{Sm}$ of the cartesian functor $\mathsf{Perf}^{\geq 0} \rtimes \mathcal{D}\mathsf{Sm} \to \mathcal{D}\mathsf{Sm}$ encoding the functor $\mathsf{Perf}^{\geq 0} : \mathcal{D}\mathsf{Sm} \to \mathsf{Cat}_\infty$. In other words, it assigns to each derived smooth manifold X a perfect complex $TX \in \mathsf{Perf}^{\geq 0}(X)$, to each morphism of derived smooth manifolds $f : X \to Y$ a morphism $TX \to f^*TY$, and coherent homotopies for every chain of morphisms $X_0 \to \cdots \to X_p$ with $p \geq 2$.

The tangent complex functor is defined uniquely up to contractible choice by the requirement that it preserve finite limits and that the following diagram commute.

$$\begin{tikzcd} \mathsf{Vect} \rtimes \mathsf{Sm} \arrow[r] \arrow[d, shift left] & \mathsf{Perf}^{\geq 0} \rtimes \mathcal{D}\mathsf{Sm} \arrow[d, shift left] \\ \mathsf{Sm} \arrow[r] \arrow[u, bend left, "T"] & \mathcal{D}\mathsf{Sm} \arrow[u, bend left, "T"] \end{tikzcd} \tag{2.9.27.1}$$

In other words, the tangent functor on derived smooth manifolds preserves finite limits and solves the following lifting problem.

$$\begin{tikzcd} \mathsf{Sm} \arrow[r] \arrow[d] & \mathsf{Perf}^{\geq 0} \rtimes \mathcal{D}\mathsf{Sm} \arrow[d] \\ \mathcal{D}\mathsf{Sm} \arrow[r, equal] \arrow[ur, dashed] & \mathcal{D}\mathsf{Sm} \end{tikzcd} \tag{2.9.27.2}$$

The tangent bundle functor on smooth manifolds $T : \mathsf{Sm} \to \mathsf{Vect} \rtimes \mathsf{Sm}$ (2.4.10) preserves finite products, as do the inclusions $\mathsf{Vect} \rtimes \mathsf{Sm} \hookrightarrow \mathsf{Vect} \rtimes \mathcal{D}\mathsf{Sm} \hookrightarrow \mathsf{Perf}^{\geq 0} \rtimes \mathcal{D}\mathsf{Sm}$, hence so does their composition $\mathsf{Sm} \to \mathsf{Perf}^{\geq 0} \rtimes \mathcal{D}\mathsf{Sm}$. Now the universal property of $\mathsf{Sm} \hookrightarrow \mathcal{D}\mathsf{Sm}$ (2.8.88), namely that it freely adjoins finite limits modulo preserving finite products within the realm of perfect topological sites, implies the space of lifts (2.9.27.2) is contractible.

Concretely, the tangent complex of a derived smooth manifold X may be described as follows. Suppose X is the limit of a finite diagram $p : K \to \mathsf{Sm}$ of smooth manifolds. The pair $(TX, X) \in \mathsf{Perf}^{\geq 0} \rtimes \mathcal{D}\mathsf{Sm}$ is then the limit of $Tp : K \to \mathsf{Vect} \rtimes \mathsf{Sm} \subseteq \mathsf{Perf}^{\geq 0} \rtimes \mathcal{D}\mathsf{Sm}$. This limit may be computed by first taking the limit in $\mathcal{D}\mathsf{Sm}$ and then taking the relative limit (1.4.153) in $\mathsf{Perf}^{\geq 0} \rtimes \mathcal{D}\mathsf{Sm} \to \mathcal{D}\mathsf{Sm}$, which in this case is the limit in a fiber (1.4.154). Thus $TX \in \mathsf{Perf}^{\geq 0}(X)$ is the limit of the diagram $K \to \mathsf{Vect}(X) \subseteq \mathsf{Perf}^{\geq 0}(X)$ obtained by pulling back the diagram Tp to X.

2.9.28 Example. Consider a finite diagram of smooth manifolds $p : K \to \mathsf{Sm}$, and consider its derived limit $\lim_K^{\mathcal{D}\mathsf{Sm}} p$. The fiber of the tangent complex of $\lim_K^{\mathcal{D}\mathsf{Sm}} p$ at a point x is the limit $\lim_K^{\mathsf{K}^{\geq 0}(\mathsf{Vect})} T_x p$ (since the 'fiber at x' functor $\mathsf{Perf}^{\geq 0}(X) \to \mathsf{Perf}^{\geq 0}(*) = \mathsf{K}^{\geq 0}(\mathsf{Vect})$ preserves finite limits (**??**)). Recall that the diagram p is called transverse at x precisely when this limit lies in $\mathsf{Vect} \subseteq \mathsf{K}^{\geq 0}(\mathsf{Vect})$ (2.9.8). Thus if p is not transverse, then the tangent complex of $\lim_K^{\mathcal{D}\mathsf{Sm}} p$ is not concentrated in degree zero, and so the derived limit $\lim_K^{\mathcal{D}\mathsf{Sm}} p$ is not a smooth manifold. Thus the result that $\mathsf{Sm} \to \mathcal{D}\mathsf{Sm}$ preserves transverse limits (2.9.22) is sharp: if p is non-transverse, then its derived limit does not lie in the full subcategory $\mathsf{Sm} \subseteq \mathcal{D}\mathsf{Sm}$.

2.9.29 Exercise. Let $X = s^{-1}(0)$ be the derived zero set of a section $s : M \to E$ of a vector bundle E over a smooth manifold M. The map of vector bundles $ds : TM \to E$ on M depends on a choice of connection on E. Fixing any choice of connection, show that the cone of this map, restricted to X, is the tangent complex TX.

2.9.30 Exercise. Show that for any point x of a derived smooth manifold X, there exists a function $(X, x) \to (\mathbb{R}, 0)$ with any prescribed derivative $T_x^0 X \to \mathbb{R}$ at x.

2.9.31 Definition (Relative tangent complex)**.** For any map of derived smooth manifolds $f : X \to Y$, the relative tangent complex $T_{X/Y}$ is the fiber product

$$\begin{tikzcd} T_{X/Y} \arrow[r] \arrow[d] & TX \arrow[d, "Tf"] \\ 0 \arrow[r] & f^*TY \end{tikzcd} \tag{2.9.31.1}$$

in $\mathsf{Perf}^{\geq 0}(X)$ (in other words, it is the cone $T_{X/Y} = [TX \to f^*TY[-1]]$).

The utility of the tangent complex ultimately comes down to the next result asserting that 'infinitesimal behavior determines local behavior' (to a certain extent at least). This is a generalization of the inverse function theorem for smooth manifolds and its consequent local normal form results (2.4.4)(2.4.5)(2.4.6).

⋆ **2.9.32 Proposition** (Minimal amplitude factorization)**.** *Every map of derived smooth manifolds $X \to Y$ is, locally near any point $x \in X$, a finite composition $X = Z_N \to \cdots \to Z_0 \to Z_{-1} = Y$ in which $Z_i \to Z_{i-1}$ is locally a pullback of the ith diagonal of $T_x^i(X/Y)$.*

Proof. Recall the argument of (2.9.16), which showed that presenting our input map $X \to Y$ by a submersive and Reedy submersive map of cosimplicial smooth manifolds $X^\bullet \to Y^\bullet$

(such a presentation always exists) gives rise to a factorization in which $Z_i \to Z_{i-1}$ is a pullback of the ith diagonal of (the vertical tangent space of) the ith relative matching map $X^i \to M^iX^\bullet \times_{M^iY^\bullet} Y^i$. The key to the present result is the fact that every Reedy submersive presentation $X^\bullet \to Y^\bullet$ can be refined to one which is *minimal* at x (2.9.21), which we recall means that the cosimplicial vector space $T_x(X^\bullet/Y^\bullet)$ maps under the Dold–Kan correspondence to a cochain complex with vanishing differential.

It suffices therefore to match the vertical tangent space of the ith relative matching map of $X^\bullet \to Y^\bullet$ at x with $T_x^i(X/Y)$ when $X^\bullet \to Y^\bullet$ is minimal at x; this is now simply a matter of unwinding definitions. The vertical tangent space of the ith matching map of $X^\bullet \to Y^\bullet$ is the kernel of the ith matching map of $T_xX^\bullet \to T_xY^\bullet$ (the pullbacks involved in the construction (2.9.14)(1.2.18) of the matching map of $X^\bullet \to Y^\bullet$ are all submersive, so they are preserved by passing to the tangent space at x). The kernel of the ith matching map of $T_xX^\bullet \to T_xY^\bullet$ is in turn identified (1.2.26) with the kernel of the map on normalized cochain complexes $N^\bullet T_xX^\bullet \to N^\bullet T_xY^\bullet$ in degree i. Now $X^\bullet \to Y^\bullet$ is levelwise submersive, so $N^\bullet T_xX^\bullet \to N^\bullet T_xY^\bullet$ is degreewise surjective (1.2.24), so its kernel is its fiber in $\mathsf{K}^{\geq 0}(\mathsf{Vect})$. Since $X^\bullet \to Y^\bullet$ is minimal, the kernel of $N^\bullet T_xX^\bullet \to N^\bullet T_xY^\bullet$ has vanishing differential, so the space in question is thus $H^i(N^\bullet T_xX^\bullet \to N^\bullet T_xY^\bullet[-1])$. The normalized cochain complex of a cosimplicial vector space is the same as its limit in $\mathsf{K}^{\geq 0}(\mathsf{Vect})$ (**??**), so this is the same as the ith cohomology of $[\lim_{\boldsymbol{\Delta}} T_xX^\bullet \to \lim_{\boldsymbol{\Delta}} T_xY^\bullet[-1]] = T_x(X/Y)$ as desired. $\square$

⋆ **2.9.33 Definition** (Derived Lie group)**.** A *derived Lie group* is a group object (1.1.128) in the ∞-category of derived smooth manifolds $\mathcal{D}\mathsf{Sm}$.

A derived Lie group is Hausdorff and paracompact for the same reason as a Lie group (2.4.13).

2.9.34 Exercise. Let G be a derived Lie group. Identify TG with the pullback along the diagonal map $G \to G \times G$ of the relative tangent bundle of the first projection map $G \times G \to G$. Conclude from the pullback diagram (1.1.129) and compatibility of the relative tangent bundle with pullback (**??**) that TG is pulled back from $*$. Conclude that TG is a direct sum of shifts of vector bundles, hence satisfies the structure result (**??**).

⋆ **2.9.35 Definition** (Universal tangent vector τ)**.** We denote by τ the derived zero set of the function x^2, namely the fiber product

$$\begin{tikzcd} \tau \arrow[r] \arrow[d] & * \arrow[d, "0"] \\ \mathbb{R} \arrow[r, "x \mapsto x^2"] & \mathbb{R} \end{tikzcd} \tag{2.9.35.1}$$

in the category $\mathcal{D}\mathsf{Sm}$.

2.9.36 Proposition. *Let M be a smooth manifold, and consider the derived zero set $s^{-1}(0) \in \mathcal{D}\mathsf{Sm}$ of a function $s : M \to \mathbb{R}$ whose (topological) zero set has empty interior. The sheaf $C^\infty_{s^{-1}(0)}$ of real valued functions on $s^{-1}(0)$ is the quotient (as a sheaf of sets) C^∞_M/s of the sheaf C^∞_M of real valued functions on M by the equivalence relation of equality modulo s.*

Proof. We present the derived zero set $s^{-1}(0)$ as a cosimplicial smooth manifold as follows. The point $*$ is the limit of the cosimplicial smooth manifold $C_0^{\mathrm{cell}}(\Delta^\bullet)$ (2.9.18), and the augmentation map $C_0^{\mathrm{cell}}(\Delta^\bullet) \to \mathbb{R}$ is surjective, hence submersive, by inspection. Hence

$$M \times_{\mathbb{R}} 0 = M \times_{\mathbb{R}} \lim_{\mathbf{\Delta}} C_0^{\mathrm{cell}}(\Delta^\bullet) = \lim_{\mathbf{\Delta}}(M \times_{\mathbb{R}} C_0^{\mathrm{cell}}(\Delta^\bullet)) \tag{2.9.36.1}$$

is a cosimplicial smooth manifold presenting the derived zero set $s^{-1}(0)$. It is truncated since $C_0^{\mathrm{cell}}(\Delta^\bullet)$ is truncated. Formation of the sheaf of smooth functions commutes with totalizations of truncated cosimplicial objects (by axiom (**??**) of derived smooth manifolds), so we obtain

$$C^\infty_{s^{-1}(0)} = \operatorname*{colim}_{\mathbf{\Delta}} C^\infty_{M\times_{\mathbb{R}} C_0^{\mathrm{cell}}(\Delta^\bullet)}|_{s^{-1}(0)}. \tag{2.9.36.2}$$

Recall that the colimit functor $\operatorname{colim}_{\mathbf{\Delta}} : \mathsf{sSet} \to \mathsf{Spc}$ simply amounts to regarding a simplicial set as a space in the obvious way (**??**).

Explicitly, the cosimplicial smooth manifold $M \times_{\mathbb{R}} C_0^{\mathrm{cell}}(\Delta^\bullet)$ is as follows.

$$M \xrightarrow{\substack{(p,s(p))\\(p,0)}} M\times\mathbb{R} \xrightarrow{\substack{(p,s(p),x)\\(p,x,x)\\(p,x,0)}} M\times\mathbb{R}^2 \xrightarrow{\substack{(p,s(p),x,y)\\(p,x,x,y)\\(p,x,y,y)\\(p,x,y,0)}} M\times\mathbb{R}^3 \xrightarrow{\substack{(p,s(p),x,y,z)\\(p,x,x,y,z)\\(p,x,y,y,z)\\(p,x,y,z,z)\\(p,x,y,z,0)}} \cdots \tag{2.9.36.3}$$

Every function on $M \times \mathbb{R}$ is uniquely of the form $f(p) + x(g(p) + (x - s(p))h(p, x)))$ by Hadamard's Lemma (2.4.22) (applied twice). Such a function pulls back under the maps $M \rightrightarrows M \times \mathbb{R}$ to a pair of functions on M of the form $(f(p), f(p) + s(p)g(p))$. Thus two elements of $C^\infty(M)$ are joined by an edge in $C^\infty(M \times_{\mathbb{R}} C_0^{\mathrm{cell}}(\Delta^\bullet))$ iff they are congruent modulo s. We conclude that $\pi_0 C^\infty(M \times_{\mathbb{R}} C_0^{\mathrm{cell}}(\Delta^\bullet)) = C^\infty(M)/s$.

Now it remains to show that $\pi_k C^\infty(M \times_{\mathbb{R}} C_0^{\mathrm{cell}}(\Delta^\bullet)) = 0$ for $k > 0$. This is a simplicial abelian group, hence a Kan complex (1.3.15), so it suffices to show that every map $(\Delta^k, \partial\Delta^k) \to (C^\infty(M \times_{\mathbb{R}} C_0^{\mathrm{cell}}(\Delta^\bullet)), 0)$ for $k > 0$ is null-homotopic. Such a map consists of a function $F : M \times \mathbb{R}^k \to \mathbb{R}$ whose the restrictions

$$F(p, s(p), y_1, \ldots, y_{k-1}) \tag{2.9.36.4}$$
$$F(p, y_1, y_1, \ldots, y_{k-1}) \tag{2.9.36.5}$$
$$\vdots$$
$$F(p, y_1, \ldots, y_i, y_i, \ldots, y_{k-1}) \tag{2.9.36.6}$$
$$\vdots$$
$$F(p, y_1, \ldots, y_{k-1}, y_{k-1}) \tag{2.9.36.7}$$
$$F(p, y_1, \ldots, y_{k-1}, 0) \tag{2.9.36.8}$$

all vanish. By Hadamard's Lemma (2.4.22), the last of these vanishing conditions implies that $F(p, x_1, \ldots, x_k) = x_k G(p, x_1, \ldots, x_k)$ for some smooth G. Now consider the $(k+1)$-simplex given by $(z_k - z_{k+1})G(p, z_1, \ldots, z_k)$. Its face $z_{k+1} = 0$ is our given simplex Δ^k, so it suffices to show that all other faces are zero. That is, we should show that

$$G(p, s(p), y_1, \ldots, y_{k-1}) \tag{2.9.36.9}$$
$$G(p, y_1, y_1, \ldots, y_{k-1}) \tag{2.9.36.10}$$
$$\vdots$$
$$G(p, y_1, \ldots, y_i, y_i, \ldots, y_{k-1}) \tag{2.9.36.11}$$
$$\vdots$$
$$G(p, y_1, \ldots, y_{k-1}, y_{k-1}) \tag{2.9.36.12}$$

all vanish. These are the same pullbacks which are known to annihilate F. The difference is that now we have divided by x_k, so it suffices to show that the inverse image of the locus $\{x_k = 0\}$ along all such pullbacks is nowhere dense. When $k > 1$, every such inverse image is simply $\{y_{k-1} = 0\}$, which is evidently nowhere dense. For $k = 1$, the inverse image is $s^{-1}(0) \subseteq M$, which is nowhere dense by hypothesis. □

2.9.37 Exercise. Deduce from (2.9.36) (formally, without repeating the proof) a similar characterization of the sheaf of maps to any smooth manifold N (use Hadamard's Lemma (2.4.22) to show that there is a well defined notion of 'equality modulo s' for maps to N, independent of the choice of coordinate charts of N used to define it).

⋆ **2.9.38 Proposition.** *The functor* $\underline{\mathrm{Hom}}(\tau, -) : \mathcal{D}\mathsf{Sm} \to \mathcal{D}\mathsf{Sm}$ *exists and is canonically isomorphic to the tangent functor* $T : \mathcal{D}\mathsf{Sm} \to \mathcal{D}\mathsf{Sm}$.

Proof. For smooth manifolds M and N, the sheaf of functions $M \times \tau \to N$ is computed in (2.9.36) to equal the sheaf of functions $M \times \mathbb{R} \to N$ modulo x^2 (indeed, $M \times \tau \to M \times \mathbb{R}$ is the derived zero set of the function $(p, x) \mapsto x^2$ and has empty interior). The sheaf of functions $M \times \mathbb{R} \to N$ modulo x^2 is in turn identified with the sheaf of functions $M \to TN$ by taking derivative in the x direction by Hadamard's Lemma (2.4.22). These identifications are functorial in M, hence exhibit TN as representing the functor $\underline{\mathrm{Hom}}(\tau, N)$ on Sm. They are also functorial in N, hence define in fact an isomorphism $\underline{\mathrm{Hom}}(\tau, -) = T$ of functors $\mathsf{Sm} \to \mathsf{Sm}$.

Now the $\underline{\mathrm{Hom}}(\tau, -)$ and $\mathcal{Y}_{\mathcal{D}\mathsf{Sm}} \circ T$ are both topological functors $\mathcal{D}\mathsf{Sm} \to \mathsf{Shv}(\mathcal{D}\mathsf{Sm})$ which preserve finite limits. The isomorphism between their restrictions to Sm extends uniquely to $\mathcal{D}\mathsf{Sm}$ by the universal property of $\mathsf{Sm} \to \mathcal{D}\mathsf{Sm}$ (**??**). □

⋆ **2.9.39 Definition** (Tangent functor $T : \mathsf{Shv}(\mathcal{D}\mathsf{Sm}) \to \mathsf{Shv}(\mathcal{D}\mathsf{Sm})$)**.** The tangent functor on derived smooth stacks $T : \mathsf{Shv}(\mathcal{D}\mathsf{Sm}) \to \mathsf{Shv}(\mathcal{D}\mathsf{Sm})$ is the unique cocontinuous extension (2.8.33) of the tangent functor on derived smooth manifolds $T : \mathcal{D}\mathsf{Sm} \to \mathcal{D}\mathsf{Sm}$ (2.9.27).

$$\begin{tikzcd} \mathcal{D}\mathsf{Sm} \arrow[r, "T_{\mathcal{D}\mathsf{Sm}}"] \arrow[d] & \mathcal{D}\mathsf{Sm} \arrow[d] \\ \mathsf{Shv}(\mathcal{D}\mathsf{Sm}) \arrow[r, dashed, "T_{\mathsf{Shv}(\mathcal{D}\mathsf{Sm})}"] & \mathsf{Shv}(\mathcal{D}\mathsf{Sm}) \end{tikzcd} \tag{2.9.39.1}$$

In other words, $T_{\mathsf{Shv}(\mathcal{D}\mathsf{Sm})}$ is the left Kan extension $(T_{\mathcal{D}\mathsf{Sm}})_!$ of $T_{\mathcal{D}\mathsf{Sm}} : \mathcal{D}\mathsf{Sm} \to \mathcal{D}\mathsf{Sm}$ (2.8.36). Alternatively, $T_{\mathsf{Shv}(\mathcal{D}\mathsf{Sm})}$ is the sheaf pullback functor $(- \times \tau)^* : \mathsf{Shv}(\mathcal{D}\mathsf{Sm}) \to \mathsf{Shv}(\mathcal{D}\mathsf{Sm})$ under multiplication by the universal tangent vector $\times\tau : \mathcal{D}\mathsf{Sm} \to \mathcal{D}\mathsf{Sm}$ (2.9.35). Indeed, the sheaf pullback functor $(- \times \tau)^*$ is cocontinuous since $\times\tau$ preserves underlying topological spaces (2.8.39), and its restriction to $\mathcal{D}\mathsf{Sm} \subseteq \mathsf{Shv}(\mathcal{D}\mathsf{Sm})$ is canonically identified with $T : \mathcal{D}\mathsf{Sm} \to \mathcal{D}\mathsf{Sm}$ (2.9.38). This description shows that $T_{\mathsf{Shv}(\mathcal{D}\mathsf{Sm})}$ is continuous (every sheaf pullback functor is continuous (2.8.36)).

2.9.40 Exercise (Tangent space of $\underline{\mathrm{Sec}}$)**.** Recall that $T_{Q/M}$ is the pullback of $TQ \to TM$ under the zero section $M \to TM$. Conclude that a map $Z \to T_{Q/M}$ from a derived smooth manifold Z is a diagram of the following shape.

$$\begin{tikzcd} Z \times \tau \arrow[r] \arrow[d, "p_Z"'] & Q \arrow[d] \\ Z \arrow[r] & M \end{tikzcd} \tag{2.9.40.1}$$

Conclude that a map $Z \to \underline{\mathrm{Sec}}(M, T_{Q/M})$ is the same as a diagram

$$\begin{tikzcd} Z \times M \times \tau \arrow[r] \arrow[d, "p_{Z\times M}"'] & Q \arrow[d] \\ Z \times M \arrow[r, "p_M"] & M \end{tikzcd} \tag{2.9.40.2}$$

which in turn is the same as a map $Z \times \tau \to \underline{\mathrm{Sec}}(M, Q)$, thus identifying $T\underline{\mathrm{Sec}}(M, Q) = \underline{\mathrm{Sec}}(M, T_{Q/M})$. Generalize this argument to show that $T(\underline{\mathrm{Sec}}_B(M, Q)/B) = \underline{\mathrm{Sec}}_B(M, T_{Q/M})$.

Left Kan extension from smooth manifolds

We now introduce a powerful technique which can be used to formally deduce statements about derived smooth manifolds from the special case of smooth manifolds. The underlying engine behind this technique is the left Kan extension functor $\mathsf{Shv}(\mathsf{Sm}) \hookrightarrow \mathsf{Shv}(\mathcal{D}\mathsf{Sm})$ (fully faithful since $\mathsf{Sm} \hookrightarrow \mathcal{D}\mathsf{Sm}$ is fully faithful (2.8.40)) and, crucially, the characterization of its essential image (2.8.89). Stated informally, a sheaf $F : \mathcal{D}\mathsf{Sm}^{\mathsf{op}} \to \mathsf{Spc}$ is left Kan extended from Sm when every morphism $Q \to F$ from a derived smooth manifold Q factors, uniquely up to contractible choice, through a smooth manifold $Q \to M \to F$. Formulated in this way, it is not so surprising that certain $F \in \mathsf{Shv}(\mathcal{D}\mathsf{Sm})$ being left Kan extended from Sm allows us to deduce results about derived smooth manifolds from the special case of smooth manifolds.

⋆ **2.9.41 Proposition.** *Let $E \to X$ be a submersion of derived smooth manifolds. If X is compact Hausdorff, then the derived smooth stack $\underline{\mathrm{Sec}}(X, E)$ is left Kan extended from* Sm*.*

This result is ultimately a reflection of the fact that smooth manifolds have bump functions (and, indeed, it fails for derived complex analytic spaces).

Proof. Left Kan extension along $\mathsf{Sm} \to \mathcal{D}\mathsf{Sm}$ is a fully faithful strict topological functor (2.8.40) of perfect topological ∞-sites (2.8.46). It follows that being in the image of left Kan extension is a local property of objects of $\mathsf{Shv}(\mathcal{D}\mathsf{Sm})$ (2.8.45). Since X is paracompact Hausdorff, the stack $\underline{\mathrm{Sec}}(X,E)$ has an open cover by stacks of sections of vector bundles (2.4.19). It therefore suffices to treat the case that $E \to X$ is a vector bundle.

To show that $\underline{\mathrm{Sec}}(X,E)$ is left Kan extended from Sm, we appeal to the criterion (2.8.89), according to which it suffices to check that $\underline{\mathrm{Sec}}(X,E)$ topologically preserves finite cosifted limits. So, fix a finite cosifted limit diagram $p : K^{\triangleleft} \to \mathcal{D}\mathsf{Sm}$, and let us show that its composition with $\underline{\mathrm{Sec}}(X,E) : \mathcal{D}\mathsf{Sm} \to \mathsf{Shv}^{\mathsf{op}} \rtimes \mathsf{Top}$ remains a limit diagram. The composition $\underline{\mathrm{Sec}}(X,E)(p) : K^{\triangleleft} \to \mathsf{Shv}^{\mathsf{op}} \rtimes \mathsf{Top}$ is the pushforward along $|X \times p \to p|$ of $\mathrm{Sec}(\mathsf{Open}(X \times p), E) : K^{\triangleleft} \to \mathsf{Shv}^{\mathsf{op}} \rtimes \mathsf{Top}$.

$$\underline{\mathrm{Sec}}(X,E)(p) = |X \times p \to p|_* \mathrm{Sec}(\mathsf{Open}(X \times p), E) \tag{2.9.41.1}$$

This diagram $\mathrm{Sec}(\mathsf{Open}(X \times p), E)$ is a relative limit diagram: this assertion is local on X, hence we may assume E is trivial, so $\mathrm{Sec}(\mathsf{Open}(X \times p), E) = \mathrm{Hom}(\mathsf{Open}(X \times p), \mathbb{R}^n)$ is the Yoneda functor of a smooth manifold, hence is a relative limit diagram (**??**). Now we would like to show that pushforward along $|X \times p \to p|$ sends this relative limit diagram $\mathrm{Sec}(\mathsf{Open}(X \times p), E)$ to a relative limit diagram. The morphism $|X \times p \to p|$ is a proper map of diagrams of locally compact Hausdorff spaces which sends edges to pullbacks, so by proper base change (2.2.35) the pushforward of a relative limit diagram is a relative diagram iff the pushforward at the basepoint $|X \times p(*) \to p(*)|$ sends its cartesian transport (1.4.154) (which is also a limit diagram) to a limit diagram. Proper pushforward preserves k-acyclic colimits (2.2.39)(2.2.40), so it suffices to show that the cartesian transport

$$\mathrm{Sec}(\mathsf{Open}(X \times p), E)|_{X \times p(*)} : (K^{\triangleleft})^{\mathsf{op}} \to \mathsf{Shv}(X \times p(*)) \tag{2.9.41.2}$$

is k-acyclic (just locally on $p(*)$ is enough). To show that the cartesian transport (2.9.41.2) is k-acyclic, we appeal to (2.2.41) according to which an ∞-sifted colimit of ring-module sheaves on a locally compact Hausdorff space is k-acyclic provided at least one of its constituent ring sheaves admits compact partitions of unity (note that $p(*)$ is only locally compact *locally* Hausdorff, so (2.2.41) only applies locally on $p(*)$, but this is enough). Specifically, we apply this result to the colimit diagram of ring-module sheaves

$$(\mathrm{Hom}(\mathsf{Open}(X \times p), \mathbb{R})|_{X \times p(*)}, \mathrm{Sec}(\mathsf{Open}(X \times p), E)|_{X \times p(*)}) \tag{2.9.41.3}$$

(note that the diagram of ring sheaves is the special case $E = \underline{\mathbb{R}}$ of the diagram of module sheaves, hence is indeed also a colimit diagram). It thus suffices to show that $\mathrm{Hom}(\mathsf{Open}(X \times p(k)), \mathbb{R})|_{X \times p(*)}$ admits compact partitions of unity for some $k \in K$. It suffices to show that $\mathrm{Hom}(\mathsf{Open}(X \times p(k)), \mathbb{R})|_{X \times p(*)}$ has bump functions. Any derived smooth manifold, in this case $X \times p(k)$, has *local* bump functions (2.9.12), which, if $X \times p(*) \to X \times p(k)$ is an embedding, pull back to local bump functions on $X \times p(*)$, which then extend by zero to true bump functions provided $p(*)$ (hence also $X \times p(*)$) is Hausdorff (and we may assume $p(*)$ is Hausdorff since we are permitted to work locally on $p(*)$ as already noted). It thus remains to ensure that $X \times p(*) \to X \times p(k)$ is an embedding for some k. To see that this is always possible, note that every finite cosifted formal limit may be realized by a cosimplicial object (1.4.242) and take $k = [0] \in \mathbf{\Delta} = K$. □

2.9.42 Corollary. *The stack of proper submersions on* $\mathcal{D}\mathsf{Sm}$ *is left Kan extended from* Sm.

Proof. Proper submersions in Sm and $\mathcal{D}\mathsf{Sm}$ are locally trivial (2.4.17)(**??**). It follows that the stack of proper submersions (on both Sm and $\mathcal{D}\mathsf{Sm}$) is the disjoint union over all diffeomorphism classes of compact Hausdorff smooth manifolds F of the stack quotient

$$*/\underline{\mathrm{Diff}}(F) = \mathrm{colim}\Big(\cdots \substack{\to\\\to\\\to} \underline{\mathrm{Diff}}(F)^2 \rightrightarrows \underline{\mathrm{Diff}}(F) \to *\Big) \tag{2.9.42.1}$$

where $\underline{\mathrm{Diff}}(F) \subseteq \underline{\mathrm{Hom}}(F, F)$ denotes the open substack of diffeomorphisms. Left Kan extension preserves $\underline{\mathrm{Hom}}(F, F)$ (2.9.41), open substacks (2.8.36), finite products (2.8.41), and colimits (since it is a left adjoint). □

2.9.43 Proposition. *Let $W \to C \to B$ be submersions of smooth manifolds. If $C \to B$ is proper, then the derived smooth stack $\underline{\mathrm{Sec}}_B(C, W)$ over B is left Kan extended from* $(\mathsf{Sm} \downarrow^{\mathsf{subm}} B)$.

Proof. We follow closely the argument for the case $B = *$ given earlier (2.9.41).

The functor $(\mathsf{Sm} \downarrow^{\mathsf{subm}} B) \to (\mathcal{D}\mathsf{Sm} \downarrow B)$ exhibits the latter as the derived site of the former (**??**). Thus by left Kan extension criterion (2.8.89), it suffices to show that $\underline{\mathrm{Sec}}_B(C, W)$ topologically preserves finite cosifted limits in $(\mathcal{D}\mathsf{Sm} \downarrow B)$.

Being left Kan extended is a local property (2.8.40)(2.8.46)(2.8.45). Thus by the local structure of $\underline{\mathrm{Sec}}_B(C, W)$ (2.4.20), we may assume wlog that $W \to C$ is a vector bundle.

Now fix a finite cosifted limit diagram $p : K^\triangleleft \to (\mathcal{D}\mathsf{Sm} \downarrow B)$, and let us show that its composition with $\underline{\mathrm{Sec}}_B(C, W) : \mathcal{D}\mathsf{Sm} \to \mathsf{Shv}^{\mathsf{op}} \rtimes \mathsf{Top}$ remains a limit diagram. The composition $\underline{\mathrm{Sec}}_B(C, W)(p) : K^\triangleleft \to \mathsf{Shv}^{\mathsf{op}} \rtimes \mathsf{Top}$ is the pushforward along $|C \times_B p \to p|$ of $\mathrm{Sec}(\mathsf{Open}(C \times_B p), W) : K^\triangleleft \to \mathsf{Shv}^{\mathsf{op}} \rtimes \mathsf{Top}$.

$$\underline{\mathrm{Sec}}_B(C, W)(p) = |C \times_B p \to p|_* \mathrm{Sec}(\mathsf{Open}(C \times_B p), W) \tag{2.9.43.1}$$

This diagram $\mathrm{Sec}(\mathsf{Open}(C \times_B p), W)$ is a relative limit diagram: this assertion is local on C, hence we may assume W is trivial, so $\mathrm{Sec}(\mathsf{Open}(C \times_B p), W) = \mathrm{Hom}(\mathsf{Open}(C \times_B p), \mathbb{R}^n)$ is the Yoneda functor of a smooth manifold, hence is a relative limit diagram (**??**) (note that p being a limit diagram just means that its underlying diagram $K^\triangleleft \to \mathcal{D}\mathsf{Sm}$ is a limit diagram (**??**)). To show that pushforward along $|C \times_B p \to p|$ sends this relative limit diagram $\mathrm{Sec}(\mathsf{Open}(C \times_B p), W)$ to a relative limit diagram, first appeal to proper base change (2.2.35) to see that it is equivalent to show that pushforward along $|C \times_B p(*) \to p(*)|$ sends the colimit diagram

$$\mathrm{Sec}(\mathsf{Open}(C \times_B p), W)|_{C \times_B p(*)} : (K^{\mathsf{op}})^\triangleright \to \mathsf{Shv}(C \times_B p(*)) \tag{2.9.43.2}$$

to a colimit diagram. Proper pushforward preserves k-acyclic colimits (2.2.39)(2.2.40), so it suffices to show that this colimit diagram is k-acyclic (just locally on $p(*)$ is enough).

To show k-acyclicity, we appeal to (2.2.41) according to which it suffices to show that $\mathrm{Hom}(\mathsf{Open}(C \times_B p(k)), \mathbb{R})|_{C\times_B p(*)}$ has bump functions for some $k \in K$. Since we are permitted to work locally on $p(*)$, we may assume that it is Hausdorff, hence so is $C \times_B p(*)$, hence it suffices to show the existence of *local* bump functions. Now $C \times_B p(k)$ (indeed, every derived smooth manifold) has local bump functions (2.9.12), which gives local bump functions in $\mathrm{Hom}(\mathsf{Open}(C \times_B p(k)), \mathbb{R})|_{C\times_B p(*)}$ provided $C \times_B p(*) \to C \times_B p(k)$ is an embedding, which we can ensure by recalling that every finite cosifted formal limit may be realized by a cosimplicial object (1.4.242) and taking $k = [0] \in \mathbf{\Delta} = K$. □

Derived smooth stacks

We now study derived smooth stacks, seeking a theory parallel to that of smooth stacks (2.5).

Here is a generalization of (2.5.14).

2.9.44 Lemma. *Let X be a derived smooth stack, and let $U \to X$ be a submersion from a derived smooth manifold U. For $x \in X$, consider the map $U \times_X x \to U$ (from a smooth manifold to a derived smooth manifold). This map factors locally as the composition of a surjective submersion with vertical tangent space $\ker((T_{U/X})_u \to T^0U)$ and a map from the resulting quotient manifold with tangent space $\mathrm{im}((T_{U/X})_u \to T^0U)$ to U acting on tangent spaces via the tautological inclusion.*

Proof. The relative tangent bundle of the map $U \times_X x \to U$ is the cone of $T_{U/X} \to TU$ and is identified with (the pullback of) the relative tangent bundle of $* \to X$, namely the constant bundle with fiber $T_xX[-1]$. It follows that the rank of the tangent cohomology T^iU is constant over $U \times_X x$ as is the rank of the map $(T_{U/X})_u = T(U \times_X u) \to T^0U$. Now we can express U (locally) as the limit of a cosimplicial smooth manifold $U^\bullet$, and the functor of maps from smooth manifolds to U is (by the universal property of limits) the functor of maps to U^0 which land inside $|U| \subseteq |U^0|$. The structure theorem for maps of smooth manifolds with constant rank derivative (2.4.4) thus applies to the map $U \times_X x \to U$. □

2.9.45 Definition (Minimal submersion)**.** Given a derived smooth stack X, a submersion $U \to X$ from $U \in \mathcal{D}\mathsf{Sm}$ is called *minimal* at $u \in U$ when the map $T_{U/X} \to TU$ vanishes at u.

2.9.46 Lemma (Existence of a minimal atlas)**.** *Let X be a derived smooth stack, and let $x \in X$ be a point. If X admits an submersive atlas, then it admits a submersive atlas which is minimal at some lift of x.*

Proof. We follow the argument of (2.5.17). Begin with an arbitrary submersive atlas $U \twoheadrightarrow X$ and a lift $u \in U$ of x. If $V \subseteq U$ is the zero set of a map $U \to \mathbb{R}^k$, then $V \to X$ is a submersion iff the relative tangent complex $T_{V/X}$ is supported in degree zero. This relative tangent complex is the cone $[T_{U/X} \to \mathbb{R}^k[-1]]$ of the composition $T_{U/X} \to TU \to \mathbb{R}^k$, so $V \to X$ is a submersion iff this composition is surjective over V. Now the image of the map $T_{U/X} \to TU$ at u is some subspace of T^0_uU. Choose a map $U \to \mathbb{R}^k$ vanishing at u whose derivative at u restricted to this subspace T^0_uU is an isomorphism (2.9.30). Now the resulting submersion $V \to X$ is minimal. □

2.9.47 Lemma (Proper atlas from proper diagonal)**.** *Let X be a derived smooth stack with proper diagonal, and let $U \to X$ be a submersion which is minimal at $p \in U$. For every sufficiently small open neighborhood $p \in V \subseteq U$, we have $p \times_X p = p \times_X V$ and the map $V \to X$ is proper over an open substack of X containing the image of p.*

Proof. We argue as in the case of smooth stacks (2.5.18). Given the purely topological result (2.3.32), it suffices to show that $p \times_X p \subseteq p \times_X U$ is open, which follows from minimality (2.9.44). □

* *

We now generalize Zung's Theorem (2.5.19) to derived smooth stacks.

⋆ **2.9.48 Theorem.** *A derived smooth stack with submersive atlas and proper diagonal is a derived Lie orbifold.*

Proof. We follow the argument used to prove the corresponding result for smooth stacks (2.5.19), emphasizing the differences.

Let X be a derived smooth stack with submersive atlas and proper diagonal, and let $x \in X$ be a point. The automorphism stack $\underline{G} = x \times_X x$ is representable since X has an atlas (2.3.28), so it is a derived Lie group. Let G be the Lie group associated to $\underline{G}$ (2.9.34). Since X has proper diagonal, $\underline{G}$ (hence also G) is compact.

Fix a submersion $U \to X$ from a derived smooth manifold U which is minimal at a lift $u \in U$ of x (2.9.46). By replacing U with an open neighborhood of u, we can ensure that $u \times_X U \to U$ has image $\{u\}$ and that $U \to X$ is proper over an open substack of X containing x (2.9.47).

We have constructed a proper submersion $U \to X$ from a derived smooth manifold U over a neighborhood of x. It suffices to equip it with the structure of a principal G-bundle.

To begin, let us identify the fiber of $U \to X$ over x with G. There is a canonical map $G \to \underline{G} = x \times_X x = u \times_X x \to U \times_X x$, which we claim is an isomorphism of smooth manifolds. Since the image of $U \times_X x \to U$ is $\{u\}$, the map $u \times_X x \to U \times_X x$ is an isomorphism of underlying topological spaces. The map $u \times_X x \to U \times_X x$ is a pullback of $u \to U$, hence has relative tangent tangent complex $T_u U[-1]$, so its action on $H^0 T$ is injective. The composition $G \to U \times_X x$ is thus a homeomorphism of smooth manifolds whose derivative is injective, hence it is an isomorphism.

Now $G = u \times_X U \subseteq U \times_X U$ is a fiber of the proper submersion of derived smooth manifolds $U \times_X U \to U$. Applying Ehresmann (**??**) to $U \times_X U \to U$, we conclude that there exists a retraction $U \times_X U \to u \times_X U = G$ defined in a neighborhood of G. The complement of this neighborhood is a closed subset of $U \times_X U$, hence has closed image in X by properness of $U \to X$. Now the inverse image of x in $U \times_X U$ is G by construction (since the image of $u \times_X U \to U$ is $\{u\}$), so this closed image in X does not contain x. By replacing X with the open complement, we may assume that the map $U \times_X U \to G$ is defined everywhere.

We thus have a proper submersion $U \to X$ together with a map $U \times_X U \to G$ whose restriction to the fiber $U \times_X x$ over $x \in X$ is a diffeomorphism (and a groupoid homomorphism).

Choose arbitrarily a smooth positive section of $|\det T^*_{U/X}|$ over U (this exists since U is Hausdorff as both $U \to X$ and X have proper diagonal).

Recall that the stack of proper submersions over derived smooth manifolds is left Kan extended from smooth manifolds (2.9.42). The same holds (for the same reason) for the stack of proper submersions $E \to B$ equipped with a map $E \times_B E \to G$ and a section of $|\det T^*_{E/B}|$ (**??**) (use the fact that $\underline{\mathrm{Hom}}(F \times F, G)$ and $\underline{\mathrm{Sec}}(F, |\det TF^*|)$ are left Kan extended from smooth manifolds (2.9.41)). The stack of principal G-bundles over derived smooth manifolds is also left Kan extended from smooth manifolds (since it is the colimit $\operatorname{colim}(* \leftleftarrows G \,\substack{\leftarrow\\\leftarrow\\\leftarrow}\, \cdots)$ of smooth manifolds). The left Kan extension of Zung's retraction (2.5.23) thus defines a principal G-bundle structure on $U \to X$ over an open substack of X containing x. □

* *

Artin morphisms have been defined for topological stacks (2.3.33) and for smooth stacks (2.5.24). The definition for smooth stacks involved additional submersivity conditions to ensure stability under pullback. Since derived smooth manifolds have all pullbacks, the definition of Artin morphisms of derived smooth stacks is identical to that for topological stacks.

⋆ **2.9.49 Definition** (n-Artin morphism)**.** A morphism of derived smooth stacks $X \to Y$ is called *n-Artin* (for integers $n \geq 0$) when for every map $U \to Y$ from a derived smooth manifold U, the pullback $X \times_Y U$ admits an $(n-1)$-Artin atlas $W \twoheadrightarrow X \times_Y U$ (this is an inductive definition, the base case being that a morphism is (-1)-Artin iff it is an isomorphism). It is immediate that n-Artin morphisms are preserved under pullback.

The basic results about Artin morphisms of topological stacks (2.3.34)(2.3.35)(**??**)(2.3.36)(2.3.37)(2.3.38) remain valid for derived smooth stacks, with the same proofs.

2.9.50 Lemma. *The left Kan extension functors* $\mathsf{Shv}(\mathsf{Sm}) \to \mathsf{Shv}(\mathcal{D}\mathsf{Sm}) \to \mathsf{Shv}(\mathsf{Top})$ *preserve n-Artin morphisms and pullbacks of n-Artin morphisms.*

Proof. The argument used to treat the case of left Kan extension $\mathsf{Shv}(\mathsf{Sm}) \to \mathsf{Shv}(\mathsf{Top})$ (2.5.28) applies without change. □

2.10 Hybrid categories

In this section, we introduce 'hybrid categories'. The simplest of these is the category we denote by TopSm, whose objects we will call topological-smooth spaces. Topological-smooth spaces are locally modelled on products $Z \times \mathbb{R}^n$ for topological spaces Z. Morphisms of topological-smooth spaces, called continuous-smooth maps, are maps $Z \times \mathbb{R}^n \to Z' \times \mathbb{R}^{n'}$ which locally preserve the decomposition into 'leaves' $z \times \mathbb{R}^n$ and whose derivatives to all orders along the leaves (i.e. in the $\mathbb{R}^n$ coordinate) exist and are continuous. This category allows one to make sense of notions such as 'a family of smooth manifolds parameterized by a topological space'. It also provides a context in which to define topological spaces $\underline{\mathrm{Hom}}(X, Y)$ of smooth maps between smooth manfolds X and Y via a universal property analogous to that used to define the topological spaces $\underline{\mathrm{Hom}}(X, Y)$ of continuous maps between topological spaces X and Y (2.3.39).

Topological-smooth spaces

Let us now introduce the category of topological-smooth spaces TopSm.

2.10.1 Definition (Continuous-smooth map)**.** Let Z be a topological space and let $n \geq 0$. We consider maps defined on the product $Z \times \mathbb{R}^n$ or any open subset thereof.

A map $f : Z \times \mathbb{R}^n \to Z'$ (any topological space Z') will be called *continuous-smooth* when it is, locally on the source, the composition of the projection $Z \times \mathbb{R}^n \to Z$ and a continuous map $Z \to Z'$. A map $f : Z \times \mathbb{R}^n \to \mathbb{R}$ will be called *continuous-smooth* iff its derivative $D^\alpha f : Z \times \mathbb{R}^n \to \mathbb{R}$ exists and is continuous for every multi-index α on $\mathbb{R}^n$. A map $f : Z \times \mathbb{R}^n \to Z' \times \mathbb{R}^{n'}$ will be called *continuous-smooth* when its coordinate factors $Z \times \mathbb{R}^n \to Z'$ and $Z \times \mathbb{R}^n \to \mathbb{R}$ are all continuous-smooth.

The notion of a continuous-smooth map manifestly depends on the expression of its source and target as the product of a topological space and a Euclidean space.

2.10.2 Exercise. Determine what are the continuous-smooth maps

$$|\mathbb{R}^n| \times * \to |\mathbb{R}^n| \times * \qquad\qquad * \times \mathbb{R}^n \to |\mathbb{R}^n| \times * \tag{2.10.2.1}$$

$$|\mathbb{R}^n| \times * \to * \times \mathbb{R}^n \qquad\qquad * \times \mathbb{R}^n \to * \times \mathbb{R}^n \tag{2.10.2.2}$$

where we write $|\mathbb{R}^n|$ to denote the topological space underlying the smooth manifold $\mathbb{R}^n$ (so as to distinguish the topological and smooth factors). Do the same with the domains replaced with arbitrary open subsets thereof.

2.10.3 Lemma. *A composition of continuous-smooth maps is continuous-smooth.*

Proof. We consider a composition $Z \times \mathbb{R}^n \to Z' \times \mathbb{R}^{n'} \to Z'' \times \mathbb{R}^{n''}$. It evidently suffices to consider the case that the target is simply Z'' or $\mathbb{R}$.

The case of the target Z'' is evident: the map $Z' \times \mathbb{R}^{n'} \to Z''$ locally factors through the projection to Z', and the map $Z \times \mathbb{R}^n \to Z'$ locally factors through the projection to Z, so the composition $Z \times \mathbb{R}^n \to Z' \times \mathbb{R}^{n'} \to Z''$ locally factors through the projection to Z.

Now consider a composition $h = f \circ (t; g_1 \ldots, g_{n'})$ with target $\mathbb{R}$.

$$Z \times \mathbb{R}^n \xrightarrow{(t;g_1,\ldots,g_{n'})} Z' \times \mathbb{R}^{n'} \xrightarrow{f} \mathbb{R} \tag{2.10.3.1}$$

By the chain rule, the derivative $D^\alpha h$ is a continuous function of the derivatives of $g_1, \ldots, g_{n'}$ and f. That is, $D^\alpha h$ is the composition of a continuous function with the product of the continuous functions

$$Z \times \mathbb{R}^n \xrightarrow{(t;g_1,\ldots,g_{n'})} Z' \times \mathbb{R}^{n'} \xrightarrow{D^\gamma f} \mathbb{R} \tag{2.10.3.2}$$

$$Z \times \mathbb{R}^n \xrightarrow{D^\beta g_i} \mathbb{R} \tag{2.10.3.3}$$

for multi-indices β and γ. It is thus continuous, as desired. □

⋆ **2.10.4 Definition** (Category of topological-smooth spaces TopSm)**.** A *topological-smooth space* is a topological space X equipped with an atlas of charts (as in (2.4.1)) from open subsets of products $Z \times \mathbb{R}^n$ for topological spaces Z and integers $n \geq 0$, whose transition maps are continuous-smooth. A morphism of topological-smooth spaces is a continuous map of topological spaces which, when viewed via the charts, is continuous-smooth. The category of topological-smooth spaces is denoted TopSm.

There are tautological fully faithful embeddings of the categories of topological spaces Top and smooth manifolds Sm into the category of topological-smooth spaces TopSm.

$$\mathsf{Top} \hookrightarrow \mathsf{TopSm} \hookleftarrow \mathsf{Sm} \tag{2.10.4.1}$$

2.10.5 Exercise. Show that $Z \times \mathbb{R}^n \in \mathsf{TopSm}$ is the categorical product of $Z \in \mathsf{Top} \subseteq \mathsf{SmTop}$ and $\mathbb{R}^n \in \mathsf{Sm} \subseteq \mathsf{SmTop}$.

2.10.6 Exercise. Show that the embedding $\mathsf{Top} \subseteq \mathsf{TopSm}$ has right adjoint given by the underlying topological space functor $|\cdot| : \mathsf{TopSm} \to \mathsf{Top}$ and left adjoint given by the 'collapse leaves' functor $\mathsf{TopSm} \to \mathsf{Top}$.

2.10.7 Exercise (Locally connected)**.** Show that for a topological space X, the following are equivalent (in which case X is called *locally connected*):

(2.10.7.1) Every open subset of X is a disjoint union of connected open subsets.

(2.10.7.2) Every point $x \in X$ has arbitrarily small connected open neighborhoods.

2.10.8 Exercise (Leaf structure)**.** Consider the presheaf on $\mathsf{Top}^{\mathsf{opemb}}$ (topological spaces and open embeddings) defined as follows. To a topological space X we associate the set of equivalence relations on X all of whose equivalence classes are connected and locally connected subspaces of X. Such an equivalence relation on X restricts to an equivalence relation on any open subset $U \subseteq X$ all of whose equivalence classes are locally connected, but not necessarily connected. Splitting each such naively restricted equivalence class into its connected components (2.10.7.1) defines the restriction operation for our presheaf. Show that this presheaf is separated. A section of its sheafification is called a *leaf structure*.

2.10.9 Definition (Submersion)**.** A map of topological-smooth spaces is called a *submersion* when it is locally on the source a pullback of $\mathbb{R}^n \to *$.

Since the category TopSm is a topological site (2.8.2), we can form the category of *topological-smooth stacks* $\mathsf{Shv}(\mathsf{TopSm})$.

Log topological-smooth spaces

Now let us define a category $\mathsf{LogTopSm}$ of log topological-smooth spaces. The local models of such spaces are fiber products $Q \times_B Z$ where $Q \to B$ is a submersion of log smooth manifolds (2.7.59) and Z is an arbitrary log topological space mapping to B. Even the case $Z = *$, but possibly with nontrivial log structure, is interesting, giving the fiber of a submersion of log smooth manifolds over a not necessarily interior point of the base (e.g. the fiber of the multiplication map ${}'\mathbb{R}^2_{\geq 0} \to {}'\mathbb{R}_{\geq 0}$ over zero). Recall that a submersion of log smooth manifolds is locally monomial (2.7.62), so this class of local models is equivalent to $X_Q \times_{X_P} Z$ for injective maps of polyhedral cones $P \to Q$ and maps $Z \to X_P$. In applications, we will only ever need the case that $Q \to B$ is an exact submersion (2.7.72), but this assumption is unnecessary for setting up the general theory (the reader is warned that submersions of log smooth manifolds are not preserved under pullback (**??**), in contrast to exact submersions (2.7.78)).

2.10.10 Definition (Elementary log topological-smooth space)**.** An *elementary log topological-smooth space* is a formal symbol $Q \times_B Z$ where $Q \to B$ is a submersion of log smooth manifolds and $Z \to B$ a map from an arbitrary log topological space Z. Associated to such a formal symbol is an 'underlying log topological space' $|Q \times_B Z|$ given by the indicated fiber product in the category of log topological spaces (though we will frequently drop the $|\cdot|$ symbol).

2.10.11 Definition (Vertical map)**.** A *vertical map* between elementary log topological-smooth spaces $Q \times_B Z \to Q' \times_{B'} Z'$ is given locally by a diagram of the following shape.

$$\begin{tikzcd} {|Q \times_B Z|} \arrow[r] \arrow[d] & {|Q' \times_{B'} Z'|} \arrow[d] \\ Z \arrow[r] & Z' \end{tikzcd} \tag{2.10.11.1}$$

More formally, the sheaf of vertical maps from $Q \times_B Z$ to $Q' \times_{B'} Z'$ is the fiber product $\mathrm{Hom}(-, |Q' \times_{B'} Z'|) \times_{\mathrm{Hom}(-,Z')} p_Z^* \mathrm{Hom}(-, Z')$ where Hom denotes morphisms in LogTop.

2.10.12 Definition (Vertical ghost sheaf)**.** For an elementary log topological-smooth space $Q \times_B Z$, its *vertical ghost sheaf* (compare (2.6.17)) is the pushout

$$\begin{aligned} \mathcal{Z}_{Q\times_B Z/Z} &= \operatorname{colim}\bigl(\{1,0\} \leftarrow \mathcal{Z}_Z \to \mathcal{Z}_{Q\times_B Z}\bigr) && (2.10.12.1)\\ &= \operatorname{colim}\bigl(\{1,0\} \leftarrow \mathcal{Z}_Z \to \mathcal{Z}_Z \leftarrow \mathcal{Z}_B \to \mathcal{Z}_Q\bigr) && (2.10.12.2)\\ &= \operatorname{colim}\bigl(\{1,0\} \leftarrow \mathcal{Z}_B \to \mathcal{Z}_Q\bigr) && (2.10.12.3) \end{aligned}$$

of sheaves of monoids on $Q \times_B Z$. Here $\{1, 0\}$ is a monoid under multiplication and all maps are sharp (a map of monoids is sharp when an element of the domain is invertible iff its image in the target is invertible); note that $\{1, 0\}$ is the terminal object in the category of sharp monoid maps. The description of the vertical ghost sheaf $\mathcal{Z}_{Q\times_B Z/Z}$ as the 'cokernel' of the map $\mathcal{Z}_Z \to \mathcal{Z}_{Q\times_B Z}$ shows that it is functorial under vertical maps (a vertical map $Q \times_B Z \to Q' \times_{B'} Z'$ induces a map $\mathcal{Z}_{Q'\times_{B'} Z'/Z'} \to \mathcal{Z}_{Q\times_B Z/Z}$). The description of the vertical ghost sheaf $\mathcal{Z}_{Q\times_B Z/Z}$ as the 'cokernel' of the map $\mathcal{Z}_B \to \mathcal{Z}_Q$ shows that the map on vertical ghost sheaves associated to a vertical map $Q \times_B Z' \to Q \times_B Z$ associated to maps $Z' \to Z \to B \leftarrow Q$ is an isomorphism.

The stalk of $\mathcal{Z}_{Q\times_B Z/Z}$ at a point $(q, b, z) \in Q \times_B Z$ is given by

$$\mathcal{Z}_{Q\times_B Z/Z,(q,b,z)} = \operatorname{colim}\bigl(\{1, 0\} \leftarrow \mathcal{Z}_{B,b} \to \mathcal{Z}_{Q,q}\bigr) \tag{2.10.12.4}$$

$$= \Bigl(1 \cdot \{a \in \mathcal{Z}_{Q,q} \mid F_a \cap \mathcal{Z}_{B,b} = 0\}\Bigr) \sqcup \Bigl(0 \cdot \mathcal{Z}_{Q,q}/\mathcal{Z}_{B,b}\Bigr) \tag{2.10.12.5}$$

where $F_a \subseteq \mathcal{Z}_{Q,q}$ denotes the minimal face containing the point $a \in \mathcal{Z}_{Q,q}$, we recall that $\mathcal{Z}_{B,b} \to \mathcal{Z}_{Q,q}$ is sharp (though not necessarily injective), and the quotient $\mathcal{Z}_{Q,q}/\mathcal{Z}_{B,b}$ is the image of $\mathcal{Z}_{Q,q}$ in the quotient of vector spaces $(\mathcal{Z}_{Q,q})^{\mathrm{gp}}/(\mathcal{Z}_{B,b})^{\mathrm{gp}}$. Indeed, this follows from inspection upon noting that for a point $a \in \mathcal{Z}_{Q,q}$, the condition $F_a \cap \mathcal{Z}_{B,b} = 0$ is equivalent to $(a - \mathcal{Z}_{B,b}) \cap \mathcal{Z}_{Q,q} = \{a\}$. We will call the $1 \cdot \{\cdots\}$ subset of $\mathcal{Z}_{Q\times_B Z/Z,(q,b,z)}$ (equivalently, the set of elements not divisible by 0) the *essential piece* $\mathcal{Z}^{\mathrm{ess}}_{Q\times_B Z/Z,(q,b,z)} \subseteq \mathcal{Z}_{Q\times_B Z/Z,(q,b,z)}$, which is evidently the union of the faces of $\mathcal{Z}_{Q,q}$ corresponding to the strata of Q near q lying over the stratum $B_b \subseteq B$ (which corresponds to the face $\{0\} \subseteq \mathcal{Z}_{B,b}$) (2.7.23). That is, the faces of $\mathcal{Z}^{\mathrm{ess}}_{Q\times_B Z/Z,(q,b,z)}$ correspond to the strata of the fiber $(Q \times_B Z)_z = Q_b$ near q (note that all strata of Q near q lying over the stratum $B_b \subseteq B$ are submersive over it (2.7.60)). This stratification may be recovered intrinsically by associating to a point $p \in (Q \times_B Z)_z$ nearby (q, b, z) the subset of $\mathcal{Z}_{Q\times_B Z/Z,(q,b,z)}$ whose evaluation at p is positive (this is a subset of the essential piece since everything divisible by 0 evaluates to zero, and it is a face since it satisfies the criterion (2.7.6.1)).

The functoriality of the vertical ghost sheaf now encodes how vertical maps act on strata. Indeed, consider a vertical map $f : Q \times_B Z \to Q' \times_{B'} Z'$ and a point $(q, b, z) \in Q \times_B Z$. For a point $p \in (Q \times_B Z)_z$ nearby (q, b, z), evaluation at $f(p)$ on $\mathcal{Z}_{Q'\times_{B'} Z'/Z',f(q,b,z)}$ is evidently the composition of pullback $f^{\flat\flat}_{(q,b,z)} : \mathcal{Z}_{Q'\times_{B'} Z'/Z',f(q,b,z)} \to \mathcal{Z}_{Q\times_B Z/Z,(q,b,z)}$ and evaluation at p. It follows immediately that f sends the stratum of $(Q \times_B Z)_z$ nearby (q, b, z) corresponding to a face $F \subseteq \mathcal{Z}^{\mathrm{ess}}_{Q\times_B Z/Z,(q,b,z)}$ to the stratum of $(Q' \times_{B'} Z')_{f(z)}$ near $f(q, b, z)$ corresponding to the inverse image $(f^{\flat\flat}_{(q,b,z)})^{-1}(F) \subseteq \mathcal{Z}^{\mathrm{ess}}_{Q'\times_{B'} Z'/Z',f(q,b,z)}$. In particular, points of the same stratum of $(Q \times_B Z)_z$ map to the same stratum of $(Q' \times_{B'} Z')_{f(z)}$.

2.10.13 Lemma. *For an elementary log topological-smooth space $Q \times_B Z$, the sheaf $\mathcal{O}^{\geq 0}_{Q\times_B Z}$*

of maps to $'\mathbb{R}_{\geq 0}$ *is the colimit of the commuting diagram*

$$\begin{tikzcd} & \mathcal{A}_Q^{>0} \arrow[dl] \arrow[r] & \mathcal{A}_Q^{\geq 0} \\ C_{Q\times_B Z}^{>0} & \mathcal{A}_B^{>0} \arrow[l] \arrow[u] \arrow[r] \arrow[d] & \mathcal{A}_B^{\geq 0} \arrow[u] \arrow[d] \\ & C_Z^{>0} = \mathcal{O}_Z^{>0} \arrow[ul] \arrow[r] & \mathcal{O}_Z^{\geq 0} \end{tikzcd} \tag{2.10.13.1}$$

where $\mathcal{A}_M^{\geq 0} \subseteq \mathcal{O}_M^{\geq 0}$ *denotes the sheaf of smooth functions on a log smooth manifold* M*.*

Proof. Recall from (2.6.13) that $\mathcal{O}_{Q\times_B Z}^{\geq 0}$ is the log structure associated to the pre-log structure $\mathcal{O}_Q^{\geq 0} \sqcup_{\mathcal{O}_B^{\geq 0}} \mathcal{O}_Z^{\geq 0}$ on $Q \times_B Z$; that is $\mathcal{O}_{Q\times_B Z}^{\geq 0}$ is (2.6.6) the pushout

$$\mathcal{O}_{Q\times_B Z}^{\geq 0} = C_{Q\times_B Z}^{>0} \bigsqcup_{C_Q^{>0} \sqcup_{C_B^{\geq 0}} C_Z^{\geq 0}} \Big(\mathcal{O}_Q^{\geq 0} \bigsqcup_{\mathcal{O}_B^{\geq 0}} \mathcal{O}_Z^{\geq 0} \Big) \tag{2.10.13.2}$$

or equivalently the colimit

$$\mathcal{O}_{Q\times_B Z}^{\geq 0} = \operatorname{colim} \left(\begin{tikzcd} & C_Q^{>0} = \mathcal{O}_Q^{>0} \arrow[dl] \arrow[r] & \mathcal{O}_Q^{\geq 0} \\ C_{Q\times_B Z}^{>0} & C_B^{>0} = \mathcal{O}_B^{>0} \arrow[l] \arrow[u] \arrow[r] \arrow[d] & \mathcal{O}_B^{\geq 0} \arrow[u] \arrow[d] \\ & C_Z^{>0} = \mathcal{O}_Z^{>0} \arrow[ul] \arrow[r] & \mathcal{O}_Z^{\geq 0} \end{tikzcd} \right). \tag{2.10.13.3}$$

To show that the map from the colimit of the very similar diagram (2.10.13.1) to $\mathcal{O}_{Q\times_B Z}^{\geq 0}$ is an isomorphism, the key fact is the following:

(2.10.13.4) For any log smooth manifold M, every section of $\mathcal{O}_M^{\geq 0}$ is (locally) a product of a section of $\mathcal{A}_M^{\geq 0}$ and a section of $\mathcal{O}_M^{>0} = C_M^{>0}$.

This fact immediately implies surjectivity of the map in question (express a section of $\mathcal{O}_Q^{\geq 0}$ as a product of a section of $\mathcal{A}_Q^{\geq 0}$ and a section of product $\mathcal{O}_Q^{>0}$, and absorb the latter into $C_{Q\times_B Z}^{>0}$). It thus remains to show injectivity.

Suppose we have two triples $(a, b, c), (a', b', c') \in C_{Q\times_B Z}^{>0} \times \mathcal{A}_Q^{\geq 0} \times \mathcal{O}_Z^{\geq 0}$ with the same image in $\mathcal{O}_{Q\times_B Z}^{\geq 0}$. To show the desired injectivity assertion, it suffices to show that (a, b, c) and (a', b, c') are related via the maps in the diagram (2.10.13.1). Since (a, b, c) and (a', b', c') represent the same section of $\mathcal{O}_{Q\times_B Z}^{\geq 0}$, then there exist (locally) sections $m, m' \in \mathcal{O}_Q^{\geq 0} \sqcup_{\mathcal{O}_B^{\geq 0}} \mathcal{O}_Z^{\geq 0}$ whose underlying continuous functions are everywhere positive, satisfying $a|m|^{-1} = a'|m'|^{-1}$

and $mbc = m'b'c'$ in $\mathcal{O}_Q^{\geq 0} \sqcup_{\mathcal{O}_B^{\geq 0}} \mathcal{O}_Z^{\geq 0}$. Since positive sections of $\mathcal{O}_Q^{\geq 0} \sqcup_{\mathcal{O}_B^{\geq 0}} \mathcal{O}_Z^{\geq 0}$ are invertible (since this is true for $\mathcal{O}_Q^{\geq 0}$ and $\mathcal{O}_Z^{\geq 0}$), we may in fact assume $m = 1$. Realize m' as the product of $(d', e') \in \mathcal{O}_Q^{>0} \times \mathcal{O}_Z^{>0}$, and let us analyze the equality $bc = b'c'd'e'$ in the pushout $\mathcal{O}_Q^{\geq 0} \sqcup_{\mathcal{O}_B^{\geq 0}} \mathcal{O}_Z^{\geq 0}$. This equality means that the pairs $(b, c), (b'd', c'e') \in \mathcal{O}_Q^{\geq 0} \times \mathcal{O}_Z^{\geq 0}$ are related by a chain of relations $(xf, y) \sim (x, fy)$ for $f \in \mathcal{O}_B^{\geq 0}$. Factor each such f as the product of a section of $\mathcal{O}_B^{>0}$ and a section of $\mathcal{A}_B^{\geq 0}$ (2.10.13.4), and note that all of the factors in $\mathcal{O}_B^{>0}$ may be eliminated by modifying the choice of factorization $m' = d'e'$. With this new choice of (d', e'), we have $bc = b'c'd'e'$ in the pushout $\mathcal{O}_Q^{\geq 0} \sqcup_{\mathcal{A}_B^{\geq 0}} \mathcal{O}_Z^{\geq 0}$, namely $(b, c), (b'd', c'e') \in \mathcal{O}_Q^{\geq 0} \times \mathcal{O}_Z^{\geq 0}$ are related by a chain of relations $(xf, y) \sim (x, fy)$ for $f \in \mathcal{A}_B^{\geq 0}$. Now b and b' are both smooth, and smoothness is preserved (in both directions) by the relations associated to $f \in \mathcal{A}_B^{\geq 0}$, so we conclude that d' is smooth. Since $m = 1$ and $m' = d'e'$ for $d' \in \mathcal{A}_Q^{>0}$ and $e' \in \mathcal{O}_Z^{>0}$, we conclude that (a, b, c) and (a', b', c') represent the same element in the colimit of (2.10.13.1), as desired. □

⋆ **2.10.14 Definition** (Vertical differentiation). The *vertical tangent bundle* of an elementary log topological-smooth space $Q \times_B Z$ is the elementary log topological-smooth space $T_{Q/B} \times_B Z$ where $T_{Q/B}$ is the kernel of the map of vector bundles $TQ \to TB$ on Q, which is surjective since $Q \to B$ is a submersion. We now define when a vertical map $f : Q \times_B Z \to Q' \times_{B'} Z'$ is *vertically differentiable* (a pointwise condition) and in this case its vertical derivative $Tf : T_{Q/B} \times_B Z \to T_{Q'/B'} \times_{B'} Z'$. More precisely, the vertical derivative is defined pointwise at the level of underlying topological spaces, and when it is continuous (in which case f is called vertically C^1), it is automatically a vertical map since $|T_{Q'/B'} \times_{B'} Z'| \to |Q' \times_{B'} Z'|$ is strict. For $k \geq 2$, we say that f is vertically C^k when it is vertically C^1 and Tf is vertically C^{k-1}; vertically smooth means vertically C^k for all $k < \infty$.

Let us first define vertical differentiation of maps $f : Q_b \to M$ (any smooth manifold M) defined on the fiber Q_b of a submersion of log smooth manifolds $Q \to B$ over a point $b \in B$. For a point $q \in Q_b$, we consider the map on strata $Q_q \to B_b$, which is a submersion (2.7.60). We restrict f to the fiber $(Q_q)_b = Q_q \cap Q_b$ of this map on strata, which is a smooth manifold. The derivative at q (if it exists) of this restriction is a map $(T_{Q_q/B_b})_q \to TM$. Pre-composing this with the map $(T_{Q/B})_q \to T_{Q_q/B_b}$ (2.7.44), we obtain a map $T_q f : (T_{Q/B})_q \to TM$ which we declare to be the vertical derivative of f at q. The chain rule for post-composition by maps of smooth manifolds $M \to M'$ is evident. There is now an induced notion of vertical differentiability for maps $f : Q \times_B Z \to M$ for M a smooth manifold (note that a map of log topological spaces $|Q \times_B Z| \to M$ is the same thing as a vertical map $Q \times_B Z \to M \times_* *$). Namely, we consider the vertical derivatives (in the above sense) of the restrictions of f to fibers $Q_z = Q \times_B z$ over points $z \in Z$, forming a map $Tf : T_{Q/B} \times_B Z \to TM$ lifting f. The chain rule for post-composition by maps $M \to M'$ remains evident.

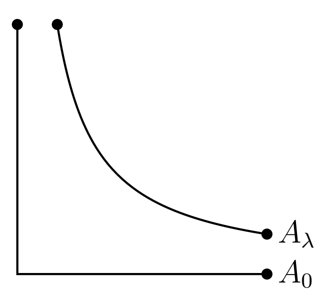
To define vertical differentiation of maps $Q \times_B Z \to {}'\mathbb{R}_{\geq 0}$ (by which we mean vertical maps $Q \times_B Z \to {}'\mathbb{R}_{\geq 0} \times_* *$ or equivalently maps of log topological spaces $|Q \times_B Z| \to {}'\mathbb{R}_{\geq 0}$), we use the colimit description (2.10.13) of the sheaf of such maps. We declare that the vertical derivative of a section of $C_{Q \times_B Z}^{>0}$ is as defined above, the vertical derivative of the pullback of a section of $\mathcal{A}_Q^{\geq 0}$ (recall $\mathcal{A}_Q^{\geq 0} \subseteq \mathcal{O}_Q^{\geq 0}$ denotes the sheaf of smooth maps to ${}'\mathbb{R}_{\geq 0}$)

is the restriction of its usual derivative to $T_{Q/B} \subseteq TQ$, and that the vertical derivative of the pullback of any section of $\mathcal{O}_Z^{\geq 0}$ is zero. Note that these declarations are consistent with the diagram (2.10.13.1), in that there they give a notion of vertical derivative for sections of all of these sheaves, consistent with the maps between them. We also declare that taking vertical derivative should send multiplication of functions valued in ${}'\mathbb{R}_{\geq 0}$ to addition of their derivatives with respect to the trivialization of $T'\mathbb{R}_{\geq 0}$ by $x\partial_x$. Thus by (2.10.13), this gives rise to a well defiend notion of vertical differentiation for sections of $\mathcal{O}_{Q\times_B Z}^{\geq 0}$ (note that the only sheaf in the diagram (2.10.13.1) with sections which are not everywhere verticall differentiable is $C_{Q\times_B Z}^{>0}$).

Having defined vertical differentiability for maps $Q \times_B Z \to {}'\mathbb{R}_{\geq 0}$, we now define a map $Q\times_B Z \to X_P$ to be vertically differentiable when its composition with every map $X_P \to {}'\mathbb{R}_{\geq 0}$ associated to a point $p \in P$ is vertically differentiable. We claim that the chain rule holds: differentiability is preserved by post-composition with differentiable maps $X_P \to X_Q$, and the derivative of the composition is the composition of the derivatives. This follows from the same argument used to prove the chain rule for ordinary log differentiation (2.7.40): it is, by definition, enough to treat the case $Q = {}'\mathbb{R}_{\geq 0}$, and expressing a differentiable map $X_P \to {}'\mathbb{R}_{\geq 0}$ as a product of an element of p and a differentiable map to $\mathbb{R}_{>0}$ reduces us to checking these two cases, where the result holds by inspection. Vertical differentiability is thus defined for maps from $Q \times_B Z$ to any log smooth manifold M.

Finally, we declare a vertical map $Q \times_B Z \to Q' \times_{B'} Z'$ to be vertically differentiable when its composition with the projection $Q' \times_{B'} Z' \to Q'$ is vertically differentiable in the above sense. For this to make sense, we must verify that the vertical derivative $T_{Q/B} \to TQ'$ of the composition lands inside $T_{Q'/B'} \subseteq TQ'$. In other words, we should show that the pullback of any cotangent vector on B' pairs to zero with $T_{Q/B}$. Any cotangent vector on B' is realized by a smooth map $B' \to {}'\mathbb{R}_{\geq 0}$, and the pullback of such a map to $Q \times_B Z$ factors through Z, hence has vanishing vertical derivative, as desired.

2.10.15 Example. Let us consider the multiplication map ${}'\mathbb{R}^2_{\geq 0} \to {}'\mathbb{R}_{\geq 0}$, and let us see what it means for a function $F : {}'\mathbb{R}^2_{\geq 0} \to \mathbb{R}$ to be vertically smooth. Let A_λ denote the fiber of the multiplication map over $\lambda \in {}'\mathbb{R}_{\geq 0}$. We consider log coordinates identifying A_λ with $\mathbb{R}$ via the maps $x = \lambda^{1/2}e^s$ and $y = \lambda^{1/2}e^{-s}$. The vector field ∂_s is thus the same as $x\partial_x - y\partial_y$, which forms a basis for the vertical tangent space $T_{'\mathbb{R}^2_{\geq 0}/'\mathbb{R}_{\geq 0}}$.



(2.10.15.1)

Now the vertical derivatives of a function $F : {}'\mathbb{R}^2_{\geq 0} \to \mathbb{R}$ are simply F, $\partial_s F$, $\partial_s\partial_s F$, etc. Vertical smoothness of F means these should all be continuous on ${}'\mathbb{R}^2_{\geq 0}$. Note that we must necessarily have $(\partial_s^i F)(0,0) = 0$ for all $i > 0$.

2.10.16 Lemma (Chain rule)**.** *The vertical derivative of a composition of vertical maps is the composition of their vertical derivatives (meaning, in particular, that the former is defined whenever the latter is).*

Proof. We consider a composition of vertical maps $Q\times_B Z \to Q'\times_{B'} Z' \to Q''\times_{B''} Z''$. Vertical differentiability for vertical maps with target $Q''\times_{B''} Z''$ is defined via their composition with the projection to Q'', so we are immediately reduced to the situation that the target $Q''\times_{B''} Z''$ is a log smooth manifold M''. By definition of vertical differentiability for maps to log smooth manifolds, we are further reduced to the case $M'' = {}'\mathbb{R}_{\geq 0}$. Now it was shown in (2.10.14) that any vertically smooth map $Q'\times_{B'} Z' \to {}'\mathbb{R}_{\geq 0}$ is a product of a vertically smooth map to $\mathbb{R}_{>0}$ and the pullback of a vertically smooth map on Q', so we are reduced to these two cases. For vertically smooth maps on Q', the present chain rule reduces to the chain rule for maps from $Q\times_B Z$ to log smooth manifolds treated in (2.10.14). Finally, for a composition of vertically smooth maps $Q\times_B Z \to Q'\times_{B'} Z' \to \mathbb{R}_{>0}$, recall that the vertical derivative of a function $f : Q\times_B Z \to \mathbb{R}_{>0}$ at a point $(q,b,z)\in Q\times_B Z$ depends only on the restriction of f to the stratum $(Q_q)_b$ of the fiber Q_b (which is also the fiber of the map $Q_q \to B_b$ on the strata of $q\in Q$ and $b\in B$). It thus suffices to recall that any vertical map $g : Q\times_B Z \to Q'\times_{B'} Z'$ maps the stratum $(Q_q)_b$ to the stratum $(Q'_{g(q)})_{g(b)}$ (2.10.12) and that this map is differentiable at (q,b,z) when g is (since locally there exist log smooth maps $Q' \to \mathbb{R}^k$ whose restriction to any given stratum of Q' is a local diffeomorphism). $\square$

2.10.17 Definition (Log topological-smooth space)**.** The category $\mathsf{LogTopSm}$ of log topological-smooth spaces is the perfection (2.8.60) of the category of elementary log topological-smooth spaces and vertically smooth maps. Concretely, this means a log topological-smooth space is a topological space equipped with an atlas of charts from open subsets of underlying log topological spaces of elementary log topological-smooth spaces $Q\times_B Z$, with specified vertically smooth transition functions satisfying the cocycle condition on triple overlaps, and a map of log topological-smooth spaces is a continuous map of topological spaces covered by specified vertically smooth maps on charts, compatible with the transition functions.

2.10.18 Exercise. Show that there are natural fully faithful inclusions of log topological spaces and log smooth manifolds into log topological-smooth spaces.

$$\mathsf{LogTop} \hookrightarrow \mathsf{LogTopSm} \qquad\qquad \mathsf{LogSm} \hookrightarrow \mathsf{LogTopSm} \tag{2.10.18.1}$$

$$Z \mapsto * \times_* Z \qquad\qquad M \mapsto M \times_* * \tag{2.10.18.2}$$

Given a submersion of log smooth manifolds $Q\to B$ and maps $Z'\to Z\to B$ of log topological spaces, define a tautological diagram

$$\begin{tikzcd} Q\times_B Z' \arrow[r] \arrow[d] & Q\times_B Z \arrow[d] \\ Z' \arrow[r] & Z \end{tikzcd} \tag{2.10.18.3}$$

in the category $\mathsf{LogTopSm}$ of log topological-smooth spaces, and show that it is a fiber product.

Chapter 3

Analysis

3.1 Topological vector spaces

We begin with some generalities about topological vector spaces. In practice, we mainly care about *complete* topological vector spaces, however these are most often described as completions of general (not necessarily complete) topological vector spaces.

⋆ **3.1.1 Definition** (Topological vector space)**.** A (real or complex) *topological vector space* is a (real or complex) vector space V whose addition $V \times V \to V$ and scalar multiplication $\mathbb{R} \times V \to V$ (or $\mathbb{C} \times V \to V$) are continuous.

3.1.2 Exercise. Show that the category of topological vector spaces and continuous linear maps has all limits and that these limits are preseved by the forgetful functor to topological spaces.

3.1.3 Exercise (Vector space topology generated by neighborhoods of the origin)**.** Let V be a real vector space, and let $\{U_\alpha \subseteq V\}_\alpha$ be a collection of subsets containing zero satisfying the following axioms:

(3.1.3.1) There is at least one α.
(3.1.3.2) For every pair α, β, there exists γ such that $U_\gamma \subseteq U_\alpha \cap U_\beta$.
(3.1.3.3) For every α, there exists β such that $U_\beta + U_\beta \subseteq U_\alpha$.
(3.1.3.4) For every α, there exists β such that $(-1, 1) \cdot U_\beta \subseteq U_\alpha$.
(3.1.3.5) For every α and every v, there exists $\varepsilon > 0$ such that $(-\varepsilon, \varepsilon) \cdot v \subseteq U_\alpha$.

Declare a set $U \subseteq V$ to be open iff for every $u \in U$ we have $u + U_\alpha \subseteq U$ for some α. Show that this defines the coarsest vector space topology on V in which every U_α is a neighborhood of the origin. Show that, conversely, given any vector space topology on V, the collection of all neighborhoods of the origin satisfies the above axioms, and generates the input topology in the above sense.

3.1.4 Definition (Semi-norm)**.** A *semi-norm* on a real (resp. complex) vector space V is a map $\|\cdot\| : V \to \mathbb{R}_{\geq 0}$ satisfying linearity $\|av\| = |a|\|v\|$ for $a \in \mathbb{R}$ (resp. $a \in \mathbb{C}$) and the triangle inequality $\|x + y\| \leq \|x\| + \|y\|$.

3.1.5 Definition (Norm)**.** A *norm* is a semi-norm for which $\|v\| = 0$ implies $v = 0$.

3.1.6 Definition (Complete topological vector space)**.** Let V be a topological vector space. A swarm (2.1.26) $v : S^* \to V$ is called *Cauchy* when for every neighborhood of zero $0 \in U \subseteq V$, there exists a neighborhood of the basepoint $A \subseteq S$ such that $v(a) - v(a') \in U$ for all $a, a' \in A^*$. A topological vector space is called *complete* when every Cauchy swarm has a unique limit.

3.1.7 Exercise. Show that a complete topological vector space is Hausdorff.

3.1.8 Definition (Smooth map to a complete topological vector space)**.** Let V be a complete topological vector space, and let us define a notion of when a map from a smooth manifold to V is smooth. A continuous map $f : Z \to V$ from a smooth manifold Z is said to be of class C^1 (continuously differentiable) when the map $Z \times Z \times (\mathbb{R} \setminus 0) \to V$ given by $(x, y, t) \mapsto t^{-1}(f(x) - f(y))$ extends continuously to the 'deformation to the tangent bundle' $\mathbb{P}(Z)$ (2.4.25), in which case the restriction to $TZ \subseteq \mathbb{P}(Z)$ of this (necessarily unique) continuous extension is called the derivative $Tf : TZ \to V$ of f. For $k \geq 2$, we say f is of class C^k when it is of class C^1 and its derivative Tf is of class C^{k-1}. Note that the derivative $Tf : TZ \to V$ is automatically linear (every relation $x + y = z$ in TZ is a limit of triples (p, q, t), (q, r, t), and (p, r, t) in $\mathbb{P}(Z)$). This notion of smoothness is respected by pre-composition with smooth maps of smooth manifolds and by post-composition with continuous linear maps of complete topological vector spaces.

3.2 Function spaces

Introducing all sorts of complicated function spaces is a sign of weakness.
—*Atle Selberg* (as told by Peter Sarnak)

We here recall various standard function spaces and their basic properties. References include [93, 5, 44, 109]. By '(smooth) manifold' we mean 'paracompact Hausdorff smooth manifold', and by 'vector bundle' we mean 'finite-dimensional smooth real (or complex) vector bundle'.

The function spaces we consider here are all completions of spaces of compactly supported smooth functions, so it is equivalent to describe the corresponding topologies on these spaces of smooth functions (and we will pass freely between topologies on compactly supported smooth functions and the corresponding complete topological vector spaces containing compactly supported functions as a dense subspace). All of the topologies we encounter here are locally convex, i.e. are generated by a set of semi-norms. While the topology only depends on the (set of semi-)norm(s) up to commensurability, it is useful to remember precisely what sort of geometric data gives rise to a relevant semi-norm, so that we can keep track of which data certain constants depend on (e.g. to prove uniformity of bounds with respect to parameters). That being said, such geometric data is usually omitted from the notation.

As a general paradigm, we first define semi-norms on functions on $\mathbb{R}^n$, we then note that they are preserved (up to commensurability) by diffeomorphisms of the domain and by multiplication by smooth functions, and finally we conclude that they make sense for functions on manifolds (possibly valued in a vector bundle). A subscript '$_c$' (resp. '$_K$') indicates functions of compact support (resp. supported inside a given compact set K), while a subscript '$_{\mathrm{loc}}$' indicates that only local constraints are imposed.

3.2.1 Definition (Multi-index notation)**.** Let V be a finite-dimensional real vector space. The symmetric algebra $\operatorname{Sym} V = \bigoplus_{r\geq 0} \operatorname{Sym}^r V$ (where $\operatorname{Sym}^r V = (V^{\otimes r})_{S_r}$) is the space of translation invariant differential operators on V. Given a basis $v_1, \ldots, v_n \in V$, there is an induced basis $\operatorname{Sym} V$ consisting of all possible monomials in $v_1, \ldots, v_n$. These monomials are in natural bijection with *multi-indices* $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n_{\geq 0}$; the *degree* of a multi-index α is denoted $|\alpha| = \sum_{i=1}^n \alpha_i$. The differential operator associated to a multi-index α is denoted D^α. Usually $V = \mathbb{R}^n$ with the standard basis.

⋆ **3.2.2 Definition** (Smooth functions C^∞_{loc} and C^∞_c)**.** Let M be a manifold, and let E/M be a vector bundle. The space of smooth sections $f : M \to E$ is denoted

$$C^\infty_{\mathrm{loc}}(M, E). \tag{3.2.2.1}$$

We denote by $C^\infty_c \subseteq C^\infty_{\mathrm{loc}}$ the subspace of functions which are compactly supported, and we denote by $C^\infty_K \subseteq C^\infty_{\mathrm{loc}}$ the subspace of functions supported inside a given compact set $K \subseteq M$. The corresponding spaces of k times continuously differentiable are denoted by C^k with the same subscripts.

⋆ **3.2.3 Definition** (C^∞_{loc}-topology and C^k_{loc}-topology)**.** For any integer $k \geq 0$, the C^k-norm of a smooth function f on $\mathbb{R}^n$ is the supremum of the sum of the absolute values of its derivatives (in the principal coordinate directions) of order $\leq k$.

$$\|f\|_{C^k(\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n} \sum_{|\alpha|\leq k} |D^\alpha f(x)| \tag{3.2.3.1}$$

Now let M be a manifold and E/M a vector bundle. Given a coordinate chart $\alpha : \mathbb{R}^n \supseteq U \hookrightarrow M$, a smooth section $\varphi : \alpha(U) \to E^*$ of compact support, and an integer $k \geq 0$, we may consider the semi-norm $f \mapsto \|\alpha^*(\varphi f)\|_{C^k(\mathbb{R}^n)}$ on $C^k_{\text{loc}}(M, E)$. A C^k_{loc}-semi-norm is a semi-norm of this form (or one which is bounded by a finite sum of semi-norms of this form). The topology generated by the family of all C^k_{loc}-semi-norms is called the C^k_{loc}*-topology* (concretely, it is the topology of uniform convergence of all derivatives of order $\leq k$ over compact subsets). A C^∞_{loc}-semi-norm is a C^k_{loc}-semi-norm for some $k < \infty$. The topology generated by the family of all C^∞_{loc}-semi-norms is called the C^∞_{loc}*-topology* (concretely, it is the topology of uniform convergence of all derivatives over compact subsets).

3.2.4 Exercise. Show that $C^k_{\text{loc}}(M, E)$ is complete with respect to the C^k_{loc}-topology for all $k \geq 0$ as well as for $k = \infty$. Show that $C^k_K \subseteq C^k_{\text{loc}}$ is a closed subspace with respect to the C^k_{loc}-topology (hence complete in the subspace topology). Show that $C^k_c \subseteq C^k_{\text{loc}}$ is a dense subspace with respect to the C^k_{loc}-topology.

3.2.5 Exercise. Let $k \geq 0$ be an integer. Fix a collection of coordinate charts $\alpha_i : \mathbb{R}^n \supseteq U_i \hookrightarrow M$ and smooth sections $\varphi_i : \alpha_i(U_i) \to E^*$ of compact support. Show that if the values $\varphi_i(x) \in E^*_x$ span for every $x \in M$, then every C^k_{loc}-semi-norm is bounded by a finite sum of the C^k_{loc}-semi-norms $f \mapsto \|\alpha_i^*(\varphi_i f)\|_{C^k(\mathbb{R}^n)}$.

⋆ **3.2.6 Lemma** (C^∞_c-topology)**.** *The directed colimit*

$$C^\infty_c = \varinjlim_{K \text{ compact}} C^\infty_K \tag{3.2.6.1}$$

exists in the category of locally convex topological vector spaces and commutes with the forgetful functor to vector spaces; we call this the C^∞_c-topology*. The* C^∞_c*-topology is complete and is generated by the family of semi-norms* $\sum_i \|\alpha_i f\|_{C^{k_i}}$ *(which we call* C^∞_c*-semi-norms) associated to locally finite collections of compactly supported smooth functions* α_i *and integers* $k_i < \infty$*.*

Proof. Existence of the colimit in locally convex topological vector spaces and its compatibility with the forgetful functor are straightforward: the colimit is given by the vector space C^∞_c equipped with the family of all semi-norms (equivalently, convex sets containing zero) whose pullback to every C^∞_K is continuous (resp. is a neighborhood of zero). What we must do is describe such semi-norms and show that the vector space topology they induce is complete.

Let $\varphi_1, \varphi_2, \ldots$ be a locally finite partition of unity with each φ_i compactly supported. Any function f is a convex combination of the functions $2^i\varphi_i f$ and zero. Now if $U \subseteq C^\infty_c$ contains zero and has open intersection with every C^∞_K, then it contains the set of functions supported inside $\operatorname{supp}\varphi_i$ with $\|\cdot\|_{C^{k_i}} \leq \varepsilon_i$ for some $k_i < \infty$ and $\varepsilon_i > 0$. Therefore if $\sum_i \|2^i\varepsilon_i^{-1}\varphi_i f\|_{C^{k_i}} \leq 1$, then $\|2^i\varphi_i f\|_{C^{k_i}} \leq \varepsilon_i$ for every i, so every $2^i\varphi_i f$ lies in U and hence so does their convex combination f provided U is also convex. Conversely, every semi-norm $\sum_i \|\alpha_i f\|_{C^{k_i}}$ for a locally finite collection of compactly supported smooth functions α_i and integers $k_i < \infty$ restricts to a C^∞_K-semi-norm. This proves the claimed description of the semi-norms on the directed colimit C^∞_c in the category of locally convext topological vector spaces.

It remains to prove completeness of C^∞_c. Let x_α be a Cauchy swarm in C^∞_c. Since $C^\infty_c \to C^\infty_{\mathrm{loc}}$ is continuous (by the universal property of colimits, since each $C^\infty_K \to C^\infty_{\mathrm{loc}}$ is continuous), the image of x_α in C^∞_{loc} is Cauchy, hence converges $x_\alpha \to x \in C^\infty_{\mathrm{loc}}$ since C^∞_{loc} is complete (3.2.4). We will show that $x \in C^\infty_c$ and that the convergence $x_\alpha \to x$ holds in the (much stronger) C^∞_c-topology. Boundedness of the C^∞_c-semi-norms on x_α (since it is Cauchy) implies x must be compactly supported (if it were not, then we could construct a C^∞_c-semi-norm whose uniform boundedness on x_α would obstruct uniform convergence $x_\alpha \to x$). Now that $x \in C^\infty_c$, we may (by translation) assume wlog that $x = 0$. That is, we have a Cauchy swarm $x_\alpha \in C^\infty_c$ with $x_\alpha \to 0$ in C^∞_{loc}, and we would like to show $x_\alpha \to 0$ in C^∞_c. Let $\|f\| = \sum_i \|\alpha_i f\|_{C^{k_i}}$ be any C^∞_c-semi-norm as above. Since $x_\alpha \to 0$ in C^∞_{loc}, we have

$$\lim_\alpha (\|x_\alpha + u\| - \|x_\alpha\|) = \|u\| \geq 0 \tag{3.2.6.2}$$

for any $u \in C^\infty_c$ (indeed, the terms in $\|\cdot\|$ which differ between x_α and $x_\alpha + u$ are just the finitely many $\|\alpha_i \cdot -\|_{C^{k_i}}$ where the support of α_i intersects the support of u, and for these we use the fact that $x_\alpha \to 0$ uniformly in all derivatives over compact sets). We thus have

$$\limsup_\alpha \|x_\alpha\| \leq \limsup_\alpha \|x_\alpha + u\| \tag{3.2.6.3}$$

for any $u \in C^\infty_c$. Now take $u = -x_\beta$ and note that $\limsup_\alpha \|x_\alpha - x_\beta\| \to 0$ as $\beta \to \infty$ since our swarm is Cauchy, thus $\limsup_\alpha \|x_\alpha\| = 0$ as desired. $\square$

3.2.7 Definition (Bundle of densities)**.** We denote by Ω_M the bundle of densities on M. It is a smooth real line bundle defined by the existence of a canonical integration map $\int_M : C^\infty_c(M, \Omega_M) \to \mathbb{R}$. In fact, it is the line bundle associated to a principal $\mathbb{R}_{>0}$-bundle, so it has powers Ω^t_M for any $t \in \mathbb{R}$.

3.2.8 Example (Delta function)**.** The delta function $\delta_p \in C^{-\infty}_c(\mathbb{R}^n)$ is the distribution given by the linear functional 'evaluate at $p \in \mathbb{R}^n$'. On a manifold, the delta function is naturally a distribution valued in densities $\delta_p \in C^{-\infty}_c(M, \Omega_M)$.

3.2.9 Definition (Schwartz functions $\mathcal{S}$)**.** The space of Schwartz functions $\mathcal{S}(\mathbb{R}^n)$ consists of those infinitely differentiable functions all of whose norms

$$\|f\|_{\mathcal{S},A,B} = \sup_{x\in\mathbb{R}^n} (1 + |x|^A) \sum_{|\alpha|\leq B} |D^\alpha_x f(x)| \tag{3.2.9.1}$$

are finite. The space $\mathcal{S}$ is complete with respect to this family of norms.

3.2.10 Definition (Fourier transform)**.** The Fourier transform is a linear map $\mathcal{S}(\mathbb{R}^n, \mathbb{C}) \to \mathcal{S}(\mathbb{R}^n, \mathbb{C})$ denoted $f \mapsto \hat{f}$ and given by the formula

$$\hat{f}(\xi) = \int e^{-2\pi i\langle \xi, x\rangle} f(x)\, dx. \tag{3.2.10.1}$$

The Fourier transform is continuous: the required decay properties of $\hat{f}$ and its derivatives follow from integration by parts. Intrinsically speaking, the Fourier transform of a function on a finite-dimensional real vector space V is a function on its dual V^* valued in $\det(V)$.

The Fourier transform also makes sense for Schwartz functions valued in a complex vector space E. When E is a real vector space, the Fourier transform maps $\mathcal{S}(\mathbb{R}^n, E)$ to the subspace of $\mathcal{S}(\mathbb{R}^n, E \otimes_{\mathbb{R}} \mathbb{C})$ consisting of those functions g satisfying $g(-\xi) = \overline{g(\xi)}$ (and conversely, the Fourier transform of such a function g lies in the subspace $\mathcal{S}(\mathbb{R}^n, E) \subseteq \mathcal{S}(\mathbb{R}^n, E \otimes_{\mathbb{R}} \mathbb{C})$).

3.2.11 Exercise. Show that $\int_{\mathbb{R}^n} e^{-\pi|x|^2}\, dx = 1$ by reducing to the case $n = 2$ and using polar coordinates.

3.2.12 Lemma (Fourier inversion)**.** *For $f \in \mathcal{S}(\mathbb{R}^n)$, we have $\hat{\hat{f}}(x) = f(-x)$.*

Proof. Note that $\hat{f}(\xi)e^{-\pi(\xi/N)^2} \to \hat{f}(\xi)$ in $\mathcal{S}(\mathbb{R}^n)$ as $N \to \infty$. It thus suffices to show that

$$\int e^{2\pi i\langle \xi, x\rangle} \hat{f}(\xi) e^{-\pi(\xi/N)^2}\, d\xi \to f(x). \tag{3.2.12.1}$$

The left hand side may be written as

$$\iint e^{2\pi i\langle \xi, x-y\rangle} f(y) e^{-\pi(\xi/N)^2}\, dy\, d\xi = \int \left(\int e^{-\pi(\xi/N)^2 + 2\pi i\langle \xi, z\rangle}\, d\xi \right) f(x+z)\, dz. \tag{3.2.12.2}$$

Now we may compute the inner integral of ξ by completing the square, moving the contour, and appealing to the identity $\int e^{-\pi x^2}\, dx = 1$ (3.2.11). The result is $N^n e^{-\pi(Nz)^2}$, making the desired convergence to $f(x)$ clear upon appealing to (3.2.11) for a second time. $\square$

3.2.13 Exercise (Fourier transform and convolution)**.** Show that for $f, g \in \mathcal{S}(\mathbb{R}^n)$, we have $\widehat{f*g} = \hat{f}\hat{g}$, where $(f*g)(x) = \int f(y)g(x-y)\, dy$ denotes convolution. Using Fourier inversion, conclude that $\widehat{fg} = \hat{f}*\hat{g}$, and specialize this to conclude that $\int fg = \int \hat{f}\hat{g}$, so in particular $\|f\|_{L^2} = \|\hat{f}\|_{L^2}$ (Plancherel's formula).

*

⋆ **3.2.14 Definition** (Sobolev spaces H^s)**.** The Sobolev H^s-norm of a function $f \in \mathcal{S}(\mathbb{R}^n)$ is

$$\|f(x)\|_{H^s(\mathbb{R}^n)} = \left\|(1 + |\xi|^2)^{s/2} \hat{f}(\xi)\right\|_{L^2(\mathbb{R}^n)}. \tag{3.2.14.1}$$

When s is a non-negative integer, differentiation under the integral sign and Plancherel (3.2.13) imply that the H^s-norm is equivalent to $\sum_{|\alpha| \le s} \|D^\alpha f\|_{L^2(\mathbb{R}^n)}$.

3.2.15 Exercise (Dual norm)**.** Recall the *dual norm* (**??**): for any norm on $C_c^\infty(M, E)$, we can take its dual $\|\cdot\|'$ on $C_c^\infty(M, E^* \otimes \Omega_M)$ given by

$$\|f\|' = \sup_{\|g\|\leq 1} \int fg. \tag{3.2.15.1}$$

Show that if $\|f\|' < \infty$ for all f, then $\|\cdot\|'$ is a norm and $\|\cdot\|'' \leq \|\cdot\|$.

3.2.16 Exercise. Show using the Cauchy–Schwarz inequality that the L^2-norm on $\mathbb{R}^n$ is self-dual (where we implicitly trivialize $\Omega_{\mathbb{R}^n}$ by $dx_1 \cdots dx_n$). More generally, show that for any positive function $w : \mathbb{R}^n \to \mathbb{R}_{>0}$, the dual of the $L^2_w(\mathbb{R}^n)$-norm $f \mapsto (\int w|f|^2)^{1/2}$ is the $L^2_{w^{-1}}$-norm. Conclude that the dual of the $H^s(\mathbb{R}^n)$-norm is the $H^{-s}(\mathbb{R}^n)$-norm.

⋆ **3.2.17 Definition** (Interpolation norm)**.** Let $a < b < c \in \mathbb{R}$. Given norms $\|\cdot\|_a$ and $\|\cdot\|_c$ on a vector space X, we define a third norm $\|\cdot\|_b$ by the formula

$$\|v\|_b = \left(N_{a,b,c}^{-1} \int_{-\infty}^{\infty} \inf_{x+y=v} \left(e^{2(b-a)t}\|x\|_a^2 + e^{2(b-c)t}\|y\|_c^2\right) dt\right)^{1/2}, \tag{3.2.17.1}$$

where $N_{a,b,c} = \int \inf_{x+y=1}\left(e^{2(b-a)t}x^2 + e^{2(b-c)t}y^2\right) dt$ is a normalization factor. This is known, more precisely, as the *K-interpolation norm at $q = 2$*.

3.2.18 Exercise. Show that

$$\int_{-\infty}^{\infty} \inf_{x+y=1} \left(e^{2(b-a)t}r^{2a}x^2 + e^{2(b-c)t}r^{2c}y^2\right) dt = N_{a,b,c}r^{2b} \tag{3.2.18.1}$$

by reparameterizing t (any $r > 0$). Use this to show that

$$\left.\begin{matrix}\|v\|_a \leq Mr^a \\ \|v\|_c \leq Mr^c\end{matrix}\right\} \implies \|v\|_b \leq Mr^b \tag{3.2.18.2}$$

or equivalently that

$$\|v\|_b \leq \|v\|_a^{\frac{c-b}{c-a}} \|v\|_c^{\frac{b-a}{c-a}} \tag{3.2.18.3}$$

with equality with $\dim X = 1$ (to prove this, consider the infimum over x and y both multiples of v).

3.2.19 Exercise. Consider interpolation triples $(\|\cdot\|_a, \|\cdot\|_b, \|\cdot\|_c)$ on vector spaces X and Y, and consider a linear map $A : X \to Y$ which is (a, a)-bounded and (c, c)-bounded. Show that A is (b, b)-bounded with (b, b)-norm bounded by

$$\|A\|_{(b,b)} \leq \|A\|_{(a,a)}^{\frac{c-b}{c-a}} \|A\|_{(c,c)}^{\frac{b-a}{c-a}}. \tag{3.2.19.1}$$

To show this, bound the infimum over $A(v) = z + w$ (appearing in $\|A(v)\|_b$) by the infinum over $v = x + y$ (taking $z = A(x)$ and $w = A(y)$), and then reparameterize the integral over t to obtain the desired constant factor times $\|v\|_b$.

⋆ **3.2.20 Lemma** (Interpolation for Sobolev norms)**.** *For $a < b < c \in \mathbb{R}$, the H^b-norm is the interpolation of the H^a-norm and H^c-norm on $\mathcal{S}(\mathbb{R}^n)$.*

Proof. The square of the H^b-norm of u may be written using the identity (3.2.18.1) as

$$N_{a,b,c}^{-1} \iint \inf_{x+y=\hat{u}(\xi)} \left(e^{2(b-a)t}(1+|\xi|^2)^a|x|^2 + e^{2(b-c)t}(1+|\xi|^2)^c|y|^2\right) dt\, d\xi. \tag{3.2.20.1}$$

The square of the interpolation of the H^a-norm and H^c-norm may be written as

$$N_{a,b,c}^{-1} \int \inf_{f+g=u} \left[\int \left(e^{2(b-a)t}(1+|\xi|^2)^a|\hat{f}(\xi)|^2 + e^{2(b-c)t}(1+|\xi|^2)^c|\hat{g}(\xi)|^2\right) d\xi\right] dt. \tag{3.2.20.2}$$

The inequality (3.2.20.1) $\leq$ (3.2.20.2) is immediate. To show equality, it suffices to note that the pointwise minimizing pair $(x(\xi), y(\xi))$ with $x + y = \hat{u}$ from (3.2.20.1) can be well approximated by pairs of the form $(\hat{f}(\xi), \hat{g}(\xi))$ with $f + g = u$. □

3.2.21 Example. Consider the operator $M_f : C_c^\infty(\mathbb{R}^n) \to C_c^\infty(\mathbb{R}^n)$ given by multiplication by a smooth function $f : \mathbb{R}^n \to \mathbb{R}$ all of whose derivatives are bounded. Direct calculation shows that M_f is bounded $H^s(\mathbb{R}^n) \to H^s(\mathbb{R}^n)$ for every integer $s \geq 0$. It follows from interpolation (3.2.19)(3.2.20) that M_f is bounded $H^s \to H^s$ for real $s \geq 0$. The adjoint of M_f is itself, so it follows from duality (3.2.16)(**??**) that M_f is bounded $H^s \to H^s$ for all s.

Now let $\phi : \mathbb{R}^n \to \mathbb{R}^n$ be a diffeomorphism whose derivatives of all positive orders are bounded, and consider the pushforward map $\phi_* : C_c^\infty(\mathbb{R}^n) \to C_c^\infty(\mathbb{R}^n)$. This operator ϕ_* is bounded $H^s(\mathbb{R}^n) \to H^s(\mathbb{R}^n)$ for integer $s \geq 0$ by direct calculation, hence for all real $s \geq 0$ by interpolation. The same applies to pullback ϕ^* since $\phi^* = (\phi^{-1})_*$. The adjoint of ϕ_* is $M_{\det D\phi} \circ \phi^*$, so by duality we conclude that ϕ_* (hence also ϕ^*) is bounded $H^s \to H^s$ for all s.

⋆ **3.2.22 Exercise** (Sobolev norm on a manifold)**.** Let M be a manifold, and let E/M be a vector bundle. Given a coordinate chart $\mathbb{R}^n \supseteq U \subseteq M$ and a smooth function of compact support $\varphi : U \to E^*$, we may consider the semi-norm $u \mapsto \|\varphi u\|_{H^s(\mathbb{R}^n)}$ on $C^\infty_{\mathrm{loc}}(M, E)$. These are called Sobolev H^s_{loc}-semi-norms on $C^\infty_{\mathrm{loc}}(M, E)$, and the topology they generate is called the H^s_{loc}-topology (also called the H^s-topology when M is compact). Show that the H^s_{loc}-semi-norms associated to a particular collection of pairs $(\mathbb{R}^n \supseteq U_i \subseteq M, \varphi_i : U_i \to E^*)$ generate the H^s_{loc}-topology provided the $\varphi_i(x)$ span E^*_x at every point $x \in M$ (note that it suffices to show that any single semi-norm $\|\psi u\|_{H^s(\mathbb{R}^n)}$ is bounded by a sum of semi-norms $\|\varphi_i u_i\|_{H^s(\mathbb{R}^n)}$, and then prove this using (3.2.21)).

⋆ **3.2.23 Definition** (Geometry)**.** Let M be a manifold. A *geometry* on M of a collection of charts $\{\alpha_i : (0,1)^{n_i} \hookrightarrow M\}_i$ for which there exists a sequence of constants $\delta > 0$ and $D, M_0, M_1, \ldots < \infty$ such that:

(3.2.23.1) Every transition function $\alpha_j^{-1}\alpha_i : \alpha_i^{-1}(\alpha_j((0,1)^{n_j})) \to \alpha_j^{-1}(\alpha_i((0,1)^{n_i}))$ has C^k-norm bounded above by M_k.

(3.2.23.2) We have $M = \bigcup_i \alpha_i((\delta, 1-\delta)^{n_i})$.

(3.2.23.3) Every $n_i \leq D$.

Such a sequence of constants is called a *bound* on the geometry $\{\alpha_i\}_i$. Two geometries are called *comparable* when their union is bounded. Comparability is transitive, and a bound on the geometries $\{\alpha_i\}_i \cup \{\beta_j\}_j$ and $\{\beta_j\}_j \cup \{\gamma_k\}_k$ determines a bound on the geometry $\{\alpha_i\}_i \cup \{\gamma_k\}_k$.

More generally, a geometry on a pair (M, E) consisting of a manifold M and a vector bundle E/M consists of a collection of charts $\{\alpha_i : (0,1)^{n_i} \hookrightarrow M\}_i$ as above, covered by trivializations $\beta_i : \mathbb{R}^{k_i} \to \alpha_i^* E$, such that:

(3.2.23.4) Every transition function $\beta_j^{-1}\alpha_j^{-1}\alpha_i\beta_i : \alpha_i^{-1}(\alpha_j((0,1)^{n_j})) \to \operatorname{Hom}(\mathbb{R}^{k_i}, \mathbb{R}^{k_j})$ has C^k-norm bounded above by M_k.

(3.2.23.5) Every $k_i \leq D$.

The generalization to multiple vector bundles is evident.

3.2.24 Exercise (Covering by small balls given bounded geometry)**.** Let $\{\alpha_i : (0,1)^{n_i} \hookrightarrow M\}_i$ be a geometry (3.2.23). Equip M with a Riemannian metric which is a convex combination of the standard metrics in each of the charts α_i (use an arbitrary partition of unity subordinate to the cover by charts); any two metrics in this class are commensurate, via a constant depending only on a bound on the geometry $\{\alpha_i\}_i$. Show that for all sufficiently small $\varepsilon > 0$ (depending on a bound on the geometry), every r-ball in M is covered by at most an exponential in r (depending on a bound on the geometry) ε-balls. Here is one way to proceed.

An ε*-net* in a metric space X is a subset $N \subseteq X$ which is maximal with respect to the property that the distance between any two distinct points of N is $\geq \varepsilon$ (so the union of the open ε-balls centered at the points of any ε-net cover X). We endow an ε-net with the structure of a graph in which two points are connected by an edge iff they have distance $\leq 3\varepsilon$. Show that if N is an ε-net in a geodesic metric space X, then for any two points $x, y \in X$ of distance $r > 0$ and any two points $\bar{x}, \bar{y} \in N$ whose respective distances to x and y are $< \varepsilon$, the distance between $\bar{x}$ and $\bar{y}$ in the graph N is at most $\lceil r/\varepsilon \rceil$. Conclude that any r-ball in X is covered by the union of the ε-balls centered at the points of an $\lceil r/\varepsilon \rceil$-ball in any ε-net in M.

Returning to our collection of charts $\{\alpha_i\}_i$ on M, show that for sufficiently small $\varepsilon > 0$ depending on a bound on the geometry, the degree of (the aforementioned graph on) any ε-net is bounded in terms of a bound on the geometry (note that by choosing $\varepsilon > 0$ sufficiently small, we can ensure all the action is confined to a single chart since M is Hausdorff).

3.2.25 Exercise (Partitions of unity associated to a geometry)**.** Let $\{\alpha_i : (0,1)^{n_i} \hookrightarrow M\}_i$ be a geometry. A bound on a subordinate partition of unity $\{\varphi_i : (0,1)^{n_i} \to \mathbb{R}_{\geq 0}\}_i$ is a sequence of constants $\eta > 0$ and $N, M_0, M_1, \ldots < \infty$ such that:

(3.2.25.1) We have $\|\varphi_i\|_{C^k} \leq M_k$ for all i.

(3.2.25.2) Every η-ball in M with respect to any convex combination of the standard flat Riemannian metrics in the charts $\{\alpha_i\}_i$ intersects $\operatorname{supp} \varphi_i$ for at most N indices i (note that this implies a bound on the same quantity for any R-ball depending only on $R < \infty$ by (3.2.24)).

More generally, we may wish to consider a 'pre-partition of unity', namely a collection of compactly supported functions $\{\varphi_i : (0,1)^{n_i} \to \mathbb{R}_{\geq 0}\}_i$ which do not necessarily sum to 1, in

which case a bound consists, in addition to the above, of a constant $\mu > 0$ such that:
(3.2.25.3) $\sum \varphi_i \geq \mu$.
It is evident that a pre-partition of unity determines a partition of unity which is bounded in terms of a bound on the input pre-partition of unity (by dividing by the sum); in practice this is how bounded partitions of unity are constructed.

Show that there always exists a partition of unity which is bounded in terms of a bound on the geometry $\{\alpha_i\}_i$. Here is one way to proceed. Choose an ε-net $N \subseteq M$ as in (3.2.24). Assign each point of N to some index i in which it lies in $\alpha_i((\delta, 1-\delta)^{n_i})$. Now take φ_i to be a sum of standard bump functions of radius $A\varepsilon$ centered at each of the points of N assigned to the index i. Show that for $A < \infty$ sufficiently large and $\varepsilon > 0$ sufficiently small (in terms of the geometry and A), this collection is a bounded pre-partition of unity in the above sense.

3.2.26 Exercise (Sobolev norm on a manifold with respect to specified geometry)**.** Let M be a manifold with a fixed choice of geometry $\{\alpha_i : (0,1)^{n_i} \hookrightarrow M\}_i$ (3.2.23). We define the $H_2^s(M)$ and $H_\infty^s(M)$ Sobolev norms

$$\|u\|_{s,2} = \Big(\sum_i \|\varphi_i \cdot \alpha_i^* u\|_{H^s(\mathbb{R}^n)}^2\Big)^{1/2} \tag{3.2.26.1}$$

$$\|u\|_{s,\infty} = \sup_i \|\varphi_i \cdot \alpha_i^* u\|_{H^s(\mathbb{R}^n)} \tag{3.2.26.2}$$

where $\{\varphi_i\}_i$ denotes any partition of unity subordinate to the geometry $\{\alpha_i\}_i$ which is bounded in terms of a bound on the $\{\alpha_i\}_i$ (3.2.25). Show that these norms are well defined up to commensurability via a constant depending only on a bound on the input geometry (and conclude from this that comparable geometries give rise to commensurate H_2^s and H_∞^s norms).

Here is one way to proceed. Consider any two such partitions of unity $\{\varphi_i\}_i$ and $\{\bar\varphi_i\}_i$. For any index i, the number of indices j for which $\operatorname{supp}\varphi_j$ intersects $\operatorname{supp}\bar\varphi_i$ is bounded. It therefore suffices to show that $\|\bar\varphi_i \cdot \alpha_i^* u\|_s$ is bounded by a bounded constant times the sum of $\|\varphi_j \cdot \alpha_j^* u\|_s$ over this set of indices j. This follows by writing $\bar\varphi_i u = \sum_j \bar\varphi_i \cdot \varphi_j u$ and appealing to boundedness of the action on Sobolev spaces by diffeomorphisms and multiplication by smooth functions (3.2.21).

3.2.27 Exercise. The manifold $\mathbb{R}^n$ has a standard flat geometry (3.2.23) given by all translations of $(0,1)^n \subseteq \mathbb{R}^n$. Show that the induced Sobolev norm H_2^s (3.2.26) is commensurate to the standard Sobolev norm $H^s(\mathbb{R}^n)$ (3.2.14.1).

Here is one way to proceed. Consider a collection of smooth functions $\varphi_i : \mathbb{R}^n \to \mathbb{R}$ and constants $N, M_0, M_1, \ldots < \infty$ such that:
(3.2.27.1) $\|\varphi_i\|_{C^k} \leq M_k$ for all i.
(3.2.27.2) Every ball of unit radius intersects $\operatorname{supp}\varphi_i$ for at most N indices i.

Show that the maps

$$C_c^\infty(\mathbb{R}^n) \to \bigoplus_i C_c^\infty(\mathbb{R}^n) \qquad\qquad \bigoplus_i C_c^\infty(\mathbb{R}^n) \to C_c^\infty(\mathbb{R}^n) \tag{3.2.27.3}$$

$$u \mapsto \big(\varphi_i u\big)_i \qquad\qquad (u_i)_i \mapsto \sum_i \varphi_i u_i \tag{3.2.27.4}$$

are bounded in terms of $N, M_0, M_1, \ldots$ and s when we equip $C_c^\infty(\mathbb{R}^n)$ with the $H^s(\mathbb{R}^n)$-norm and we equip $\bigoplus_i C_c^\infty(\mathbb{R}^n)$ with the norm $\|(u_i)_i\|^2 = \sum_i \|u_i\|_s^2$ (verify explicitly for integer $s \geq 0$, then use interpolation and duality). Now show that for any such collection of functions φ_i, there exists another collection ψ_i (same index set) with $\psi_i \equiv 1$ over $\operatorname{supp}\varphi_i$, which is also bounded in the above sense. Conclude that $\|u_s\|^2 \asymp \sum_i \|\varphi_i u\|_s^2$ for any partition of unity on $\mathbb{R}^n$ of bounded geometry (with constants depending on such bounds on geometry).

* *

⋆ **3.2.28 Lemma** (Sobolev embedding)**.** *For integer $k \geq 0$ and real $s > k + \frac{n}{2}$, we have $\|u\|_{C^k} \leq \mathrm{const}_{K,k,s}\|u\|_{H^s}$ for $\operatorname{supp} u \subseteq K$ (K compact).*

Proof. Differentiation D^α on $\mathbb{R}^n$ is bounded $H^s \to H^{s-|\alpha|}$ by direct calculation for integer $s \geq |\alpha|$, hence for all real $s \geq |\alpha|$ by interpolation (3.2.19)(3.2.20). It therefore suffices to treat the case $k = 0$, where we are supposed to show that

$$|u(0)|^2 = \left|\int e^{2\pi i\langle\xi,x\rangle}\hat{u}(\xi)\,d\xi\right|^2 \leq \mathrm{const}_s \int |\hat{u}(\xi)|^2(1+|\xi|)^{2s}\,d\xi. \tag{3.2.28.1}$$

This follows from Cauchy–Schwarz provided $\int(1+|\xi|)^{-2s}\,d\xi < \infty$, which is the case for $s > \frac{n}{2}$ (using polar coordinates, it is equivalent to $\int_1^\infty r^{-2s}r^{n-1}\,dr < \infty$). □

3.2.29 Lemma (Sobolev restriction)**.** *For a codimension d submanifold $N \subseteq M$, we have $\|u|_N\|_s \leq \mathrm{const}_{K,s}\|u\|_{s+d/2}$ for $\operatorname{supp} u \subseteq K$ (K compact) and $s > 0$.*

Proof. It suffices to show that restriction of smooth functions $\mathbb{R}^n$ to $\mathbb{R}^{n-1} \times 0$ is bounded $H^{s+1/2}(\mathbb{R}^n) \to H^s(\mathbb{R}^{n-1})$ provided $s > 0$.

Fix coordinates $(x, y) \in \mathbb{R}^{n-1} \times \mathbb{R}$ and dual coordinates $(\xi, \eta) \in \mathbb{R}^{n-1} \times \mathbb{R}$. The Fourier transform of the restriction $f(x, 0)$ is the integral $\int \hat{f}(\xi, \eta)\,d\eta$. Our desired estimate is thus

$$\int (1+|\xi|^2)^s \left|\int \hat{f}(\xi,\eta)\,d\eta\right|^2 d\xi \leq \mathrm{const}_s \int (1+|\xi|^2+|\eta|^2)^{s+\frac{1}{2}}|\hat{f}(\xi,\eta)|^2\,d\eta\,d\xi. \tag{3.2.29.1}$$

Cauchy–Schwarz gives

$$\left|\int \hat{f}(\xi,\eta)\,d\eta\right|^2 \leq \int (1+|\xi|^2+|\eta|^2)^{-s-\frac{1}{2}}\,d\eta \int (1+|\xi|^2+|\eta|^2)^{s+\frac{1}{2}}|\hat{f}(\xi,\eta)|^2\,d\eta, \tag{3.2.29.2}$$

so it suffices to show that

$$\int (1+|\xi|^2+|\eta|^2)^{-s-\frac{1}{2}}\,d\eta \leq \mathrm{const}_s(1+|\xi|^2)^{-s}. \tag{3.2.29.3}$$

Writing $1+|\xi|+|\eta| = (1+|\xi|)(1+\frac{|\eta|}{1+|\xi|})$ and performing the change of variables $\eta = (1+|\xi|)t$, we are reduced to the inequality $\int (1+|t|)^{-2s-1}\,dt \leq \mathrm{const}_s$, which holds since $s > 0$. □

The following result is a quantitative version of 'compactness' of the inclusion of Sobolev spaces $H^s \to H^t$ for $s > t$.

3.2.30 Rellich Lemma ([95, 61])**.** *For every $s < t \in \mathbb{R}$ and every $\varepsilon > 0$, there exists a finite list of functions $\rho_1, \ldots, \rho_N \in C_c^\infty(\mathbb{R}^n)$ such that we have*

$$\|u\|_s \leq \varepsilon\|u\|_t + \sum_i \left|\int \rho_i u\right| \tag{3.2.30.1}$$

for all $u \in C^\infty(\mathbb{R}^n)$ supported inside the unit ball. The same holds for any manifold M and functions supported in any given compact set $K \subseteq M$.

Proof. For functions on the torus $M = \mathbb{R}^n/\mathbb{Z}^n$, the Sobolev norm is expressible in Fourier series $\|u\|_s^2 = \sum_m |\hat{u}(m)|^2(1+|m|^2)^s$, so we may take $\rho_1, \ldots, \rho_N$ to be large multiples of the Fourier modes $e^{2\pi imx}$ for $|m|^2 \leq M$ for suitable $M < \infty$ depending on $\varepsilon > 0$. By embedding the unit ball into the torus, we conclude the case of the unit ball as well. The Sobolev norm on a general manifold is defined in terms of the Sobolev norm on the unit ball using a partition of unity (3.2.22), so the case of the unit ball implies the general case. □

3.2.31 Lemma. *For any compact manifold M, there exists a sequence of finite rank endomorphisms $\pi_k : C^\infty(M) \to C^\infty(M)$ with $\|\pi_k\|_{(s,s)} \leq \mathrm{const}_s$ and $\pi_k \to \mathbf{1}$ in $H^s \to H^t$ operator norm for all $s > t$.*

Proof. Cover M by finitely many open sets $U_i \subseteq M$, and let $\varphi_i : M \to \mathbb{R}$ be a subordinate partition of unity. Further fix $\psi_i : M \to \mathbb{R}$ supported inside U_i so that $\psi_i \equiv 1$ over $\operatorname{supp}\varphi_i$. We will take

$$\pi_k = \sum_i \psi_i \pi_{k,i} \varphi_i \tag{3.2.31.1}$$

for certain $\pi_{k,i} : C_c^\infty(U_i) \to C_{\mathrm{loc}}^\infty(U_i)$. Choose an embedding $U_i \subseteq \mathbb{R}^n/\mathbb{Z}^n$, and take $\pi_{k,i}$ to be (the restriction of) the map $C^\infty(\mathbb{R}^n/\mathbb{Z}^n) \to C^\infty(\mathbb{R}^n/\mathbb{Z}^n)$ given on Fourier series by multiplication by the characteristic function of $[-k,k]^n$. Certainly each $\pi_{k,i}$ has finite rank, hence so does π_k. The operator norm of $\pi_{k,i}$ acting on $H^s(\mathbb{R}^n/\mathbb{Z}^n)$ is precisely 1 when the H^s-norm is defined in the usual way via Fourier series $\|u\|_s = (\sum_n (1+|n|^2)^s|\hat{u}(n)|^2)^{1/2}$, so we have $\|\pi_k\|_{(s,s)}$ bounded uniformly on k. Finally, to show that $\pi_k \to \mathbf{1}$ in $H^s \to H^t$ operator norm for $s > t$, write $\pi_k - \mathbf{1} = \sum_i \psi_i(\pi_{k,i} - \mathbf{1})\varphi_i$ for

$$\begin{array}{ccccc}
C^\infty(M) & \xrightarrow{\varphi_i} & C^\infty_{\operatorname{supp}\varphi_i}(U_i) & \hookrightarrow & C^\infty(\mathbb{R}^n/\mathbb{Z}^n) \\
 & & & & \Big\downarrow{\scriptstyle \pi_{k,i}-\mathbf{1}} \\
C^\infty(M) & \hookleftarrow & C^\infty_{\operatorname{supp}\psi_i}(U_i) & \xleftarrow{\psi_i} & C^\infty(\mathbb{R}^n/\mathbb{Z}^n)
\end{array} \tag{3.2.31.2}$$

and note that $\|\pi_{k,i} - \mathbf{1}\|_{(t,s)} \to 0$ as $k \to \infty$ for all $s > t$ (while everything else is independent of k). $\square$

* *

The next result allows us to define Sobolev spaces of maps to non-linear targets (i.e. manifolds) whenever $H^s \subseteq C^0$ (3.2.28) and s is an integer.

3.2.32 Proposition (Moser [85, §2])**.** *Let $s \geq 0$ be an integer for which $H^s \subseteq C^0$, and suppose $F : \mathbb{R}^n \to \mathbb{R}$ is smooth and vanishes at the origin. For compact $K \subseteq \mathbb{R}^m$, we have*

$$\|F(g)\|_{H^s} \leq \mathrm{const}_{K,s} \|F\|_{C^s(g(K))} \|g\|_{H^s} \tag{3.2.32.1}$$

for $\operatorname{supp} g \subseteq K$.

Proof. Derivatives of $F(g)$ of order $\leq s$ are sums of terms of the form

$$(D^\alpha F)(g) \prod_{i=1}^{|\alpha|} D^{\beta_i} g \tag{3.2.32.2}$$

with $|\beta_i| \geq 1$ and $\sum_i |\beta_i| \leq s$ (in particular, $|\alpha| \leq s$). In the case $\alpha = 0$, we note that $\|F(g)\|_{L^2}$ is bounded by a constant times $\|g\|_{L^2}$ since $F(0) = 0$. For $|\alpha| \geq 1$, the factor $(D^\alpha F)(g)$ is bounded uniformly by $\|F\|_{C^s(K)}$, so it suffices to show that

$$\left\| \prod_{i=1}^{|\alpha|} D^{\beta_i} g \right\|_{L^2} \leq \mathrm{const}_{K,s} \|g\|_{H^s}. \tag{3.2.32.3}$$

Using Hölder's inequality (**??**) and compactness of K, it suffices to show $\|D^\beta g\|_{L^{2s/|\beta|}} \leq \mathrm{const}_{K,s} \|g\|_{H^s}$. That is, we should show that $H^{s-r} \to L^{2s/r}$ is bounded for integers $r = 1, \ldots, s$. For $r = 0$, this is the assumption that $H^s \to C^0$ is bounded, and for $r = s$ this is the definition $H^0 = L^2$. It thus follows for general $r \in [0, s]$ by interpolation (3.2.19)(3.2.20)(**??**). $\square$

3.2.33 Corollary. *In the setup of* (3.2.32)*, if F vanishes to order $m \geq 1$ at the origin and $\|g\|_{C^0} \leq 1$, then $\|F(g)\|_{H^s} \leq \mathrm{const}_{K,s,m} \|F\|_{C^{\max(s,m)}(B(\|g\|_{C^0}))} \|g\|_{C^0}^{m-1} \|g\|_{H^s}$.*

Proof. Let $F_\varepsilon(x) = \varepsilon^{-m} F(\varepsilon x)$, and note that

$$\|F_\varepsilon\|_{C^s(B(1))} \leq \mathrm{const}_{s,m} \|F\|_{C^{\max(s,m)}(B(\varepsilon))} \tag{3.2.33.1}$$

for all $0 < \varepsilon \leq 1$ since F vanishes to order m at the origin. Now take $\varepsilon = \|g\|_{C^0}$ and write

$$\|F(g)\|_{H^s} = \varepsilon^m \|F_\varepsilon(\varepsilon^{-1} g)\|_{H^s} \overset{(3.2.32)}{\leq} \mathrm{const}_s \cdot \varepsilon^m \|F_\varepsilon\|_{C^s(B(1))} \|\varepsilon^{-1} g\|_{H^s} \tag{3.2.33.2}$$

which combines with (3.2.33.1) to give the desired bound. $\square$

3.2.34 Exercise ($H^s \subseteq C^0$ is an algebra)**.** Let $s \geq 0$ be an integer for which $H^s \subseteq C^0$. Use (3.2.32) and rescaling as in (3.2.33) to show that $\|fg\|_{H^s} \leq \mathrm{const}_{K,s}(\|f\|_{C^0}\|g\|_{H^s} + \|f\|_{H^s}\|g\|_{C^0})$ for $\operatorname{supp} f, \operatorname{supp} g \subseteq K$.

3.2.35 Exercise. Let $s \geq 0$ be an integer for which $H^s \subseteq C^0$. Show that if A vanishes along $\mathbb{R}^n \times \mathbb{R}^m \times 0$ and vanishes to order two along $\mathbb{R}^n \times 0 \times 0$, then

$$\|A(x, f(x), g(x))\|_{H^s} \leq \mathrm{const}_{K,s}\|A\|_{C^{\max(s,2)}(B(1))} \cdot (\|f\|_{H^s}\|g\|_{C^0} + (\|f\|_{C^0} + \|g\|_{C^0})\|g\|_{H^s}) \quad (3.2.35.1)$$

for $\|f\|_{C^0}, \|g\|_{C^0} \leq 1$ and $\operatorname{supp} f, \operatorname{supp} g \subseteq K$ (split into the two cases $\|f\|_{C^0} \geq \|g\|_{C^0}$ and $\|f\|_{C^0} \leq \|g\|_{C^0}$, and use (3.2.32) and rescaling as in (3.2.33)). Make a change of variables to conclude that if B vanishes along $\mathbb{R}^n \times \Delta_{\mathbb{R}^m}$ and to order two along $\mathbb{R}^n \times 0 \times 0$, then

$$\|B(x, f(x), g(x))\|_{H^s} \leq \mathrm{const}_{K,s}\|B\|_{C^{\max(s,2)}(B(1))} \cdot ((\|f\|_{H^s} + \|g\|_{H^s})\|f - g\|_{C^0} + (\|f\|_{C^0} + \|g\|_{C^0})\|f - g\|_{H^s}) \quad (3.2.35.2)$$

for $\|f\|_{C^0}, \|g\|_{C^0} \leq 1$ and $\operatorname{supp} f, \operatorname{supp} g \subseteq K$.

3.3 Differential operators

⋆ **3.3.1 Definition** (Differential operator)**.** On a manifold M carrying vector bundles E and F, a differential operator $L : C^\infty_{\mathrm{loc}}(M, E) \to C^\infty_{\mathrm{loc}}(M, F)$ of order $\leq m$ is a map which is given in local coordinates $M \supseteq U \subseteq \mathbb{R}^n$ by an expression of the form

$$Lf = \sum_{|\alpha| \leq m} c_\alpha D^\alpha f \tag{3.3.1.1}$$

where c_α are smooth functions taking values in $\mathrm{Hom}(E, F)$ (an operator being of this form is evidently preserved by diffeomorphisms).

The order m terms transform under diffeomorphisms independently of the others, so a differential operator L of order $\leq m$ has a well-defined order m term. Intrinsically, this order m term is an element of $\mathrm{Hom}(E, F) \otimes (TM^{\otimes m})_{S_m}$ or, equivalently, a homogeneous degree m polynomial map $T^*M \to \mathrm{Hom}(E, F)$ called the *order m symbol* of L.

It is evident that for a differential operator L of order $\leq m$, we have for all compact $K \subseteq M$ that

$$\|Lu\|_{C^k} \leq \mathrm{const}_{L,K,k} \|u\|_{C^{k+m}} \tag{3.3.1.2}$$

for $\operatorname{supp} u \subseteq K$. To interpret this estimate, we remind the reader of our convention for norms of functions on manifolds. Such a norm depends on a choice of covering family of charts and subordinate partition of unity, and is well-defined only up to constant factor. The constant appearing in estimates such as (3.3.1.2) thus depends on the choice of data used to define the C^k_K-norm on M, although we systematically omit this dependence from the notation (it can be regarded as part of the explicitly indicated dependence on k).

3.3.2 Definition (Formal adjoint)**.** For a differential operator $L : C^\infty_{\mathrm{loc}}(M, E) \to C^\infty_{\mathrm{loc}}(M, F)$, its formal adjoint is the differential operator

$$L^* : C^\infty_{\mathrm{loc}}(M, F^* \otimes \Omega_M) \to C^\infty_{\mathrm{loc}}(M, E^* \otimes \Omega_M) \tag{3.3.2.1}$$

defined by the property $\int_M \langle u, Lv \rangle = \int_M \langle L^* u, v \rangle$ (say for u and v of compact support), where Ω_M denotes the bundle of densities on M. In other words, L^* is obtained from L by formally integrating by parts.

3.3.3 Exercise. Show that a differential operator $L : C^\infty_{\mathrm{loc}}(M, E) \to C^\infty_{\mathrm{loc}}(M, F)$ admits a unique continuous extension $L : C^{-\infty}_{\mathrm{loc}}(M, E) \to C^{-\infty}_{\mathrm{loc}}(M, F)$.

3.3.4 Exercise. Let $w \in C^{-\infty}_{\mathrm{loc}}(\mathbb{R}^n)$ be a distribution supported (in the sense of (**??**)) at the origin. Show that w is a linear combination of the delta function (3.2.8) and its derivatives.

3.3.5 Exercise. Let $\Delta = \partial_x^2 + \partial_y^2$ on $\mathbb{R}^2$, and show that $\Delta(\log r) = 2\pi\delta_0$ (after first making precise how the function $\log r$ defines a distribution on $\mathbb{R}^2$).

3.3.6 Exercise. Let $\partial_{\bar z} = \frac{1}{2}(\partial_x + i\partial_y)$ on $\mathbb{C}$, and show that $\partial_{\bar z}(z^{-1}) = \pi\delta_0$ (after first making precise how the function z^{-1} defines a distribution on $\mathbb{C}$).

⋆ **3.3.7 Proposition.** *Let $L : E \to F$ be a differential operator on M of order $\leq m$. For any compact $K \subseteq M$, we have $\|Lu\|_s \leq \mathrm{const}_{L,K,s}\|u\|_{s+m}$ for $\operatorname{supp} u \subseteq K$.*

Proof. The statement is local, so it suffices to consider the case of differential operators on $M = \mathbb{R}^n$ and functions u supported inside the unit ball.

We begin by considering the case that L has *constant coefficients*, namely $L : C^\infty_{\mathrm{loc}}(\mathbb{R}^n, E) \to C^\infty_{\mathrm{loc}}(\mathbb{R}^n, F)$ for vector spaces E and F takes the form $Lf = \sum_{|\alpha|\leq m} c_\alpha D^\alpha f$ for constants $c_\alpha \in \mathrm{Hom}(E, F)$. In this case, we have

$$\widehat{Lu}(\xi) = P(2\pi i\xi)\hat{u}(\xi) \tag{3.3.7.1}$$

for $P(\xi) = \sum_{|\alpha|\leq m} c_\alpha \xi^\alpha$. Since P is a polynomial of degree $\leq m$, we conclude that $\|Lu\|_s \leq \mathrm{const}_{L,s}\|u\|_{s+m}$ for all u on $\mathbb{R}^n$.

We now consider the general case of variable coefficient operators L on $\mathbb{R}^n$. Since we are considering functions u supported inside a fixed compact set, we may assume the same for L. Write $Lf = \sum_{|\alpha|\leq m} c_\alpha D^\alpha f$ for functions c_α supported in the unit ball, and let $P(x, \xi) = \sum_{|\alpha|\leq m} c_\alpha(x)\xi^\alpha$, so we have

$$Lu(x) = \int e^{2\pi i\langle \xi,x\rangle} P(x, 2\pi i\xi)\hat{u}(\xi)\, d\xi \tag{3.3.7.2}$$

by differentiating under the integral sign. We may thus calculate

$$\begin{aligned}
\widehat{Lu}(\zeta) &= \int e^{-2\pi i\langle \zeta,x\rangle} \int e^{2\pi i\langle \xi,x\rangle} P(x, 2\pi i\xi)\hat{u}(\xi)\, d\xi\, dx && (3.3.7.3)\\
&= \int \hat{u}(\xi) \int e^{2\pi i\langle \xi-\zeta,x\rangle} P(x, 2\pi i\xi)\, dx\, d\xi && (3.3.7.4)\\
&= \int K_P(\zeta, \xi)\hat{u}(\xi)\, d\xi \quad \text{for } K_P(\zeta, \xi) = \int e^{2\pi i\langle \xi-\zeta,x\rangle} P(x, 2\pi i\xi)\, dx && (3.3.7.5)
\end{aligned}$$

where the interchange of integrals is justified by the fact that the x-support of P is compact and $\hat{u}(\xi)$ is rapidly decaying. Note that the kernel K_P is bounded by

$$|K_P(\zeta, \xi)| = \mathrm{const}_{L,N} \cdot (1 + |\xi - \zeta|)^{-N} \cdot (1 + |\xi|)^m \tag{3.3.7.6}$$

for any $N < \infty$ (integrate by parts N times in the direction of $\xi - \zeta$ if $|\xi - \zeta| \geq 1$).

We now claim that the desired bound

$$\int |\widehat{Lu}(\zeta)|^2(1 + |\zeta|^2)^s\, d\zeta \leq \mathrm{const}_{L,s} \int |\hat{u}(\xi)|^2(1 + |\xi|^2)^{s+m}\, d\xi \tag{3.3.7.7}$$

follows from the estimate (3.3.7.6) on the kernel $K_P(\zeta, \xi)$. First of all, we have

$$\begin{aligned}
|\widehat{Lu}(\zeta)|^2 &= \left(\int K_P(\zeta, \xi)\hat{u}(\xi)\, d\xi\right)^2 && (3.3.7.8)\\
&\leq \mathrm{const}_{L,N}\left(\int |\hat{u}(\xi)|(1 + |\xi|)^m(1 + |\zeta - \xi|)^{-N}\, d\xi\right)^2 && (3.3.7.9)\\
&\leq \mathrm{const}_{L,N} \int |\hat{u}(\xi)|^2(1 + |\xi|)^{2m}(1 + |\xi - \zeta|)^{-N}\, d\xi && (3.3.7.10)
\end{aligned}$$

by Cauchy–Schwarz. Now multiply this by $(1+|\zeta|^2)^s$ and integrate $\int d\zeta$. Then apply the bound $\int(1+|\xi-\zeta|)^{-N}(1+|\zeta|^2)^s\,d\zeta \leq \text{const}_{s,N}(1+|\xi|^2)^s$ on the right to obtain (3.3.7.7). $\square$

3.3.8 Exercise. Prove (3.3.7) using interpolation and duality as in (3.2.21).

3.4 Elliptic operators

We review the theory of elliptic operators. References include Lawson–Michelsohn [66, III], Atiyah–Bott [10, §§3,6], Wells [109, IV 3–4], and Egorov–Schulze [24, §§1–4].

⋆ **3.4.1 Definition** (Elliptic operator)**.** A differential operator of order $\leq m$ is called *elliptic of order m* when its order m symbol $T^*M \to \operatorname{Hom}(E,F)$ sends nonzero elements $\xi \in T^*M$ to invertible elements of $\operatorname{Hom}(E,F)$.

3.4.2 Example. The operator $\sum_i \partial_{x_i}^2$ on functions $\mathbb{R}^n \to \mathbb{R}$ is elliptic (its symbol is $\sum_i \xi_i^2$). The operators $\partial_x \pm i\partial_y$ on functions $\mathbb{R}^2 \to \mathbb{C}$ are elliptic (their symbols are $\xi_1 \pm i\xi_2$).

Roughly speaking, an operator is elliptic when it is 'invertible on high frequencies' (indeed, the symbol of an operator describes, to leading order, its action on a given frequency). We shall see below that this implies elliptic operators are 'almost invertible', in the sense that every order m elliptic operator L has a *parametrix* Q, which is an operator of 'order $-m$' such that both operators $\mathbf{1} - LQ$ and $\mathbf{1} - QL$ have 'order $-\infty$'. Having 'order r' in the relevant sense, to be made precise below, means, in particular, being bounded $H^s \to H^{s-r}$. The existence of parametrices leads to many strong results about elliptic operators.

* *

Parametrices are not differential operators, rather they belong to the broader class of 'semi-local' operators which we now define.

3.4.3 Definition (Support of an operator)**.** Let $Q : C_c^\infty(M) \to C_{\mathrm{loc}}^\infty(M')$ be a linear operator. The *support* of Q is the closed subset $\operatorname{supp} Q \subseteq M' \times M$ defined by the property that $(p',p) \notin \operatorname{supp} Q$ iff there exist neighborhoods U, U' of p, p' such that $\operatorname{supp} u \subseteq U$ implies $Qu|_{U'} \equiv 0$.

3.4.4 Exercise. Show that the support of a differential operator on M is contained in the diagonal of $M \times M$.

3.4.5 Exercise. Let $Q : C_c^\infty(M) \to C^\infty(M')$ be a linear operator. Show that $\operatorname{supp} Qu$ is contained in the image of $Q \times_M \operatorname{supp} u \to M'$.

3.4.6 Exercise. Let $Q : C_c^\infty(M) \to C_{\mathrm{loc}}^\infty(M')$ be a linear operator. Show that:
(3.4.6.1) If $\operatorname{supp} Q$ is proper over M, then $Q : C_c^\infty \to C_c^\infty$.
(3.4.6.2) If $\operatorname{supp} Q$ is proper over M', then Q has a canonical extension $C_{\mathrm{loc}}^\infty \to C_{\mathrm{loc}}^\infty$.
(3.4.6.3) If $\operatorname{supp} Q$ is compact, then $Q : C_{\mathrm{loc}}^\infty \to C_c^\infty$.
We say that Q is *semi-local* when $\operatorname{supp} Q$ is proper over both M and M'.

3.4.7 Exercise. Let $P : C_c^\infty(M) \to C_{\mathrm{loc}}^\infty(M')$ and $Q : C_c^\infty(M') \to C_{\mathrm{loc}}^\infty(M'')$ be operators for which $\operatorname{supp} Q \times_{M'} \operatorname{supp} P \to M'' \times M$ is proper. Show that there is a canonical 'composition' $QP : C_c^\infty(M) \to C_{\mathrm{loc}}^\infty(M'')$ whose support is contained in the image of this map. Conclude that any composition of semi-local operators is defined and semi-local.

⋆ **3.4.8 Definition** (Operator of order $\le m$)**.** A linear operator $Q : C_c^\infty(M) \to C_{\mathrm{loc}}^\infty(M')$ is said to have order $\le m$ iff for every compact set $K \subseteq M$, every compactly supported smooth $\varphi : M' \to \mathbb{R}$, and every $s \in \mathbb{R}$, we have

$$\|\varphi \cdot Qu\|_s \le \mathrm{const}_{K,\varphi,s}\|u\|_{s+m} \tag{3.4.8.1}$$

for $\operatorname{supp} u \subseteq K$. A *smoothing operator* is an operator of order $-\infty$, meaning order $\le -N$ for every $N < \infty$.

3.4.9 Exercise. Let L be a differential operator. We saw in (3.3.7) that if L has order $\le m$ as a differential operator, then it has order $\le m$ in the sense of (3.4.8). Show the converse, namely that if L has order $\le m$ in the sense of (3.4.8), then it has order $\le m$ as a differential operator.

3.4.10 Exercise. Show that order is subadditive under composition, namely that if Q and Q' have order $\le m$ and $\le m'$ and satisfy the criterion for existence of the composition QQ' (3.4.7), then QQ' has order $\le m + m'$.

⋆ **3.4.11 Definition** (Parametrix)**.** Let L be an elliptic operator of order m. A *parametrix* for L is a semi-local operator Q of order $\le -m$ for which $\mathbf{1} - QL$ and $\mathbf{1} - LQ$ are both smoothing operators (thus a parametrix is an 'inverse modulo smoothing operators'). The analogous notion of a left (resp. right) parametrix requires that only $\mathbf{1} - QL$ (resp. $1 - LQ$) be a smoothing operator.

3.4.12 Exercise. Fix an elliptic operator L of order m. Show that if Q is a left parametrix for L and $Q' - Q$ is a smoothing operator, then Q' is a left parametrix for L. Show that if Q is a left parametrix and Q' is a right parametrix, then $Q - Q'$ is a smoothing operator (consider $Q(\mathbf{1} - LQ') - (\mathbf{1} - QL)Q'$), and hence both Q and Q' are parametrices for L.

* *

To construct parametrices for elliptic operators with variable coefficients, we will study the following general class of operators.

⋆ **3.4.13 Definition** (Pseudo-differential operator)**.** A pseudo-differential operator $T_A : C_c^\infty(\mathbb{R}^n) \to C_c^\infty(\mathbb{R}^n)$ is an operator of the form

$$(T_A u)(x) = \int e^{2\pi i\langle \xi, x\rangle} A(x,\xi)\hat{u}(\xi)\, d\xi. \tag{3.4.13.1}$$

where A has compact x-support and is a *symbol of order* $\le m$, meaning that $|D_x^\alpha D_\xi^\beta A(x,\xi)| \le \mathrm{const}_{\alpha,\beta} \cdot (1 + |\xi|)^{m-|\beta|}$.

3.4.14 Example. Compactly supported differential operators on $\mathbb{R}^n$ of order $\le m$ are precisely the pseudo-differential operators (3.4.13.1) in which $A = \sum_{|\alpha|\le k} c_\alpha(x)(2\pi i\xi)^\alpha$ is a polynomial of degree $\le k$ in ξ with coefficients which are smooth compactly supported functions of x.

3.4.15 Lemma. *If A is a symbol of order $\le m$, then T_A is an operator of order $\le m$.*

Proof. The reasoning used to treat the special case of differential operators (3.3.7) applies with little change. We have

$$\widehat{T_A u}(\zeta) = \int K_A(\zeta,\xi)\hat{u}(\xi)\,d\xi \quad \text{where } K_A(\zeta,\xi) = \int e^{2\pi i\langle \xi-\zeta, x\rangle} A(x,\xi)\,dx. \tag{3.4.15.1}$$

The fact that A is a symbol of order $\le m$ with compact x-support implies that

$$|K_A(\zeta,\xi)| \le \mathrm{const}_{A,N}\cdot(1+|\xi-\zeta|)^{-N}\cdot(1+|\xi|)^m \tag{3.4.15.2}$$

for all $N<\infty$ by integrating by parts. This bound on K_A implies the desired estimate as in (3.3.7). □

⋆ **3.4.16 Proposition** (Composition of pseudo-differential operators)**.** *Fix operators T_A and T_B of the form* (3.4.13.1) *where A and B are symbols of order $\le m_A$ and $\le m_B$, respectively, of compact spatial support. We have $T_A\circ T_B = T_C$ where C is a symbol of order $\le m_A+m_B$ and has asymptotic expansion*

$$C(x,\xi) \sim \sum_\alpha \frac{D_\xi^\alpha A(x,\xi) D_x^\alpha B(x,\xi)}{\alpha!(2\pi i)^{|\alpha|}} \tag{3.4.16.1}$$

where $\alpha! = \prod_i \alpha_i!$. The meaning of this asymptotic expansion (3.4.16.1) *is that the difference between C and the sum of terms on the right with $|\alpha|<N$ is a symbol of order m_A+m_B-N.*

Proof. As we have seen, the action of an operator of the form T_A on Fourier transforms is given by integration against a corresponding kernel K_A (3.4.15.1). The decay properties of these kernels (3.4.15.2) justify the exchange of integrals needed to show that a composition of such operators is given by the composition of their kernels:

$$(T_A T_B u)^\wedge(\eta) = \int K_C(\eta,\xi)\hat{u}(\xi)\,d\xi \quad \text{for } K_C(\eta,\xi) = \int K_A(\eta,\zeta)K_B(\zeta,\xi)\,d\zeta. \tag{3.4.16.2}$$

Now let us write the composed kernel K_C as follows.

$$K_C(\eta,\xi) = \iiint e^{2\pi i\langle\zeta-\eta,y\rangle}A(y,\zeta)e^{2\pi i\langle\xi-\zeta,x\rangle}B(x,\xi)\,dy\,dx\,d\zeta \tag{3.4.16.3}$$

$$= \int e^{2\pi i\langle\xi-\eta,y\rangle}\left[\iint e^{2\pi i\langle\zeta-\xi,y-x\rangle}A(y,\zeta)B(x,\xi)\,dx\,d\zeta\right]dy \tag{3.4.16.4}$$

$$= \int e^{2\pi i\langle\xi-\eta,y\rangle}\left[\iint e^{-2\pi i\langle\beta,t\rangle}A(y,\xi+\beta)B(y+t,\xi)\,dt\,d\beta\right]dy \tag{3.4.16.5}$$

At least formally, the bracketed expression in the middle will be our new symbol C, however we still need to justify the interchange of order of integration. We begin with the first triple integral (3.4.16.3). It is not absolutely convergent: it is defined (so that it equals the integral of K_A against K_B) by first integrating with respect to x and y (in which the integrand has compact support) to get something with rapid decay in ζ, and then integrating $d\zeta$. However, merely doing first the integral dx already gives us rapid decay in ζ, so we can interchange the y integral and ζ integral. This justifies the integral manipulation above. Now we define

$$C(y,\xi) = \iint e^{-2\pi i\langle \beta,t\rangle} A(y,\xi+\beta)B(y+t,\xi)\,dt\,d\beta \tag{3.4.16.6}$$

(note the order of integration: after doing the integral dt, we have rapid decay in β), so we have $T_A \circ T_B = T_C$.

It remains to show that C admits the asymptotic expansion (3.4.16.1) (and thus is a symbol of order $\leq m_A + m_B$). The key to proving the asymptotic expansion of C is to consider the Taylor expansion

$$A(y,\xi+\beta) \sim \sum_\alpha \frac{D_\xi^\alpha A(y,\xi)}{\alpha!}\beta^\alpha. \tag{3.4.16.7}$$

If in the definition of C we replace $A(y,\xi+\beta)$ by this Taylor expansion, we obtain precisely the asymptotic expansion (3.4.16.1) (we have $\iint e^{-2\pi i\langle\beta,t\rangle}\beta^\alpha B(y+t,\xi)\,dt\,d\beta = (2\pi i)^{-\alpha} D_y^\alpha B(y,\xi)$ by Fourier inversion and integration by parts). It thus suffices to show that the error in the Taylor expansion above contributes a symbol of order $\leq m_A + m_B - N$ to C.

We wish to show that the expression

$$R(y,\xi) = \int \left[\int e^{-2\pi i\langle\beta,t\rangle} B(y+t,\xi)\,dt\right]\left(A(y,\xi+\beta) - \sum_{|\alpha|<N} \frac{D_\xi^\alpha A(y,\xi)}{\alpha!}\beta^\alpha\right)d\beta, \tag{3.4.16.8}$$

is a symbol of order $\leq m_A + m_B - N$. That is, we should show that $D_y^\gamma D_\xi^\delta R(y,\xi)$ is bounded by $\mathrm{const}_{\gamma,\delta}\cdot(1+|\xi|)^{m_A+m_B-N-|\delta|}$. Now note that the derivatives D_y^γ and D_ξ^δ fall on A and B, producing symbols whose orders sum to $m_A+m_B-|\delta|$. The estimate for general (γ,δ) thus follows from the special case of $\gamma=\delta=0$ (for different A and B). It therefore suffices to show that $|R(y,\xi)| \leq \mathrm{const}(1+|\xi|)^{m_A+m_B-N}$.

The function $R(y,\xi)$ is an integral $d\beta$ of a product of two factors, which we bound separately. The first factor (bracketed integral dt) is bounded by $\mathrm{const}_M(1+|\xi|)^{m_B}(1+|\beta|)^{-M}$ for any $M<\infty$ since B is a symbol of order m_B (3.4.15.2). The second factor (Taylor remainder in parentheses) is bounded by $\mathrm{const}_N|\beta|^N(1+|\xi|+|\beta|)^{m_A-N}$ by the Taylor remainder theorem since A is a symbol of order m_A. We are therefore left with showing that

$$\int (1+|\beta|)^{-M}(1+|\xi|+|\beta|)^{m_A-N}\,d\beta \leq \mathrm{const}_N(1+|\xi|)^{m_A-N} \tag{3.4.16.9}$$

for some $M<\infty$. Over the locus $|\beta|\geq|\xi|$, the integrand is bounded by $(1+|\beta|)^{m_A-N-M}$ (up to constant factor), hence the integral decays faster than any power of $|\xi|$ by taking M large. Over the locus $|\beta|\leq|\xi|$, the integrand is bounded by $(1+|\beta|)^{-M}(1+|\xi|)^{m_A-N}$ (up to constant factor), hence has integral bounded by $(1+|\xi|)^{m_A-N}$. $\square$

* *

We now explain how the existence of parametrices for elliptic operators is straightforward given the asymptotic composition formula (3.4.16). We consider the generality of symbols defined not on all of $\mathbb{R}^n$, rather on open subsets thereof.

3.4.17 Definition (Spaces of symbols S^m, S, $S^{-\infty}$)**.** For any domain $\Omega \subseteq \mathbb{R}^n$, denote by $S^m(\Omega)$ the space of symbols of order $\leq m$, namely smooth functions A on $\Omega \times \mathbb{R}^n$ satisfying $|D_x^\alpha D_\xi^\beta A(x,\xi)| \leq \mathrm{const}_{\alpha,\beta} \cdot (1+|\xi|)^{m-|\beta|}$. Denote by $S_c^m(\Omega) \subseteq S^m(\Omega)$ the symbols supported inside $K \times \mathbb{R}^n$ for some compact $K \subseteq \Omega$. Let $S = \bigcup_m S^m$ be the ascending union of the spaces S^m, and let $S^{-\infty} := \bigcap_m S^m$ be their intersection.

Associated to a symbol in $S(\Omega)$ (resp. $S_c(\Omega)$) is a pseudo-differential operator $C_c^\infty(\Omega) \to C_{\mathrm{loc}}^\infty(\Omega)$ (resp. $C_c^\infty(\Omega) \to C_c^\infty(\Omega)$. The operator associated to a symbol of order $\leq m$ has order $\leq m$ by (3.4.15). Composition of compactly supported symbols is defined by composition of operators (3.4.16).

3.4.18 Definition (φ-parametrix)**.** Let L be an elliptic operator of order m and let φ be a smooth function of compact support. A *left (resp. right) φ-parametrix* for L is a compactly supported (3.4.6.3) operator Q of order $\leq -m$ for which $\varphi - QL$ (resp. $\varphi - LQ$) is a smoothing operator.

3.4.19 Corollary. *Every elliptic operator L on an open set $\Omega \subseteq \mathbb{R}^n$ has left and right φ-parametrices for every $\varphi \in C_c^\infty(\Omega)$.*

Proof. By (**??**), there exists a symbol $Q \in S^{-m}(\Omega)$ which is inverse to L modulo smoothing operators. Also denote by Q the associated pseudo-differential operator $C_c^\infty(\Omega) \to C_{\mathrm{loc}}^\infty(\Omega)$. Now for $\psi \in C_c^\infty(\Omega)$ satisfying $\psi \equiv 1$ over a neighborhood of $\operatorname{supp}\varphi$, we claim that the operators $\varphi Q \psi, \psi Q \varphi : C_{\mathrm{loc}}^\infty(\Omega) \to C_c^\infty(\Omega)$ are our desired left and right parametrices. Indeed, the identities $\varphi Q \psi L \sim \varphi \sim L \psi Q \varphi$ follow by inspecting composition of symbols. □

3.4.20 Corollary. *Every elliptic operator L of order m has a parametrix Q.*

Proof. Let $M = \bigcup_i U_i$ be a locally finite open cover by Euclidean charts, and let $\varphi_i : M \to \mathbb{R}$ be a subordinate partition unity. By (3.4.19), there exist left and right φ_i-parametrices $Q_i, Q_i' : C_{\mathrm{loc}}^\infty(U_i) \to C_c^\infty(U_i)$ of order $\leq -m$. Their sums $Q = \sum_i Q_i$ and $Q' = \sum_i Q_i'$ are thus left and right parametrices for L. It follows formally (see (3.4.12)) that their difference $Q - Q'$ is a smoothing operator and hence that both Q and Q' are parametrices for L. □

* *

We now explore the consequences of the existence of parametrices for elliptic operators (3.4.20).

3.4.21 Corollary (Elliptic estimate)**.** *Let L be an elliptic operator of order m. We have*

$$\|u\|_s \leq \mathrm{const}_{L,K,s} \|Lu\|_{s-m} + \mathrm{const}_{L,K,N,s} \|u\|_{s-N} \qquad (3.4.21.1)$$

for u supported inside compact $K \subseteq M$ and any $N < \infty$.

Proof. Let Q be a parametrix for L (3.4.20). Write $u = QLu + (\mathbf{1} - QL)u$, and note that $\|Q\|_{(s-m,s)} \leq \mathrm{const}_{L,K,s}$ and $\|\mathbf{1} - QL\|_{(s-N,s)} \leq \mathrm{const}_{L,K,N,s}$. □

⋆ **3.4.22 Corollary** (Elliptic regularity)**.** *Let L be an elliptic operator of order m. If $Lu \in H^s_{\mathrm{loc}}$ then $u \in H^{s+m}_{\mathrm{loc}}$.*

Proof. Suppose $u \in C^{-\infty}_{\mathrm{loc}}$ and $Lu \in H^s_{\mathrm{loc}}$. Fix any Euclidean chart $U \subseteq M$ and smooth function φ supported inside U. By (3.4.19), there exists a left φ-parametrix for L, that is a semi-local operator Q' of order $\leq -m$ for which $\varphi - Q'L$ is a smoothing operator. Thus the identity $\varphi u = (\varphi - Q'L)u + Q'Lu$ implies that $\varphi u \in H^{s+m}_{\mathrm{loc}}$. Since U and φ where arbitrary, we conclude that $u \in H^{s+m}_{\mathrm{loc}}$. □

⋆ **3.4.23 Corollary** (Kernel and cokernel of an elliptic operator)**.** *For an elliptic operator L of order m, the natural inclusions between the two-term complexes*

$$C^\infty_{\mathrm{loc}}(M,E) \xrightarrow{L} C^\infty_{\mathrm{loc}}(M,F) \tag{3.4.23.1}$$

$$H^s_{\mathrm{loc}}(M,E) \xrightarrow{L} H^{s-m}_{\mathrm{loc}}(M,F) \tag{3.4.23.2}$$

$$C^{-\infty}_{\mathrm{loc}}(M,E) \xrightarrow{L} C^{-\infty}_{\mathrm{loc}}(M,F) \tag{3.4.23.3}$$

are all quasi-isomorphisms. We denote by $\ker L$ *and* $\operatorname{coker} L$ *the kernel and cokernel of these operators; we have* $\ker L \subseteq C^\infty_{\mathrm{loc}}(M,E)$ *and* $C^\infty_{\mathrm{loc}}(M,F) \twoheadrightarrow \operatorname{coker} L$*. The same holds for the action of L on $C^\infty_c \subseteq H^s_c \subseteq C^{-\infty}_c$, giving spaces* $\ker_c L$ *and* $\operatorname{coker}_c L$*.*

Proof. Consider the case $H^s \hookrightarrow H^t$ for $s \geq t$ (the others are identical). It suffices to show that the total complex of the double complex

$$\begin{tikzcd} H^s_{\mathrm{loc}}(M,E) \arrow[r, "L"] \arrow[d, "\mathbf{1}"'] & H^{s-m}_{\mathrm{loc}}(M,F) \arrow[d, "\mathbf{1}"] \\ H^t_{\mathrm{loc}}(M,E) \arrow[r, "L"] & H^{t-m}_{\mathrm{loc}}(M,F) \end{tikzcd} \tag{3.4.23.4}$$

is acyclic. The endomorphism of this double complex given by

$$\begin{tikzcd} H^s_{\mathrm{loc}}(M,E) & H^{s-m}_{\mathrm{loc}}(M,F) \arrow[l, "Q"'] \\ H^t_{\mathrm{loc}}(M,E) \arrow[u, "\mathbf{1}-QL"] & H^{t-m}_{\mathrm{loc}}(M,F) \arrow[l, "Q"'] \arrow[u, "\mathbf{1}-LQ"'] \end{tikzcd} \tag{3.4.23.5}$$

is a chain homotopy between the identity map and the zero map (which implies acyclicity). Note that in writing the vertical arrows above, we are appealing to the fact that $\mathbf{1} - LQ$ and $\mathbf{1} - QL$ are smoothing operators. In the case of compactly supported functions, note that Q is semi-local. □

3.4.24 Corollary. *Let L be an elliptic operator of order m on a compact manifold M. If L is an isomorphism, then*

$$\|L^{-1}\|_{(s,s+m)} \leq \mathrm{const}_{M,L,s,a,b}\|L^{-1}\|_{(a,b)} \tag{3.4.24.1}$$

for any $s, a, b \in \mathbb{R}$. The same holds for a right inverse P provided $a \leq s$ and for a left inverse P' provided $b \geq s + m$.

Proof. Let Q be a parametrix for L, and write $L^{-1} = (\mathbf{1} - QL)L^{-1}(\mathbf{1} - LQ) + 2Q - QLQ$ or $P = (\mathbf{1} - QL)P + Q$ or $P' = P'(\mathbf{1} - LQ) + Q$. □

3.4.25 Corollary (Openness of isomorphism)**.** *Let L be an elliptic operator of order m on a compact manifold. If L is an isomorphism, then so is every L' sufficiently close to L in the C^∞-topology, and in fact $\|(L')^{-1}\|_{(s,s+m)} \leq \mathrm{const}_{L,s}$ for every such L'.*

Proof. Fix $s \in \mathbb{R}$ arbitrarily. We have $\|\mathbf{1} - L'L^{-1}\|_{(s,s)} = \|(L - L')L^{-1}\|_{(s,s)} \leq \|L^{-1}\|_{(s,s+m)}\|L - L'\|_{(s+m,s)}$. Taking L' sufficiently close to L (in terms of s) in the C^∞-topology ensures that $\|L - L'\|_{(s+m,s)}$ becomes arbitrarily small. In particular, this produces a neighborhood over which $\|\mathbf{1} - L'L^{-1}\|_{(s,s)} \leq \frac{1}{2}$, hence the usual inverse series $\sum_{i=0}^\infty L^{-1}(\mathbf{1} - L'L^{-1})^i$ (**??**) converges and is inverse to L'. This explicit construction of $(L')^{-1}$ makes the bound $\|(L')^{-1}\|_{(s+m,s)} \leq \mathrm{const}_{L,s}$ evident *this particular value of s*. To conclude the same for *all* s (without needing to change the chosen neighborhood of L in the C^∞-topology), apply (3.4.24). □

3.4.26 Remark (Matrix operators)**.** Given an elliptic operator $L : E \to F$ on a manifold M, it is often useful to consider 'matrix operators' of the form

$$\begin{pmatrix} L & \alpha \\ \beta & \gamma \end{pmatrix} : C_c^\infty(M, E) \oplus V \to C_c^\infty(M, F) \oplus W \tag{3.4.26.1}$$

for finite-dimensional vector spaces V and W and matrix entries $\alpha \in C_c^\infty(M, F \otimes V^*)$, $\beta \in C_c^\infty(M, E^* \otimes \Omega_M \otimes W)$, and $\gamma \in V^* \otimes W$. More generally, it can make sense to consider $\alpha \in C_{\mathrm{loc}}^{-\infty}$ and/or $\beta \in C_{\mathrm{loc}}^{-\infty}$, with the caveat that this imposes restrictions on the topologies we can consider on the domain and codomain of our matrix operator.

If Q is a parametrix for L, then the operator $\left(\begin{smallmatrix} Q & 0 \\ 0 & 0 \end{smallmatrix}\right)$ is a parametrix for $\left(\begin{smallmatrix} L & \alpha \\ \beta & \gamma \end{smallmatrix}\right)$. Using such parametrices, the basic properties of elliptic operators and their proofs may be generalized quite directly to these sorts of matrix operators, and we will feel free to apply the same in this generality.

When studying a given elliptic operator (or, more generally, a matrix operator (3.4.26.1)) L, it is often helpful to consider an auxiliary operator $\left(\begin{smallmatrix} L & \alpha \\ \beta & \gamma \end{smallmatrix}\right)$ for particular choices of α, β, and γ which ensure this auxiliary operator has certain convenient properties (typically injectivity, surjectivity, or bijectivity).

3.4.27 Corollary. *Let L be an elliptic operator of order m on M. For every $s \in \mathbb{R}$ and every compact $K \subseteq M$, there exist finitely many smooth functions $\rho_1, \ldots, \rho_N$ such that*

$$\|u\|_s \leq \mathrm{const}_{L,K,s}\|Lu\|_{s-m} + \sum_{i=1}^N \left| \int \rho_i u \right| \tag{3.4.27.1}$$

for u supported inside K.

Proof. Begin with the elliptic estimate $\|u\|_s \leq \text{const}_{L,K,s}\|Lu\|_{s-m} + \text{const}_{L,K,s}\|u\|_{s-1}$ (3.4.21), and apply the Rellich Lemma (3.2.30) to bound $\text{const}_{L,K,s}\|u\|_{s-1}$ by $\frac{1}{2}\|u\|_s + \sum_{i=1}^N \left|\int \rho_i u\right|$. □

⋆ **3.4.28 Corollary** (Kernel finiteness)**.** *The kernel of an elliptic operator on a compact manifold is finite-dimensional.*

Proof #1. The estimate (3.4.27) implies that $\|u\|_s \leq \sum_{i=1}^N \left|\int \rho_i u\right|$ for all $u \in \ker L$, so the map $\ker L \to \mathbb{R}^N$ given by $u \mapsto (\int \rho_i u)_i$ is injective. □

Proof #2. It suffices to show that there exists a finite collection of points $P \subseteq M$ such that an element of $\ker L$ which vanishes on P must be zero. Take P to be any set of points such that the ε-balls centered at P cover M (we will choose $\varepsilon > 0$ later). Thus $\varphi|_P = 0$ implies that $\|\varphi\|_{C^0} \leq \varepsilon\|\varphi\|_{C^1}$. On the other hand, $\varphi \in \ker L$ implies that $\|\varphi\|_{C^1} \leq \text{const}_L\|\varphi\|_{C^0}$ since L is elliptic (3.4.21). Thus if $\varphi|_P = 0$ and $\varphi \in \ker L$, then these combine to give $\|\varphi\|_{C^0} \leq \varepsilon \cdot \text{const}_L\|\varphi\|_{C^0}$, which implies $\varphi = 0$ provided we choose $\varepsilon > 0$ sufficient small. □

3.4.29 Exercise. Explain the relation between the two proofs of kernel finiteness (3.4.28).

⋆ **3.4.30 Corollary** (Cokernel finiteness)**.** *The cokernel of an elliptic operator on a compact manifold is finite-dimensional.*

Proof. Let $L : E \to F$ over M, and fix a parametrix Q. It suffices to show that there exists a map $\alpha : \mathbb{R}^k \to C^\infty(M, F)$ for which $L \oplus \alpha : C^\infty(M, E) \oplus \mathbb{R}^k \to C^\infty(M, F)$ is surjective. To show $L \oplus \alpha$ is surjective, it suffices to construct a map $\nu : C^\infty(M, F) \to \mathbb{R}^k$ for which $Q \oplus \nu : C^\infty(M, F) \to C^\infty(M, E) \oplus \mathbb{R}^k$ is an approximate right inverse to $L \oplus \alpha$, in the sense that $\|\mathbf{1} - (L \oplus \alpha)(Q \oplus \nu)\|_{(s,s)} < 1$ for some $s \in \mathbb{R}$. We are free to choose α and ν as we wish, however it is evident that all that really matters is their composition $\sigma = \alpha\nu : C^\infty(M, F) \to C^\infty(M, F)$, which can choose to be anything of finite rank. We will take $\sigma = \pi \circ (\mathbf{1} - LQ)$ for a finite rank endomorphism $\pi : C^\infty(M, F) \to C^\infty(M, F)$, so we have $\mathbf{1} - (L \oplus \alpha)(Q \oplus \nu) = (\mathbf{1} - \pi)(\mathbf{1} - LQ)$. The (s, s)-operator norm of this quantity is thus bounded by $\|\mathbf{1} - LQ\|_{(s,s+1)}\|\mathbf{1} - \pi\|_{(s+1,s)}$, so it suffices to show that π may be taken so that $\mathbf{1} - \pi$ has arbitrarily small $H^{s+1} \to H^s$ operator norm. The existence of such π was proven in (3.2.31). □

3.4.31 Corollary (Semi-continuity of kernel and cokernel)**.** *Let L be an elliptic operator of order m on a compact manifold. For every L' sufficiently close to L in the C^∞-topology, we have* $\dim\ker L' \leq \dim\ker L$ *and* $\dim\operatorname{coker} L' \leq \dim\operatorname{coker} L$*.*

Proof. Consider a matrix operator $\hat{L} = \left(\begin{smallmatrix} L & \alpha \\ \beta & 0 \end{smallmatrix}\right)$ as in (3.4.26), where α and β induce isomorphisms $V \to \operatorname{coker} L$ and $\ker L \to W$ (such α and β exist by kernel finiteness (3.4.28) and cokernel finiteness (3.4.30)). This ensures that $\hat{L}$ is an isomorphism, and hence $\hat{L}' = \left(\begin{smallmatrix} L' & \alpha \\ \beta & 0 \end{smallmatrix}\right)$ is an isomorphism for all L' sufficiently close to L (3.4.25). The fact that $\hat{L}'$ is an isomorphism implies that α and β induce (respectively) a surjection $V \twoheadrightarrow \operatorname{coker} L'$ and an injection $\ker L' \hookrightarrow W$. □

3.4.32 Definition (Index)**.** The *index* of an elliptic operator $L : E \to F$ on a compact manifold M is given by $\operatorname{ind} L = \chi(M, L) = \dim \ker L - \dim \operatorname{coker} L$ (kernel and cokernel are both finite-dimensional (3.4.28)(3.4.30) and independent of the function spaces under consideration (3.4.23)).

3.4.33 Exercise (Index is locally constant)**.** Let L be an elliptic operator on a compact manifold. Show that for all L' sufficiently close to L in the C^∞-topology, we have $\operatorname{ind} L' = \operatorname{ind} L$ (use the proof of (3.4.31)).

3.5 Rough coefficients

In the study of *smooth non-linear* elliptic equations, it is often necessary to have estimates for *linear* elliptic equations with *non-smooth* coefficients. We now generalize some of the results from (3.3)–(3.4) about linear differential operators with smooth coefficients to the setting of coefficients in some Sobolev space.

3.6 Ellipticity in cylindrical ends

In this section, we study cylindrical and asymptotically cylindrical elliptic operators. This means operators on a cylinder $\mathbb{R} \times N$ which are $\mathbb{R}$-equivariant; more generally, on a manifold M with ends modelled asymptotically on $\mathbb{R} \times N$, an asymptotically cylindrical operator is one which is asymptotically $\mathbb{R}$-equivariant in the ends. The reference for this section is Lockhart–McOwen [71].

3.6.1 Definition (Cylinder)**.** A 'cylinder' is a product $\mathbb{R} \times N$. The adjective 'cylindrical' when applied to objects living on a cylinder means $\mathbb{R}$-equivariant. For example, a cylindrical vector bundle on $\mathbb{R}\times N$ is one identified with the pullback of a vector bundle on N, and on such vector bundles we can consider cylindrical (i.e. $\mathbb{R}$-equivariant) operators $C^\infty_{\mathrm{loc}}(\mathbb{R} \times N, E) \to C^\infty_{\mathrm{loc}}(\mathbb{R} \times N, F)$.

⋆ **3.6.2 Definition** (Manifold with asymptotically cylindrical ends)**.** A manifold with asymptotically cylindrical ends M is a paracompact Hausdorff space with an atlas of charts from open subsets of $(0, \infty] \times \mathbb{R}^n$ whose transition functions take the form

$$(t, x) \mapsto (t + a(x) + o(1)_{C^\infty}, \phi(x) + o(1)_{C^\infty}) \quad \text{as } t \to \infty \tag{3.6.2.1}$$

for smooth $\phi : N \to N$ and $a : N \to \mathbb{R}$. Points of M at infinity (in the t coordinate) are called 'ideal points' and form a closed subset $M^{\mathrm{id}} \subseteq M$, which is a manifold. The complement $M \setminus M^{\mathrm{id}}$ is called the 'non-degenerate locus' $M^\circ \subseteq M$. A cylinder $\mathbb{R}\times N$ is the non-degenerate locus of a manifold with asymptotically cylindrical ends $[-\infty, \infty] \times N$.

The adjective 'asymptotically cylindrical' means built using functions of the form $f(x) + o(1)_{C^\infty}$ on charts $(0, \infty]\times\mathbb{R}^n$. Asymptotically cylindrical objects on M 'restrict' to cylindrical objects on $\mathbb{R} \times M^{\mathrm{id}}$ (their 'asymptotic limit'). Objects (vector bundles, almost complex structures, etc.) on asymptotically cylindrical manifolds are by default asymptotically cylindrical unless specified otherwise.

Asymptotic cylindricity is a special case of 'log smoothness' (2.7.46), so a manifold with asymptotically cylindrical ends is the same thing as a log smooth manifold of depth one (2.7.27), and asymptotically cylindrical objects (functions, vector bundles, etc.) are the same as log smooth objects.

Beware that one must be careful with the term 'compact' in the context of manifolds with asymptotically cylindrical ends. For example, if M is a manifold with asymptotically cylindrical ends, then compactness of M is distinct from compactness of M° (which implies $M^{\mathrm{id}} = \varnothing$, hence is a rather vacuous setting for our present discussion). Also contrast compactly supported functions on M with compactly supported functions on M°.

3.6.3 Example. Let C be a Riemann surface, and let $p \in C$ be a point. Given any local holomorphic chart $(D^2, 0) \to (C, p)$, we may glue $C \setminus p$ together with $(0, \infty] \times S^1$ via the identification of $z = e^{-t-i\theta} \in D^2$ with $(t, \theta) \in (0, \infty]\times S^1$. The coordinate change between any two such local holomorphic charts has the form $(t, \theta) \mapsto (t + a + O(e^{-t})_{C^\infty}, \theta + b + O(e^{-t})_{C^\infty})$ as $t \to \infty$ (by analyticity of holomorphic functions). These charts thus define a manifold with asymptotically cylindrical ends $\mathrm{Bl}_p C$ with interior $(\mathrm{Bl}_p C)^\circ = C \setminus p$.

3.6.4 Definition (Spaces C^∞_∞)**.** On a manifold with asymptotically cylindrical ends M, we denote by $C^\infty_{\infty,\mathrm{loc}}(M)$ the space of log smooth functions $M \to \mathbb{R}$ (which in local cylindrical coordinates means functions of the form $(t,x) \mapsto g(x) + o(1)_{C^\infty}$ for smooth g). We denote by $C^\infty_{\infty,\mathrm{loc}}(M, M^{\mathrm{id}}) \subseteq C^\infty_{\infty,\mathrm{loc}}(M)$ those functions which vanish 'at infinity' (meaning on $M^{\mathrm{id}} \subseteq M$). These are complete locally convex topological vector spaces when equipped with the semi-norms $\sum_i \|\varphi_i u\|_{C^{k_i}(\mathbb{R}^n \times \mathbb{R})}$ associated to collections of charts $\mathbb{R}^n \times (0,\infty] \supseteq U_i \hookrightarrow M$, integers $k_i < \infty$, and compactly supported smooth functions $\{\varphi_i : U_i \to \mathbb{R}\}_i$ which are locally finite on M.

* *

Let us now recall Sobolev norms on manifolds with asymptotically cylindrical ends.

⋆ **3.6.5 Definition** (Sobolev spaces H^s_2)**.** Let M be a manifold with asymptotically cylindrical ends carrying a vector bundle E. Given a coordinate chart $\alpha : (0,\infty] \times \mathbb{R}^n \supseteq U \hookrightarrow M$ and a smooth function of compact support $\varphi : \alpha(U) \to E^*$, we may consider the semi-norm $u \mapsto \|\alpha^*(\varphi u)\|_{H^s(\mathbb{R}\times\mathbb{R}^n)}$ on $C^\infty_c(M, M^{\mathrm{id}})$. We call these $H^s_{2,\mathrm{loc}}$-semi-norms and the induced topology the $H^s_{2,\mathrm{loc}}$-topology with completion $H^s_{2,\mathrm{loc}}(M, M^{\mathrm{id}}; E)$. The description of generating families of semi-norms from (3.2.22) continues to hold, for the same reason.

⋆ **3.6.6 Definition** (Spaces C^∞_2)**.** Let M be a manifold with asymptotically cylindrical ends. We define

$$C^\infty_{2,\mathrm{loc}}(M, M^{\mathrm{id}}) = \bigcap_s H^s_{2,\mathrm{loc}}(M, M^{\mathrm{id}}) \subseteq C^\infty_{\mathrm{loc}}(M, M^{\mathrm{id}}) \tag{3.6.6.1}$$

to be the space of smooth functions on M which vanish on M^{id} and such that in local cylindrical coordinates on M, all their derivatives are square integrable. The $C^\infty_{2,\mathrm{loc}}$-topology is that generated by all $C^\infty_{2,\mathrm{loc}}$-semi-norms, which are simply all the $H^s_{2,\mathrm{loc}}$-semi-norms for all s.

3.6.7 Example. The function $(1+x^2)^{-1/5}$ lies in $C^\infty_\infty([-\infty,\infty], [-\infty,\infty]^{\mathrm{id}})$ but not in C^∞_2.

⋆ **3.6.8 Proposition.** *Let L be an asymptotically cylindrical differential operator of order $\leq m$ on a manifold with asymptotically cylindrical ends M. For any compact $K \subseteq M$, we have $\|Lu\|_s \leq \mathrm{const}_{L,K,s}\|u\|_{s+m}$ for $u \in C^\infty_K(M, M^{\mathrm{id}})$.*

Proof. Express the H^s-norm squared as a sum of local pieces using a partition of unity of bounded geometry (in cylindrical coordinates) as in (**??**). This reduces us to the local case on $\mathbb{R}^n$ (3.3.7) since L has bounded geometry in cylindrical coordinates. □

* *

The definition of ellipticity (3.4.1) makes sense as written for asymptotically cylindrical operators.

3.6.9 Exercise. Show that an asymptotically cylindrical operator L on a manifold with asymptotically cylindrical ends M is elliptic of order m iff its restriction to the interior M° is elliptic of order m and its asymptotic limit L^{id} on $\mathbb{R} \times M^{\mathrm{id}}$ is elliptic of order m.

* *

3.6.10 Definition (Reduction)**.** Let $L : E \to F$ be a cylindrical operator on $\mathbb{R} \times N$. If we restrict L to $\mathbb{R}$-invariant sections, we obtain an operator

$$L_0 : C^\infty(N, E) = C^\infty(\mathbb{R} \times N, E)^{\mathbb{R}} \to C^\infty(\mathbb{R} \times N, F)^{\mathbb{R}} = C^\infty(N, F) \tag{3.6.10.1}$$

called the *reduction* of L.

More generally, we may consider those sections on $\mathbb{R}\times N$ which transform under translation by the character e^{zt} for any complex number z. This defines operators $L_z : C^\infty(N, E) \to C^\infty(N, F)$ called *twisted reductions* of L. If $L = \sum_{i,\alpha} c_{i,\alpha}(x)D_t^i D_x^\alpha$ in coordinates $(t, x) \in \mathbb{R}\times N$, then $L_z = \sum_{i,\alpha} z^i c_{i,\alpha}(x)D_x^\alpha$ (hence this gives a bijection between cylindrical differential operators on $\mathbb{R} \times N$ and differential operators on N depending polynomially on a parameter z).

An asymptotically cylindrical operator L on M has an associated cylindrical operator L^{id} on $\mathbb{R} \times M^{\mathrm{id}}$, whose reductions L_z^{id} on M^{id} may also simply be denoted L_z.

3.6.11 Exercise. Show that any twisted reduction of an elliptic operator is elliptic.

3.6.12 Definition (Twist)**.** Given a cylindrical vector bundle V on $\mathbb{R} \times N$, its *twist* $\tau_z V$ by a complex number $z \in \mathbb{C}$ is obtained by multiplying the $\mathbb{R}$-translation action on V by e^{zt}. Multiplication by e^{zt} thus defines an isomorphism $V \to \tau_z V$. The twist of a cylindrical differential operator $L : E \to F$ is given by $\tau_z L = e^{zt}Le^{-zt} : E_z \to F_z$ (explicitly, if $L = \sum_{i,\alpha} c_{i,\alpha}(x)D_t^i D_x^\alpha$ then $\tau_z L = \sum_{i,\alpha} c_{i,\alpha}(x)(D_t - z)^i D_x^\alpha$). Twisting and reduction are compatible in the evident way: $(\tau_z L)_w = L_{w-z}$.

The twist of an asymptotically cylindrical vector bundle V on M may be defined by the property that $\tau_z V = V$ over M° and a map $V \to \tau_z V$ over M° extends smoothly to M if it is given over M° by multiplication by a function f which in cylindrical charts $(0, \infty] \times U$ has the form $f(x, t) = e^{zt}(m(x) + o(1)_{C^\infty})$ for some nonzero smooth function m. To ensure such $\tau_z V$ exists, we should note that coordinate changes between cylindrical charts (3.6.2.1) preserve the class of functions of the form $(x, t) \mapsto e^{zt}(m(x) + o(1)_{C^\infty})$ and that for any two such functions f and g, their ratio f/g extends smoothly on M. An asymptotically cylindrical operator $L : E \to F$ evidently induces a twisted operator $\tau_z L : \tau_z E \to \tau_z F$ by conjugating L by the isomorphisms $E = \tau_z E$ and $F = \tau_z F$ over M° (noting that the result of such conjugation on M° extends smoothly to M).

Beware that twisting a cylindrical vector bundle or operator by z corresponds to twisting by z at $+\infty$ and by $-z$ at $-\infty$.

⋆ **3.6.13 Definition** (Non-degenerate cylindrical elliptic operator)**.** Let L be a cylindrical elliptic operator on $\mathbb{R} \times N$ where N is compact. We say L is *non-degenerate* when its twisted reductions $L_{i\xi}$ are invertible for all $\xi \in \mathbb{R}$. An asymptotically cylindrical elliptic operator L is called non-degenerate when its asymptotic limit L^{id} is non-degenerate.

⋆ **3.6.14 Proposition.** *A non-degenerate cylindrical elliptic operator is invertible.*

3.7 Elliptic boundary conditions

3.8 Families of elliptic operators

In this section, we study how the results of the previous sections on elliptic operators (3.4)–(3.7) apply in families. Our objects of study are proper submersions $\pi : Q \to B$ equipped with vertical (i.e. fiberwise) elliptic operators L. The main result is that such a family determines a (homotopically) canonical two-term complex of vector bundles $\pi_* L$ on B whose cohomology at $b \in B$ is identified with the kernel and cokernel of L_b (**??**). In particular, the set of $b \in B$ for which L_b is surjective is open, and over this open set $\pi_* L$ is a vector bundle. These results come in a number of different flavors depending on the category in which the base B lives.

* *

Due to the categorical nature of the following definition, it makes sense in quite a number of different contexts (namely any setting with a reasonable notion of submersion and vertical differential operator).

⋆ **3.8.1 Definition** (Pushforward)**.** Let $\pi : Q \to B$ be a submersion, and let $L : E \to F$ be a vertical elliptic operator on Q. The pushforward $\pi_* L$ is the fiber product

$$\pi_* L = \underline{\mathrm{Sec}}_B(Q, E) \times_{\underline{\mathrm{Sec}}_B(Q,F)} 0, \tag{3.8.1.1}$$

where $\underline{\mathrm{Sec}}$ is the stack of sections (2.3.52) and the map $\underline{\mathrm{Sec}}_B(Q, E) \to \underline{\mathrm{Sec}}_B(Q, F)$ is given by applying L. It is evident that $(\pi_* L) \times_B B' = \pi'_* L'$, where $\pi' : Q' \to B'$ and $L' : E' \to F'$ denote the pullback of (π, L) under a map $B' \to B$.

* *

We first study families of elliptic operators on smooth manifolds. In this context, the most immediate interpretation of the pushforward (3.8.1) is as a smooth stack $(\pi_* L)_{\mathsf{Sm}}$. Namely, for a vertical elliptic operator $L : E \to F$ on a submersion $\pi : Q \to B$ of smooth manifolds, a map $Z \to \pi_* L$ from a smooth manifold Z is, by definition (3.8.1.1)(2.3.52), a pair (f, u) consisting of a map $f : Z \to B$ and a section $u : Q \times_B Z \to E$ satisfying $Lu = 0$.

The following is the key analytic result about such families of operators. Once we prove it, the rest of the reasoning is formal/categorical.

⋆ **3.8.2 Proposition** (Fiberwise isomorphism implies isomorphism)**.** *Let $L : E \to F$ be a vertical elliptic operator on a proper submersion $Q \to B$ of smooth manifolds.*

(3.8.2.1) *The set of $b \in B$ for which L_b is an isomorphism is open.*

(3.8.2.2) *If L_b is an isomorphism for every $b \in B$, then $L : \underline{\mathrm{Sec}}_B(Q, E) \to \underline{\mathrm{Sec}}_B(Q, F)$ is an isomorphism of smooth stacks.*

Proof. The desired assertion is local on B. By Ehresmann (2.4.17), the family $Q \to B$ is locally trivial on the base. The same argument applies moreover to the vector bundles E and F on the total space. We are therefore in the setting of a compact Hausdorff smooth manifold M and a family of elliptic operators $L_b : E \to F$ on M depending smoothly on $b \in B$. In this context, the set of $b \in B$ for which L_b is an isomorphism is open by (3.4.25). It thus remains to prove that if L_b is an isomorphism for every $b \in B$, then $L : \underline{\mathrm{Sec}}_B(Q, E) \to \underline{\mathrm{Sec}}_B(Q, F)$ is an isomorphism of smooth stacks.

Now a map $Z \to \underline{\mathrm{Sec}}_B(Q, E)$ is the same thing as a map $Z \to B$ and a section of E over $Z \times M$. By replacing B with Z, we reduce (3.8.2.2) to the following concrete assertion.

(3.8.2.3) If $f : B \times M \to F$ is smooth, then $L^{-1} f : B \times M \to E$ is also smooth (under the assumption that every L_b is an isomorphism, which thus defines $L^{-1} f$ fiberwise).

Let us now prove (3.8.2.3), which we note is a local assertion on B. Fix a basepoint $0 \in B$, and note that L_b^{-1} may be described in terms of L_0^{-1} by the usual series

$$L_b^{-1} f = \sum_{i=0}^{\infty} L_0^{-1} (\mathbf{1} - L_b L_0^{-1})^i f, \tag{3.8.2.4}$$

provided we can appropriately estimate its convergence. The series (3.8.2.4) converges in the fiberwise H^s-norm over a given b provided $\|\mathbf{1} - L_b L_0^{-1}\|_{(s,s)} < 1$. We have $\mathbf{1} - L_b L_0^{-1} = (L_0 - L_b) L_0^{-1}$, so this norm is bounded by $\|L_0^{-1}\|_{(s,s+m)}$ (constant) times $\|L_b - L_0\|_{(s+m,s)}$ (small provided b is sufficiently close to 0 (3.3.7)). We thus conclude that (3.8.2.4) converges exponentially in $H^s(M)$ for any fixed b, uniformly over a neighborhood of $0 \in B$ (depending on s). By Sobolev embedding (3.2.28), this implies the same for $C^k(M)$ in place of $H^s(M)$. It follows that arbitrary derivatives of $L^{-1} f$ in the M direction are continuous on $B \times M$.

To treat the derivatives in the B direction, we simply differentiate each term of the series (3.8.2.4) with respect to b. A derivative with respect to b applied to a term $(\mathbf{1} - L_b L_0^{-1})^i f$ will hit either L_b or f. Differentiating ℓ times leaves at least $i - \ell$ factors of $\mathbf{1} - L_b L_0^{-1}$, so the series remains exponentially convergent in fiberwise $C^k(M)$ for every $k < \infty$, uniformly over a neighborhood of $0 \in B$ depending on k. It follows that $L^{-1} f$ is smooth on $B \times M$. □

3.8.3 Lemma. *Let L be a vertical elliptic operator on a proper submersion $Q \to B$ of smooth manifolds. If L_b is surjective, then there exists (near b) a smooth section $Q \to E^* \otimes \Omega_{Q/B} \otimes \mathbb{R}^k$ (inducing via integration a map $\beta : \underline{\mathrm{Sec}}_B(Q, E) \to \mathbb{R}^k$) for which $L_b \oplus \beta_b$ is an isomorphism. The same holds for matrix operators as in* (**??**).

Proof. Since L_b is elliptic and Q_b is compact, the kernel of L_b is finite-dimensional (3.4.28). Fix a smooth section $Q_b \to E^* \otimes \Omega_{Q/B} \otimes \mathbb{R}^k$ whose associated map $C^\infty(Q_b, E) \to \mathbb{R}^k$ restricts to an isomorphism on $\ker L_b$. Now extend this section to a neighborhood of $Q_b \subseteq Q$ arbitrarily (e.g. by taking a linear combination of local extensions according to a partition of unity). □

We now unfold the consequences of the key analytic result (3.8.2).

3.8.4 Corollary. *Let L be a vertical elliptic operator on a proper submersion $\pi : Q \to B$ of smooth manifolds.*

(3.8.4.1) *The set of $b \in B$ for which L_b is surjective is open.*

(3.8.4.2) *If L_b is surjective for every $b \in B$, then $\pi_* L$ is locally isomorphic to $\mathbb{R}^k \times B \to B$ as a smooth manifold over B (in particular, it is representable).*

The same holds for matrix operators as in (**??**).

Proof. The desired conclusions are local on B, so we may fix a basepoint $b \in B$ for which L_b is surjective and replace B with a neighborhood of b at will. Fix a map $\beta : \underline{\mathrm{Sec}}_B(Q, E) \to \mathbb{R}^k$ for which $L_b \oplus \beta_b$ is an isomorphism (3.8.3). Thus (after replacing B with a neighborhood of b) every $L_{b'} \oplus \beta_{b'}$ is an isomorphism (3.8.2.1) (which implies $L_{b'}$ is surjective (3.8.4.1)) and $L \oplus \beta$ is an isomorphism of smooth stacks (3.8.2.2) (which implies $\pi_* L$ is isomorphic to $\mathbb{R}^k \times B \to B$ (3.8.4.2)). □

Now let us upgrade (3.8.4.2) to the assertion that $\pi_* L$ is a vector bundle over B whenever L is fiberwise surjective. To keep track of the linear structure on $\pi_* L$, recall the category $\mathsf{Vect} \rtimes \mathsf{Sm}$ (**??**), whose objects are pairs (M, V) where $M \in \mathsf{Sm}$ and $V \in \mathsf{Vect}(\mathsf{Sm})$ and whose morphisms $(M, V) \to (M', V')$ are pairs (f, g) consisting of a smooth map $f : M \to M'$ and a linear map $g : V \to f^* V'$. The sections functor $\underline{\mathrm{Sec}}_B(Q, E)_{\mathsf{Vect} \rtimes \mathsf{Sm}} = \underline{\mathrm{Sec}}_{(B,0)}((Q, 0), (Q, E))$ assigns to $(Z \in \mathsf{Sm}, V \in \mathsf{Vect}(Z))$ the set of pairs (f, u) where $f : Z \to B$ and $u : \pi^* V \to E$ (linear) over $Q \times_B Z$ (2.3.52). Applying L to a linear map $u : \pi^* V \to E$ produces a linear map $Lu : \pi^* V \to F$ (note the importance of the domain of u being pulled back from B); this defines a map $L : \underline{\mathrm{Sec}}_B(Q, E) \to \underline{\mathrm{Sec}}_B(Q, F)$ of stacks on $\mathsf{Vect} \rtimes \mathsf{Sm}$, hence a pushforward $\pi_* L \in \mathsf{Shv}(\mathsf{Vect} \rtimes \mathsf{Sm})$. Concretely, this pushforward $(\pi_* L)_{\mathsf{Vect} \rtimes \mathsf{Sm}}$ assigns the same sort of configurations (f, u) but with the additional condition that $Lu = 0$. The forgetful ('total space') functor $\mathsf{Vect} \rtimes \mathsf{Sm} \to \mathsf{Sm}$ induces a tautological forgetful map $\underline{\mathrm{Sec}}_B(Q, E)_{\mathsf{Vect} \rtimes \mathsf{Sm}} \to (\mathsf{Vect} \rtimes \mathsf{Sm} \to \mathsf{Sm})^* \underline{\mathrm{Sec}}_B(Q, E)_{\mathsf{Sm}}$, hence a forgetful map $(\pi_* L)_{\mathsf{Vect} \rtimes \mathsf{Sm}} \to (\mathsf{Vect} \rtimes \mathsf{Sm} \to \mathsf{Sm})^* (\pi_* L)_{\mathsf{Sm}}$. Of more geometric significance is the associated (by adjunction) map $(\mathsf{Vect} \rtimes \mathsf{Sm} \to \mathsf{Sm})_! (\pi_* L)_{\mathsf{Vect} \rtimes \mathsf{Sm}} \to (\pi_* L)_{\mathsf{Sm}}$.

3.8.5 Remark (Representability on $\mathsf{Vect} \rtimes \mathsf{Sm} \downarrow B$ vs representability on $\mathsf{Vect}(B)$)**.** A morphism in $(\mathsf{Shv}(\mathsf{Vect} \rtimes \mathsf{Sm}) \downarrow B) = \mathsf{Shv}(\mathsf{Vect} \rtimes \mathsf{Sm} \downarrow B)$ (note that $(\mathsf{Vect} \rtimes \mathsf{Sm}) \downarrow B = \mathsf{Vect} \rtimes (\mathsf{Sm} \downarrow B)$) is an isomorphism iff its restriction the fiber $\mathsf{Vect}(Z) \subseteq (\mathsf{Vect} \rtimes \mathsf{Sm} \downarrow B) \to (\mathsf{Sm} \downarrow B)$ over every $(f : Z \to B) \in (\mathsf{Sm} \downarrow B)$ is an isomorphism. Therefore a morphism $\mathsf{Vect} \rtimes \mathsf{Sm} \ni (B, X) \to (\pi_* L)_{\mathsf{Vect} \rtimes \mathsf{Sm}}$ over B is an isomorphism iff for every map $f : Z \to B$, the induced map $f^* X \to (\pi_* f^* L)_{\mathsf{Vect}(Z)}$ is an isomorphism. In other words, for $X \in \mathsf{Vect}(B)$ and $\xi : X \to \pi_* L$, the following are equivalent:

(3.8.5.1) (X, ξ) represents $\pi_* L$ on $\mathsf{Vect} \rtimes \mathsf{Sm}$.

(3.8.5.2) $(f^* X, f^* \xi)$ represents $\pi_* L$ on $\mathsf{Vect}(Z)$ for every map $f : Z \to B$.

Representability on $\mathsf{Vect} \rtimes \mathsf{Sm}$ thus encodes representability on every $\mathsf{Vect}(Z)$ by objects compatible with pullback.

3.8.6 Corollary. *In the setup of* (3.8.2.2) *(L a fiberwise isomorphism), the map* $L : \underline{\mathrm{Sec}}_B(Q, E) \to \underline{\mathrm{Sec}}_B(Q, F)$ *is an isomorphism of stacks on* $\mathsf{Vect} \rtimes \mathsf{Sm}$*. The same holds for matrix operators as in* (**??**).

Proof. By (3.8.5), it is equivalent to show that L is an isomorphism of presheaves on $\mathsf{Vect}(Z)$ for every $f : Z \to B$; we may also replace B with Z. We are thus reduced to showing that $L : \underline{\mathrm{Sec}}_B(Q, E) \to \underline{\mathrm{Sec}}_B(Q, F)$ is an isomorphism of presheaves on $\mathsf{Vect}(B)$.

Here is a trick. We know that the map $L : \underline{\mathrm{Sec}}_B(Q, E) \to \underline{\mathrm{Sec}}_B(Q, F)$ is an isomorphism of smooth stacks (3.8.2.2). A retract of an isomorphism is an isomorphism, so it suffices to show that $L : \underline{\mathrm{Sec}}_B(Q,E)_{\mathsf{Vect}(B)} \to \underline{\mathrm{Sec}}_B(Q,F)_{\mathsf{Vect}(B)}$ is a retract of $(\mathsf{Vect}(B) \to (\mathsf{Sm} \downarrow B))^*L$. The presheaf $\underline{\mathrm{Sec}}_B(Q,E)_{\mathsf{Vect}(B)}$ assigns to $V \in \mathsf{Vect}(B)$ the set of *linear* maps $\pi^*V \to E$ over Q, while the pullback presheaf $(\mathsf{Vect}(B) \to (\mathsf{Sm} \downarrow B))^*\underline{\mathrm{Sec}}_B(Q,E)_{\mathsf{Sm}}$ assigns the set of *smooth* maps (of total spaces) $\pi^*V \to E$ over Q. A smooth map $\pi^*V \to E$ over Q determines a linear map over Q by vertical differentiation along the zero section. This defines a retraction of the forgetful map $\underline{\mathrm{Sec}}_B(Q,E)_{\mathsf{Vect}(B)} \to (\mathsf{Vect}(B) \to (\mathsf{Sm} \downarrow B))^*\underline{\mathrm{Sec}}_B(Q,E)_{\mathsf{Sm}}$ which is evidently compatible with the action of L. □

3.8.7 Corollary. *In the setup of* (3.8.4.2) *(L fiberwise surjective), the stack* $(\pi_*L)_{\mathsf{Vect}\rtimes\mathsf{Sm}}$ *is representable by a vector bundle over B, and the comparison map* $(\mathsf{Vect}\rtimes\mathsf{Sm} \to \mathsf{Sm})_!(\pi_*L)_{\mathsf{Vect}\rtimes\mathsf{Sm}} \to (\pi_*L)_{\mathsf{Sm}}$ *is an isomorphism. The same holds for matrix operators as in* (**??**)*.*

Proof. The argument of (3.8.4) applies, with (3.8.6) in place of (3.8.2.2), to show that $(\pi_*L)_{\mathsf{Vect}\rtimes\mathsf{Sm}}$ is representable by a vector bundle over B. Passing to its total space yields the smooth manifold which represents $(\pi_*L)_{\mathsf{Sm}}$ according to (3.8.4), which is what it means for the comparison map $(\mathsf{Vect} \rtimes \mathsf{Sm} \to \mathsf{Sm})_!(\pi_*L)_{\mathsf{Vect}\rtimes\mathsf{Sm}} \to (\pi_*L)_{\mathsf{Sm}}$ to be an isomorphism. □

3.8.8 Exercise. Here is an alternative proof of (3.8.6) and (3.8.7) which is more intuitive though also more technical. Recall that a (real) vector space object in a category C is a functor $\mathsf{Vect}_{\mathbb{R}}^{\mathsf{op}} \to \mathsf{C}$ (where $\mathsf{Vect}_{\mathbb{R}}$ here denotes finite-dimensional real vector spaces) sending finite coproducts (direct sums) of vector spaces to products in C. Show that $\underline{\mathrm{Sec}}_B(Q,-)$ sends limits of smooth manifolds over Q to limits of smooth stacks over B (compare (2.3.54)). Conclude that a vector space object structure on $E \to Q$ (in particular, a vector bundle structure) determines a vector space object structure on $\underline{\mathrm{Sec}}_B(Q,E) \to B$. Show that $\underline{\mathrm{Sec}}_B(Q,E)_{\mathsf{Vect}\rtimes\mathsf{Sm}}$ is determined functorially from $\underline{\mathrm{Sec}}_B(Q,E)_{\mathsf{Sm}} \to B$ as a vector space object, by noting that a map $(Z,V) \to \underline{\mathrm{Sec}}_B(Q,E)_{\mathsf{Vect}\rtimes\mathsf{Sm}}$ is a map $Z \to B$ together with a morphism of vector space objects $(V \to Z) \to (\underline{\mathrm{Sec}}_B(Q,E)_{\mathsf{Sm}} \times_B Z \to Z)$. Show that $L : \underline{\mathrm{Sec}}_B(Q,E)_{\mathsf{Sm}} \to \underline{\mathrm{Sec}}_B(Q,F)_{\mathsf{Sm}}$ lifts naturally to a morphism of vector space objects in $(\mathsf{Shv}(\mathsf{Sm}) \downarrow B)$ and that the induced morphism $L : \underline{\mathrm{Sec}}_B(Q,E)_{\mathsf{Vect}\rtimes\mathsf{Sm}} \to \underline{\mathrm{Sec}}_B(Q,F)_{\mathsf{Vect}\rtimes\mathsf{Sm}}$ agrees with that defined earlier. Conclude that (3.8.2.2) implies (3.8.6). Now show that π_*L is a vector space object over B (since it is a limit of such) and that the argument of (3.8.4) identifies it locally with $\mathbb{R}^k \times B \to B$ as vector space objects (and hence that $\pi_*L \to B$ is a vector bundle).

Our next goal is to describe the pushforward π_*L in the general case (i.e. not assuming L is fiberwise surjective). To do this, we consider the ∞-category $\mathsf{Perf} \rtimes \mathsf{Sm}$ (**??**) in place of $\mathsf{Vect} \rtimes \mathsf{Sm}$, and we wish to show that $(\pi_*L)_{\mathsf{Perf}\rtimes\mathsf{Sm}} \to B$ is representable by an object of $\mathsf{Perf}^{[0\ 1]}(B)$. To define π_*L as a stack on $\mathsf{Perf} \rtimes \mathsf{Sm}$, we must make sense of $L : \underline{\mathrm{Sec}}_B(Q,E) \to \underline{\mathrm{Sec}}_B(Q,F)$ as a map of stacks on $\mathsf{Perf} \rtimes \mathsf{Sm}$. Concretely, this means we should specify the action $L : \mathrm{Hom}(\pi^*V^\bullet, E) \to \mathrm{Hom}(\pi^*V^\bullet, F)$ for $V^\bullet \in \mathsf{Perf}(B)$, which we should certainly take to be given degreewise by the maps $L : \mathrm{Hom}(\pi^*V^i, E) \to \mathrm{Hom}(\pi^*V^i, F)$ defined earlier and encoded by the map L of stacks on $\mathsf{Vect} \rtimes \mathsf{Sm}$.

3.8.9 Corollary. *In the setup of* (3.8.2.2) *(L a fiberwise isomorphism), the map* $L : \underline{\mathrm{Sec}}_B(Q,E) \to \underline{\mathrm{Sec}}_B(Q,F)$ *is an isomorphism of stacks on* $\mathsf{Perf} \rtimes \mathsf{Sm}$*. The same holds for matrix operators as in* (**??**)*.*

Proof. The action of L over $\mathsf{Perf} \rtimes \mathsf{Sm}$ is defined functorially in terms of its action over $\mathsf{Vect} \rtimes \mathsf{Sm}$ (see above), which is an isomorphism by (3.8.6). □

3.8.10 Corollary. *In the setup of* (3.8.4.2)(3.8.7) *(L fiberwise surjective), the comparison map*

$$(\mathsf{Vect} \rtimes \mathsf{Sm} \to \mathsf{Perf} \rtimes \mathsf{Sm})_!(\pi_* L)_{\mathsf{Vect}\rtimes\mathsf{Sm}} \to (\pi_* L)_{\mathsf{Perf}\rtimes\mathsf{Sm}} \tag{3.8.10.1}$$

is an isomorphism. The same holds for matrix operators as in (**??**).

Proof. The argument of (3.8.7) produces a representing object for $(\pi_* L)_{\mathsf{Vect}\rtimes\mathsf{Sm}}$, and this object represents $(\pi_* L)_{\mathsf{Perf}\rtimes\mathsf{Sm}}$ (via the same map) by substituting (3.8.9) in place of (3.8.6) in the argument of (3.8.7). □

3.8.11 Lemma. *Let L be a vertical elliptic operator on a proper submersion $Q \to B$ of smooth manifolds. There exists (near any point $b \in B$) a smooth section $Q \to F \otimes (\mathbb{R}^k)^*$ (inducing a map $\alpha : \mathbb{R}^k \to \underline{\mathrm{Sec}}_B(Q, F)$) for which $L_b \oplus \alpha_b$ is surjective. The same holds for matrix operators as in* (**??**).

Proof. This is similar to (3.8.3). Since L_b is elliptic and Q_b is compact, the cokernel of L_b is finite-dimensional (3.4.28). Fix a finite collection of smooth sections $Q_b \to F_b$ spanning this cokernel, and extend each of them to a neighborhood of $Q_b \subseteq Q$ arbitrarily (e.g. by taking a linear combination of local extensions according to a partition of unity). □

We now come to the main result, namely the existence of the derived pushforward of a family of elliptic operators. The main analytic ingredient underlying this result is (3.8.2) (plus the easy lemmas (3.8.3)(3.8.11)); the rest of the reasoning has been purely formal.

* *

We now consider vertical elliptic operators on simply-broken submersions of log smooth manifolds (2.7.82). Recall that these are maps $Q \to B$ which are locally (on the source) pulled back from $\mathbb{R}^k \to *$, $'\mathbb{R}_{\geq 0} \times \mathbb{R}^k \to *$, or $'\mathbb{R}^2_{\geq 0} \times \mathbb{R}^k \to {}'\mathbb{R}_{\geq 0}$ (the map $(x, y, t) \mapsto xy$). The fibers of such a submersion over interior points of B are thus manifolds with asymptotically cylindrical ends (3.6.2).

⋆ **3.8.12 Proposition** (Fiberwise isomorphism implies isomorphism)**.** *Let L be a vertical elliptic operator on a proper simply-broken submersion $Q \to B$ of log smooth manifolds.*

(3.8.12.1) *If L_b is an isomorphism and non-degenerate for some $b \in B$, then $L_{b'}$ is an isomorphism and non-degenerate for all b' in a neighborhood of b.*

(3.8.12.2) *If L_b is an isomorphism and non-degenerate for every $b \in B$, then $L : \underline{\mathrm{Sec}}_B(Q, E) \to \underline{\mathrm{Sec}}_B(Q, F)$ is an isomorphism of log smooth stacks.*

In comparison with the case of smooth manifolds (3.8.2), the new feature in the present setting is that a proper simply-broken submersion $Q \to B$ of log smooth manifolds need not be locally trivial. The proof of (3.8.2) relied crucially on measuring how the fiberwise operators L_b and sections u_b vary as functions of $b \in B$ *with respect to a choice of trivialization* *of the family* $Q \to B$. To adapt this argument to the present simply-broken setting, the main point is to express $C^\infty(Q_b)$ as a direct summand of $C^\infty(Q_0)$ for b in a small neighborhood of any given basepoint $0 \in B$. More precisely, Hofer's anti-gluing construction [42] identifies $C^\infty(Q_0)$ as the direct sum of $C^\infty(Q_b)$ and $C^\infty(\mathbb{R} \times N)$ for N the singular locus of Q_0. This allows us to run the same sort of argument as in (3.8.2).

Proof. It suffices (compare the discussion surrounding (3.8.2.3)) to show that if L_0 is an isomorphism and non-degenerate for some basepoint $0 \in B$, then after replacing B with a neighborhood of said basepoint, every L_b is an isomorphism and non-degenerate and for smooth $f : Q \to F$, the map $L^{-1}f : Q \to E$ (defined fiberwise since every L_b is an isomorphism) is also smooth. This assertion is manifestly local on B. Recall that openness of non-degeneracy is straightforward (**??**).

Part I: Gluing coordinates on the family $Q \to B$. To begin, let us fix coordinates on our proper simply-broken submersion $Q \to B$ using (2.7.85), which says $Q \to B$ is (locally near $0 \in B$) a pullback of a standard gluing family $M \to {}'\mathbb{R}_{\geq 0}^{\pi_0 N/\sigma}$ (2.7.84) associated to a tuple $(M_0^{\mathrm{pre}}, N, i, \sigma)$ along a map $\lambda : B \to {}'\mathbb{R}_{\geq 0}^{\pi_0 N/\sigma}$. The same argument shows that every vector bundle on Q is the pullback of a vector bundle on M obtained via gluing from a vector bundle on M_0^{pre} which over the image of i is the pullback of a σ-equivariant vector bundle on N (2.7.88).

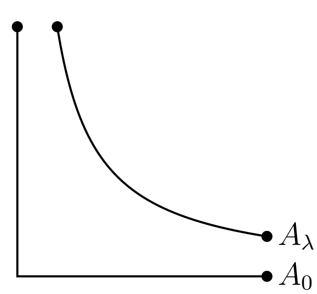
Part II: The Hofer gluing isomorphism. We next fix an isomorphism

$$C^\infty(M_0) = C^\infty(M_\lambda) \oplus C^\infty([-\infty,\infty] \times N/\sigma, [-\infty,\infty]^{\mathrm{id}} \times N/\sigma) \oplus C^\infty(N/\sigma) \qquad (3.8.12.3)$$

for every $\lambda > 0$ (for simplicity of notation we assume a single gluing component $\pi_0 N/\sigma = *$); we consider here sections of E and F, though they are omitted from the notation for reasons which we now explain. Away from a small neighborhood of the 'singular locus' $N \times (0,0) \subseteq N \times {}'\mathbb{R}^2_{\geq 0} \subseteq M$, the fibers M_0 and M_λ are identified by construction. Near the singular locus, the gluing identification (3.8.12.3) will depend only on the ${}'\mathbb{R}^2_{\geq 0}$ coordinate, hence the vector bundles E and F may be ignored since they are pulled back from N over this region. For the purposes of writing formulae, it thus suffices to construct a ($\mathbb{Z}/2$-equivariant) isomorphism

$$C^\infty(A_0) = C^\infty(A_\lambda) \oplus C^\infty([-\infty,\infty], [-\infty,\infty]^{\mathrm{id}}) \oplus \mathbb{R} \qquad (3.8.12.4)$$

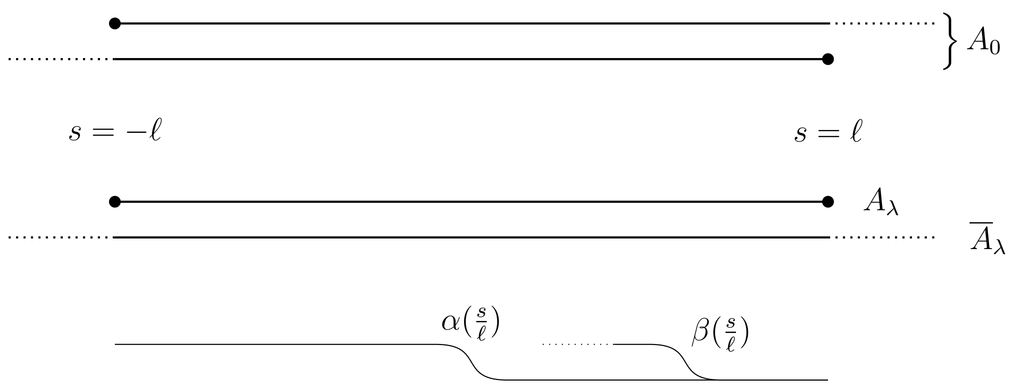
where A_λ denotes the fiber of the multiplication map ${}'\mathbb{R}^2_{\geq 0} \to {}'\mathbb{R}_{\geq 0}$ over $\lambda \in {}'\mathbb{R}_{\geq 0}$.



(3.8.12.5)

We have coordinates $(x,y) \in {}'\mathbb{R}^2_{\geq 0}$ (say $x, y \leq 1$) satisfying $xy = \lambda$ over A_λ. It is convenient to introduce the shifted log (aka cylindrical) coordinates $s_x = -\ell - \log x$ and $s_y = -\ell - \log y$

(where $\ell = -\log \lambda^{1/2} > 0$). Thus the two 'branches' of A_0 have separate coordinates $s_x, s_y \in [-\ell, \infty)$, while on A_λ we have coordinates $s_x, s_y \in [-\ell, \ell]$ which are negatives of each other. In fact, we will unify these into a single coordinate $s = s_x = -s_y$ on the two branches of A_0 and on A_λ; note that this coordinate is compatible with the identification of M_0 and M_λ away from the singular locus. We also introduce the notation $\overline{A}_\lambda$ for the cylinder $[-\infty, \infty]$ with this same coordinate s (more intrinsically, we may regard $\overline{A}_\lambda$ as obtained from A_0 by normalizing (2.7.79) and identifying $(x, 0)$ and $(0, y)$ when $xy = \lambda$).



(3.8.12.6)

Fix smooth functions α and β with the following properties.

$$\alpha : \mathbb{R} \to [0,1] \qquad \alpha(s) = \begin{cases} 1 & s \leq -\frac{1}{5} \\ 0 & s \geq \frac{1}{5} \end{cases} \qquad \alpha(s) + \alpha(-s) = 1 \tag{3.8.12.7}$$

$$\beta : \mathbb{R} \to [0,1] \qquad \beta(s) = \begin{cases} 1 & s \leq \frac{3}{5} \\ 0 & s \geq \frac{4}{5} \end{cases} \tag{3.8.12.8}$$

Note that $\alpha(s)\beta(s) = \alpha(s)$. Now we consider the following maps identifying $C^\infty(A_0)$ with $\mathbb{R} \oplus C^\infty(A_\lambda) \oplus C^\infty(\overline{A}_\lambda, \overline{A}_\lambda^{\mathrm{id}})$. The introduction of the 'anti-gluing' component $u_{\mathrm{anti}} \in C^\infty(\overline{A}_\lambda, \overline{A}_\lambda^{\mathrm{id}})$ is due to Hofer [42].

$$(u_\infty, u_x, u_y) \in C^\infty(A_0) \tag{3.8.12.9}$$
$$(u_\infty, u_{\mathrm{glue}}, u_{\mathrm{anti}}) \in \mathbb{R} \oplus C^\infty(A_\lambda) \oplus C^\infty(\overline{A}_\lambda, \overline{A}_\lambda^{\mathrm{id}}) \tag{3.8.12.10}$$
$$(u_\infty, u_x, u_y) \mapsto (u_\infty,\ \alpha(\tfrac{s}{\ell})u_x(s) \qquad + \alpha(\tfrac{-s}{\ell})u_y(s), \tag{3.8.12.11}$$
$$\beta(\tfrac{-s}{\ell})(u_x(s) - u_\infty) - \ \beta(\tfrac{s}{\ell})(u_y(s) - u_\infty)) \tag{3.8.12.12}$$
$$(u_\infty, u_{\mathrm{glue}}, u_{\mathrm{anti}}) \mapsto (u_\infty,\ \beta(\tfrac{s}{\ell})u_{\mathrm{glue}}(s) + (1 - \ \beta(\tfrac{s}{\ell}))u_\infty + \alpha(\tfrac{-s}{\ell})u_{\mathrm{anti}}(s), \tag{3.8.12.13}$$
$$\beta(\tfrac{-s}{\ell})u_{\mathrm{glue}}(s) + (1 - \beta(\tfrac{-s}{\ell}))u_\infty - \ \alpha(\tfrac{s}{\ell})u_{\mathrm{anti}}(s)) \tag{3.8.12.14}$$

The notation $(u_\infty, u_x, u_y) \in C^\infty(A_0)$ indicates that u_x and u_y are the restrictions of a function on A_0 to the two axes $'\mathbb{R}_{\geq 0} \times 0$ and $0 \times \mathbb{R}_{\geq 0}$ inside A_0 and that u_∞ is its value at the singular point $(0, 0) \in A_0$ (thus $u_x(s), u_y(s) \to u_\infty$ as $s \to \infty$). This point bears emphasis: the coordinates (u_x, u_y, u_∞) are not independent of each other (in contrast to $(u_\infty, u_{\mathrm{glue}}, u_{\mathrm{anti}})$), in particular the inclusion of u_∞ is entirely redundant, however it makes the formulae for the gluing isomorphism simpler. Here are matrices for the gluing isomorphism and its inverse.

$$G=\begin{pmatrix} 1 & 0 & 0 \\ 0 & \alpha(\frac{s}{\ell}) & \alpha(-\frac{s}{\ell}) \\ \beta(\frac{s}{\ell})-\beta(-\frac{s}{\ell}) & \beta(-\frac{s}{\ell}) & -\beta(\frac{s}{\ell}) \end{pmatrix} \quad G^{-1}=\begin{pmatrix} 1 & 0 & 0 \\ 1-\beta(\frac{s}{\ell}) & \beta(\frac{s}{\ell}) & \alpha(-\frac{s}{\ell}) \\ 1-\beta(-\frac{s}{\ell}) & \beta(-\frac{s}{\ell}) & -\alpha(\frac{s}{\ell}) \end{pmatrix} \tag{3.8.12.15}$$

It is a straightforward inspection to see that these maps are well defined, splice together with the tautological identification of M_0 and M_λ away from the singular locus, and are inverse to each other (given the specificed properties on α and β).

Part III: Sobolev norms. To further analyze the situation, we need to fix Sobolev norms on M_0 and M_λ which are well defined up to commensurability *uniform in λ near zero*. We will use the H^s_∞-norms (3.2.26) associated to the natural cylindrical geometry on M_0 and M_λ. To be precise, note that away from the singular locus, all M_λ are identified with M_0^{pre}, so we may simply take the 'same' geometry (and hence H^s_∞-norm) on this part. Near the singular locus $A_\lambda \times N \subseteq M_\lambda$, we define the H^s_∞-norm using the geometry given by cylindrical coordinates $[-\ell,\ell]$ on A_λ times any fixed geometry on N.

It is evident that the Hofer gluing isomorphism (3.8.12.3) is bounded $H^s_\infty \to H^s_\infty$ uniformly in λ. It is also evident that the operators L_b are bounded uniformly for b near $0 \in B$. Indeed, write L in cylindrical coordinates

$$L = \sum_{i,\alpha} c_{i,\alpha}(s,n)\partial_s^i\partial_n^\alpha \tag{3.8.12.16}$$

where $s \in \mathbb{R}$ and $n \in N$ are the coordinates on the partial cylinders $[-\ell,\ell]\times N \subseteq M_\lambda$ and $c_{i,\alpha}$ are log smooth functions on ${}'\mathbb{R}^2_{\geq 0} \times N$. Log smoothness of the coefficients $c_{i,\alpha}$ means their derivatives in cylindrical coordinates (i.e. with respect to our chosen geometry on M_λ) are uniformly bounded, hence the operator they define is bounded $H^s_\infty \to H^s_\infty$ by the local bound (3.3.7). Similarly, for any smooth section $u : Q \to E$ or F, the H^s_∞ norm of its restriction u_b to Q_b is bounded uniformly in b near zero.

Part IV: Anti-glued extensions $\tilde f$ of f and $\tilde L$ of L. Our next step is to extend f_b and L_b from $C^\infty(Q_b)$ to $C^\infty(Q_0)$ (inside which $C^\infty(Q_b)$ is a direct summand (3.8.12.3)), for some smooth section $f : Q \to F$. To define the extension $\tilde f_b$, it suffices to specify its components in $C^\infty(N_b/\sigma_b)$ and $C^\infty(\overline{Q}_b)$. We take the component of $\tilde f_b$ in $C^\infty(N_b/\sigma_b)$ to be the weighted average $\big(\int_{\mathbb{R}}\beta(\frac{s}{\ell})\beta(-\frac{s}{\ell})\cdot f_b\,ds\big)/\big(\int_{\mathbb{R}}\beta(\frac{s}{\ell})\beta(-\frac{s}{\ell})\,ds\big)$ in the $\mathbb{R}$-coordinate direction. We take the 'anti-glued' component of $\tilde f_b$ in $C^\infty(\overline{Q}_b)$ to be zero. Now $\tilde f : B \to C^\infty(Q_0)$ is continuous with respect to the H^s_∞-topology for any $s < \infty$ (this is immediate on the inverse image under $\lambda : B \to {}'\mathbb{R}^{\pi_0 N/\sigma}_{\geq 0}$ of any stratum of ${}'\mathbb{R}^{\pi_0 N/\sigma}_{\geq 0}$, and across strata it follows from vertical smoothness of f via an explicit inspection of the Hofer gluing isomorphism (3.8.12.3)).

The extension $\tilde L_b$ of L_b will be diagonal with respect to the Hofer gluing isomorphism (3.8.12.3), so it suffices to specify its action via endomorphisms of $C^\infty(N_b/\sigma_b)$ and $C^\infty(\overline{Q}_b)$. To define these actions, we write L_b in cylindrical coordinates (3.8.12.16), and we again take the $\beta(\frac{s}{\ell})\beta(-\frac{s}{\ell})\,ds$ weighted average of its coefficients to obtain a cylindrical differential operator on $\mathbb{R}\times N_b$. Such an operator descends to an operator on $C^\infty(\overline{Q}_b)$ and an operator on

$C^\infty(N_b/\sigma_b)$ (by restricting to $(\mathbb{R}\rtimes(\mathbb{Z}/2))$-invariant sections (3.6.10), i.e. setting $\partial_s=0$). Now we claim that map $\tilde{L}:B\to\operatorname{End}(C^\infty(Q_0))$ is continuous with respect to the H^s_∞ operator norm. This is again immediate on the inverse image of any stratum of ${}'\mathbb{R}_{\geq 0}^{\pi_0 N/\sigma}$. To verify it across strata, we should transport $\tilde{L}_b$ and $\tilde{L}_{b'}$ to the same fiber Q_0 via the Hofer gluing isomorphism. The result now follows by inspection, from the fact that the coefficients of L_b are vertically smooth (the point is that the commutator of a differential operator with either of the functions $\alpha(\frac{s}{\ell})$ and $\beta(\frac{s}{\ell})$ carries a factor of ℓ^{-1}, hence approaches zero uniformly in all derivatives as $\ell\to\infty$).

Part V: Vertical smoothness of $L^{-1}f$. Since $\tilde{L}:B\to\operatorname{Hom}(C^\infty(Q_0,E),C^\infty(Q_0,F))$ is continuous in H^s_∞-operator-norm and $\tilde{L}_0=L_0$ is invertible, there exists a neighborhood of $0\in B$ (*a priori* depending on s) over which every $\tilde{L}_b$ is invertible by the usual formula $\tilde{L}_b^{-1}=\sum_{i\geq 0}\tilde{L}_0^{-1}(\mathbf{1}-\tilde{L}_b\tilde{L}_0^{-1})^i=\sum_{i\geq 0}(\mathbf{1}-\tilde{L}_0^{-1}\tilde{L}_b)^i\tilde{L}_0^{-1}$ (**??**) (thus its direct summand L_b is also invertible over this neighborhood, giving (3.8.12.1)). Since $\tilde{f}:B\to C^\infty(Q_0,F)$ is continuous in the H^s_∞-topology, it follows that $\tilde{L}^{-1}\tilde{f}:B\to C^\infty(Q_0,E)$ is continuous in the H^s_∞-topology. It follows from inspecting the gluing map (3.8.12.11)–(3.8.12.12) that the section $(\tilde{L}^{-1}\tilde{f})_{\mathrm{glue}}:Q\to E$ obtained from $\tilde{L}^{-1}\tilde{f}$ by gluing is vertically C^k (2.10.14) whenever $H^s\subseteq C^k$ (3.2.28). This section $(\tilde{L}^{-1}\tilde{f})_{\mathrm{glue}}$ is nothing other than $L^{-1}f$ (recall that by definition $\tilde{L}_b$ is diagonal and that the components of $\tilde{L}_b$ and $\tilde{f}_b$ in the direct summand $C^\infty(Q_b)\subseteq C^\infty(Q_0)$ are simply L_b and f_b). Thus $L^{-1}f$ is vertically C^k (over a neighborhood of $0\in B$ depending on k, hence over the entire open locus where L is fiberwise surjective) for every $k<\infty$, hence vertically smooth.

Part VI: Smoothness of $L^{-1}f$. Finally, let us show that $L^{-1}f:Q\to E$ is smooth. This proof will in fact be independent of the proof of vertical smoothness given just above (and hence that discussion could have been omitted, though it is a good warm up for the present one).

In outline, the argument goes as follows. We deduce smoothness of $L^{-1}f=(\tilde{L}^{-1}\tilde{f})_{\mathrm{glue}}$ from smoothness of $\tilde{L}^{-1}\tilde{f}:B\to C^\infty(Q_0,E)$ in the H^s_∞-topology. This, in turn, is deduced from smoothness of $\tilde{L}$ and $\tilde{f}$ in the H^s_∞-operator-topology and H^s_∞-topology, respectively. Finally, smoothness of $\tilde{L}$ and $\tilde{f}$ will be verified explicitly using smoothness of L and f.

To implement this outline, we should first define a notion of smoothness for maps from log smooth manifolds to (complete) topological vector spaces. Recall that a notion of smoothness for maps from smooth manifolds to topological vector spaces was given above in (3.1.8). We now say that a map from a log smooth manifold M to a topological vector space V is of class C^1 when its restriction to every stratum of M (each of which is a smooth manifold) is C^1 and the resulting derivative map $TM\to TV$ is continuous; then C^k is defined inductively as usual.

It is straightforward to check that if $f:M\to V$ and $g:M\to W$ are C^k and $h:V\times W\to Z$ is continuous bilinear, then $h(f,g):M\to Z$ is C^k (identify $T(h(f,g))$ with $(Th)(Tf,Tg)$ for a certain continuous bilinear map $Th:TV\times TW\to TZ$ and use induction). It is also straightforward to check that if V has a norm topology and $f:M\to\operatorname{Hom}(V,V)$ is C^k in the operator norm topology and invertible pointwise, then $f^{-1}:M\to\operatorname{Hom}(V,V)$ is C^k (express f^{-1} locally in terms of the series (**??**) and show it converges uniformly in C^k).

Given this notion of smoothness for maps from log smooth manifolds to complete topological vector spaces and its properties, it suffices to show the following:

(3.8.12.17) $\tilde{f}$ is smooth in the H^s_∞-topology.

(3.8.12.18) $\tilde{L}$ is smooth in the H^s_∞-operator-topology.

(3.8.12.19) If $u : B \to C^\infty(Q_0, E)$ is smooth in the H^s_∞-topology, then $u_{\mathrm{glue}} : Q \to E$ is C^k for $H^s \subseteq C^k$.

To prove these assertions, we should analyze how derivatives interact with the Hofer gluing isomorphism (3.8.12.3).

$\square$

Chapter 4

Riemann surfaces

We assume some basic familiarity with complex analysis, say Ahlfors [6, 7].

4.1 Basic notions

4.1.1 Definition (Riemann surface)**.** A *Riemann surface* C is a topological space with an atlas of charts from open subsets of $\mathbb{C}$ whose transition functions are holomorphic (compare (2.4.1)). Equivalently, a Riemann surface S is a smooth manifold of dimension two equipped with a smooth endomorphism $j : TS \to TS$ with $j^2 = -1$ (called an *almost complex structure*) (4.1.2).

4.1.2 Exercise (Integrability of almost complex structures in dimension two)**.** To distinguish between the two notions of a Riemann surface (4.1.1) (before we prove them to be equivalent), let us call them 'holomorphic Riemann surface' and 'smooth Riemann surface'. Define a tautological forgetful functor from holomorphic Riemann surfaces to smooth Riemann surfaces by regarding $\mathbb{C}$ as $\mathbb{R}^2$ with the almost complex structure $j_{\mathrm{std}}(\partial_x) = \partial_y$ where $z = x+iy$. Show that this forgetful functor is fully faithful. Show that it is essentially surjective iff for every smooth Riemann surface S and every point $p \in S$, there exists a (locally defined) function $f : (S, p) \to \mathbb{C}$ whose derivative is $\mathbb{C}$-linear (everywhere) and is non-zero at p. Show that the operator $f \mapsto idf - df \circ j_S$ is elliptic (3.4.1). Conclude from the local existence theory for linear elliptic equations (**??**) that the desired local functions exist.

4.1.3 Definition (Bordered Riemann surface)**.** A *bordered Riemann surface* C is a topological space with an atlas of charts from open subsets of $\mathbb{C}_{\mathrm{Im}\geq 0} = \{z \in \mathbb{C} : \operatorname{Im} z \geq 0\}$ whose transition functions are holomorphic (**??**). Equivalently, a bordered Riemann surface S is a smooth manifold-with-boundary of dimension two equipped with a smooth endomorphism $j : TS \to TS$ with $j^2 = -1$ (**??**).

Chapter 5

Pseudo-holomorphic maps

5.1 Symplectic and almost complex structures

Linear theory

We begin with the linear story.

5.1.1 Definition. Let V be a finite-dimensional real vector space.

(5.1.1.1) A *complex structure* on V is an endomorphism $J : V \to V$ satisfying $J^2 = -\mathbf{1}$ (equivalently, it is a lift of V to a complex vector space).

(5.1.1.2) A *symplectic form* on V is an anti-symmetric pairing $\omega : V \otimes V \to \mathbb{R}$ which is *non-degenerate*, meaning that the induced map $V \to V^*$ is an isomorphism.

(5.1.1.3) A *metric* on V is a symmetric pairing $g : V \otimes V \to \mathbb{R}$ which is *positive definite*, meaning that $g(v, v) > 0$ for $v \neq 0$.

(5.1.1.4) A pair (J, ω) is called *compatible* when $g(v, w) = \omega(v, Jw)$ is a metric (i.e. is symmetric and positive definite).

(5.1.1.5) A pair (J, ω) is called *tame* when $\omega(v, Jv) > 0$ for $v \neq 0$. Compatible pairs are evidently tame. A tame pair also determines a metric $g(v, w) = \omega(v, Jw) + \omega(w, Jv)$.

More generally, these notions apply (fiberwise) to any vector bundle V over a smooth manifold.

5.1.2 Exercise. Let V be a symplectic vector space. Show that if a subspace $P \subseteq V$ is symplectic, then its ω-orthogonal subspace $P^\perp$ ($v \in P^\perp$ iff $\omega(v, p) = 0$ for all $p \in P$) is also symplectic and is a complement of P. Conclude that there exists a basis $v_1, \ldots, v_n, w_1, \ldots, w_n$ of V satisfying $\omega(v_i, v_j) = \omega(w_i, w_j) = 0$ and $\omega(v_i, w_j) = \delta_{ij}$.

5.1.3 Lemma. *Let V be a finite-dimensional real vector space. The maps*

$$\begin{tikzcd} & \{(\omega, J) \text{ compatible}\} \arrow[dl] \arrow[d] \arrow[dr] & \\ \{\omega\} & & \{J\} \\ & \{(\omega, J) \text{ tame}\} \arrow[ul] \arrow[ur] & \end{tikzcd} \tag{5.1.3.1}$$

are homotopy equivalences.

Proof. For fixed J, the space of tame (resp. compatible) ω is convex and non-empty, hence contractible. It thus suffices to show that for fixed ω, the space of compatible J is contractible, which goes as follows. Fix $v \in V$ arbitrarily. The value of $J(v)$ must lie in $\{w \in V : \omega(v, w) > 0\}$, which is convex and non-empty, hence contractible. For $v, w \in V$ with $\omega(v, w) > 0$, a compatible J satisfying $J(v) = w$ stabilizes the ω-orthogonal complement of $\operatorname{span}(v, w) \subseteq V$. The space of compatible almost complex structures on this orthogonal complement is contractible by induction on the dimension of V. $\square$

5.1.4 Definition. Let V be a symplectic vector space. For a subspace $P \subseteq V$, denote by $P^\perp \subseteq V$ its *ω-orthogonal*, consisting of the vectors v for which $\omega(v, p) = 0$ for all $p \in P$.

(5.1.4.1) P is called *isotropic* when $P \subseteq P^\perp$.

(5.1.4.2) P is called *co-isotropic* when $P \supseteq P^\perp$.
(5.1.4.3) P is called *Lagrangian* when $P = P^\perp$.

5.1.5 Definition. Let V be a complex vector space. A subspace $P \subseteq V$ is called *totally real* when $P \cap JP = 0$ and $P + JP = V$ (equivalently, when the natural map $P \otimes_{\mathbb{R}} \mathbb{C} \to V$ is an isomorphism).

5.1.6 Exercise. Let V be a symplectic vector space equipped with a tame almost complex structure. Show that a Lagrangian subspace $P \subseteq V$ is totally real.

Non-linear theory

We now continue on to the setting of manifolds.

5.1.7 Definition. Let M be a smooth manifold.
(5.1.7.1) A *metric* on M is a smooth fiberwise metric on TM.
(5.1.7.2) An *almost symplectic form* on M is a smooth fiberwise symplectic form on TM. A *symplectic form* is an almost symplectic form ω satisfying $d\omega = 0$.
(5.1.7.3) A submanifold $L \subseteq M$ of a symplectic manifold (M, ω) is called *Lagrangian* when $TL \subseteq TM$ is Lagrangian at every point of L.
(5.1.7.4) An *almost complex structure* on M is a smooth fiberwise complex structure on TM. A *complex structure* is an almost complex structure which is locally isomorphic to $(\mathbb{C}^n, J_{\mathrm{std}} = i)$; such almost complex structures are also called *integrable*.
(5.1.7.5) A submanifold $L \subseteq M$ of an almost complex manifold (M, J) is called *totally real* when $TL \subseteq TM$ is totally real at every point of L.

5.2 Pseudo-holomorphic moduli problems

In this section, we recall the pseudo-holomorphic map equation and the various geometric settings in which this equation and its variants are defined.

⋆ **5.2.1 Definition** (Pseudo-holomorphic map)**.** A map $u : C \to X$ from a Riemann surface C to an almost complex manifold X is called *pseudo-holomorphic* when its differential $du : TC \to u^*TX$ is complex linear.

A pair (C, X) as above is the simplest instance of what we will call a pseudo-holomorphic map *problem*. The pseudo-holomorphic maps $u : C \to X$ are called the *solutions* of this pseudo-holomorphic map problem. We will see pseudo-holomorphic map problems which involve sections, allow domains with boundary (paired with appropriate boundary conditions), impose point constraints, and allow varying domains and targets.

5.2.2 Definition (Complex conjugate vector space)**.** For a complex vector space V, we denote by $\overline{V}$ its complex conjugate, namely its pullback under the conjugation automorphism of $\mathbb{C}$. Concretely, $\overline{V} = V$ as sets; for $v \in V$, the corresponding element of $\overline{V}$ is denoted $\overline{v}$; and the vector space structure on $\overline{V}$ is that suggested by the notation, namely $\overline{v} + \overline{w} = \overline{v + w}$ and $\lambda\overline{v} = \overline{\overline{\lambda}v}$. There is an evident identification $\overline{\overline{V}} = V$. A complex linear map $\overline{V} \to W$ is the same as a complex conjugate linear map $V \to W$.

5.2.3 Definition (Decomposition of the complexification of a complex vector space)**.** Let V be a complex vector space, and consider $V \otimes_{\mathbb{R}} \mathbb{C}$, regarded as a complex vector space via the second factor. There is a canonical map

$$V \otimes_{\mathbb{R}} \mathbb{C} \to V \oplus \overline{V}, \tag{5.2.3.1}$$
$$v \otimes \lambda \mapsto \lambda v \oplus \overline{\lambda}\overline{v}, \tag{5.2.3.2}$$

of complex vector spaces. In fact, this map is an isomorphism, with inverse given by

$$V \oplus \overline{V} \to V \otimes_{\mathbb{R}} \mathbb{C}, \tag{5.2.3.3}$$
$$v \oplus \overline{w} \mapsto \tfrac{1}{2}(v \otimes 1 - iv \otimes i) + \tfrac{1}{2}(w \otimes 1 + iw \otimes i). \tag{5.2.3.4}$$

5.2.4 Definition (Identifying the real dual and the complex dual)**.** Let V be a complex vector space. It has both a complex dual and a real dual

$$V^{*_{\mathbb{C}}} = \operatorname{Hom}_{\mathbb{C}}(V, \mathbb{C}) \quad \text{and} \quad V^{*_{\mathbb{R}}} = \operatorname{Hom}_{\mathbb{R}}(V, \mathbb{R}). \tag{5.2.4.1}$$

The complex dual $V^{*_{\mathbb{C}}}$ is evidently a complex vector space. We equip the real dual $V^{*_{\mathbb{R}}}$ with the complex structure $\xi \mapsto J^*\xi$. We identify $V^{*_{\mathbb{R}}} = V^{*_{\mathbb{C}}}$ via the inverse pair of complex linear isomorphisms given by

$$\operatorname{Hom}_{\mathbb{R}}(V, \mathbb{R}) \to \operatorname{Hom}_{\mathbb{C}}(V, \mathbb{C}), \qquad \operatorname{Hom}_{\mathbb{C}}(V, \mathbb{C}) \to \operatorname{Hom}_{\mathbb{R}}(V, \mathbb{R}), \tag{5.2.4.2}$$
$$\xi \mapsto \frac{1}{2}(\xi - iJ^*\xi), \qquad \zeta \mapsto 2\operatorname{Re}\zeta, \tag{5.2.4.3}$$

and henceforth we will simply write V^* for $V^{*_{\mathbb{R}}} = V^{*_{\mathbb{C}}}$ except when there is a need to distinguish between the two.

5.2.5 Exercise. Let V and W be complex vector spaces. Show that writing a real linear map $f : V \to W$ as the sum $f = f^{1,0} + f^{0,1}$ of the complex linear map $f^{1,0} = \frac{1}{2}(f - i \circ f \circ i)$ and the complex conjugate linear map $f^{0,1} = \frac{1}{2}(f + i \circ f \circ i)$ defines a direct sum decomposition

$$\operatorname{Hom}_{\mathbb{R}}(V, W) = \operatorname{Hom}_{\mathbb{C}}(V, W) \oplus \operatorname{Hom}_{\mathbb{C}}(\overline{V}, W). \tag{5.2.5.1}$$

Moreover, show that the following diagram of isomorphisms defined thus far commutes.

$$\begin{tikzcd}
\operatorname{Hom}_{\mathbb{R}}(V,W) \arrow[r, equal] \arrow[ddd, equal] & V^{*_{\mathbb{R}}} \otimes_{\mathbb{R}} W \arrow[d, equal] \\
 & V^{*_{\mathbb{C}}} \otimes_{\mathbb{R}} W \arrow[d, equal] \\
 & V^{*_{\mathbb{C}}} \otimes_{\mathbb{R}} \mathbb{C} \otimes_{\mathbb{C}} W \arrow[d, equal] \\
\operatorname{Hom}_{\mathbb{C}}(V,W) \oplus \operatorname{Hom}_{\mathbb{C}}(\overline{V},W) \arrow[r, equal] & (V^{*_{\mathbb{C}}} \oplus \overline{V^{*_{\mathbb{C}}}}) \otimes_{\mathbb{C}} W
\end{tikzcd} \tag{5.2.5.2}$$

Conclude that the equation asserting pseudo-holomorphicity of a smooth map $u : C \to X$ thus reads $(du)^{0,1} = 0$ in the space of sections of $u^*TX \otimes \overline{T^*C}$ over C.

Basic moduli problems

⋆ **5.2.6 Definition** (Pseudo-holomorphic section)**.** Let $\pi : W \to C$ be an almost complex submersion over a Riemann surface C (meaning W is an almost complex manifold and $d\pi$ is complex linear). Such a map $W \to C$ is a pseudo-holomorphic map problem whose solutions are the pseudo-holomorphic sections $u : C \to W$.

5.2.7 Exercise. Show that for any smooth section $u : C \to W$ of an almost complex submersion, the anti-holomorphic derivative $(du)^{0,1}$ takes values in $u^*T_{W/C} \otimes \overline{T^*C}$.

5.2.8 Exercise (Adding an inhomogeneous term)**.** Let C be a Riemann surface and X an almost complex manifold. Consider almost complex structures on $C \times X$ for which the projection to C and the inclusions of the fibers $c \times X$ are pseudo-holomorphic. Show that such almost complex structures are in natural bijection with sections $\gamma : C \times X \to TX \otimes \overline{T^*C}$. Show that the graph of $u : C \to X$ is pseudo-holomorphic for such an almost complex structure iff $(du)^{0,1} + \gamma(u) = 0$.

5.2.9 Exercise (Linear almost complex structures are real Cauchy–Riemann operators)**.** Let $E \to C$ be a complex vector bundle over a Riemann surface. Show that real Cauchy–Riemann operators $D : C^\infty(C, E) \to C^\infty(C, E \otimes \overline{T^*C})$ (**??**) are in natural bijection with almost complex structures on E for which the vector bundle structure maps $E \to C$, $\cdot : \mathbb{C} \times E \to E$, and $+ : E \times_C E \to E$ are pseudo-holomorphic. Show that a section of E lies in the kernel of D iff it is pseudo-holomorphic.

5.2.10 Definition (Point constraint)**.** A single point constraint on a pseudo-holomorphic section problem $W \to C$ is a smooth manifold A with a map $f : A \to W$. A solution to the constrained section problem $(W \to C, A \to W)$ is a pseudo-holomorphic section $u : C \to W$ along with a point $a \in A$ such that $u(C)$ contains $f(a)$.

More generally, we can consider constraints on the derivatives of the map u. Such a constraint is a map $f : A \to J^k(W/C)$ (recall the jet bundle (**??**)), to which a solution is a pseudo-holomorphic section $u : C \to W$ along with a point $a \in A$ such that $J^k u : C \to J^k(W/C)$ passes through $f(a)$. A constraint $A \to J^k(W/C)$ is evidently 'equivalent' to the constraint $A \times_{J^k(W/C)} J^{k+1}(W/C) \to J^{k+1}(W/C)$.

Multiple simultaneous point constraints may be specified by a map $A \to J^k(W/C)^n$ whose composition to C^n lands inside the locus of n-tuples of distinct points.

Here is a slight generalization of the pseudo-holomorphic curve equation which we will be relevant later as an auxiliary tool for reasons explained in (**??**). Recall the notion of the *jet space* of a submersion (**??**).

5.2.11 Definition (Quasi-holomorphic sections)**.** Consider a triple $(W \to C, H, \varphi)$ where C is a smooth surface, $\pi : W \to C$ is a submersion, H/W is a real vector bundle, and

$$\varphi : J^1(W/C) \to H \tag{5.2.11.1}$$

is an affine linear map over W (recall that $J^1(W/C) \to W$ is a torsor for $\operatorname{Hom}(\pi^*TC, T_{W/C})$) which is 'elliptic' in the following sense (5.2.11.2).

(5.2.11.2) An affine linear map $\varphi : J^1(W/C) \to H$ is called *elliptic* when its linear part $\operatorname{Hom}(\pi^*TC, T_{W/C}) \to H$ sends nonzero elements of π^*TC to isomorphisms $T_{W/C} \to H$.

A section $u : C \to W$ is then called *quasi-holomorphic* iff $\varphi(du) = 0$. The quasi-holomorphic sections are the solutions of the quasi-holomorphic map problem $(W \to C, H, \varphi)$.

5.2.12 Example (Pseudo-holomorphicity as quasi-holomorphicity)**.** If $W \to C$ is an almost complex fibration and we set $H = T_{W/C} \otimes \overline{T^*C}$ and $\varphi(\alpha) = \alpha^{0,1}$, then the quasi-holomorphic section equation $\varphi(du) = 0$ becomes the pseudo-holomorphic section equation $(du)^{0,1} = 0$.

The relevant generalization of the notion of a pseudo-holomorphic map to the setting of bordered Riemann surfaces is that of a pseudo-holomorphic map satisfying totally real boundary conditions.

⋆ **5.2.13 Definition** (Totally real boundary conditions)**.** Given a bordered Riemann surface $(C, \partial C)$ and an almost complex manifold X with a totally real submanifold $L \subseteq X$, we may consider pseudo-holomorphic maps $u : (C, \partial C) \to (X, L)$ (the pair notation indicating that $u : C \to X$ and $u(\partial C) \subseteq L$). More generally, given a totally real immersion $L \to X$, we may consider diagrams

$$\begin{array}{ccc} \partial C & \xrightarrow{\partial u} & L \\ \downarrow & & \downarrow \\ C & \xrightarrow{u} & X \end{array} \tag{5.2.13.1}$$

in which u is pseudo-holomorphic. Such pairs $(u, \partial u) : (C, \partial C) \to (X, L)$ are the solutions of the problem $(C, \partial C, X, L)$.

Similarly, a pseudo-holomorphic section problem over a bordered Riemann surface $(C, \partial C)$ consists of an almost complex submersion $\pi : W \to C$ and a submersion $K \to \partial C$ along with a totally real immersion $K \to \partial W = \pi^{-1}(\partial C)$ over ∂C. A solution of such a problem is then a diagram

$$\begin{tikzcd} \partial C \arrow[r, "\partial u"] \arrow[d] & K \arrow[d] \\ C \arrow[r, "u"] & W \end{tikzcd} \tag{5.2.13.2}$$

in which u is pseudo-holomorphic and both u and ∂u are sections.

5.2.14 Exercise. Show that, in the context of the definition of a pseudo-holomorphic section problem over a bordered Riemann surface, the immersion $K \to \partial W$ is totally real iff its fibers $K_p \to W_p$ (for $p \in \partial C$) are totally real.

Parameterized moduli problems

There is a natural notion (which we now make precise) of a *family* of pseudo-holomorphic map problems (in any of the senses considered thus far) parameterized by a smooth manifold B. Such a family $\{\mathcal{P}_b\}_{b \in B}$ is itself a pseudo-holomorphic map problem, which we call a *parameterized* pseudo-holomorphic map problem, a solution to which is a pair (b, u) consisting of a point $b \in B$ and a solution u of $\mathcal{P}_b$.

⋆ **5.2.15 Definition** (Parameterized pseudo-holomorphic map problem)**.** Let B be a smooth manifold. We introduce various sorts of pseudo-holomorphic map problems over B.

A pseudo-holomorphic section problem over by B consists of a pair of submersions $W \to C \to B$ where $C \to B$ has fiber dimension two, both $C \to B$ and $W \to B$ are equipped with relative almost complex structures (i.e. $T_{C/B}$ and $T_{W/B}$ have complex structures), and the map $W \to C$ is almost complex relative B (i.e. its derivative $T_{W/B} \to T_{C/B}$ is complex linear). A solution of the problem $W \to C \to B$ is a point $b \in B$ along with a pseudo-holomorphic section $u : C_b \to W_b$ (where $C_b = C \times_B b$ and $W_b = W \times_B b$ denote the fibers over b).

A quasi-holomorphic section problem over B is a pair of submersions $W \to C \to B$ along with a vector bundle H/W and an affine linear map $\varphi : J^1_B(W/C) \to H$ (recall the relative jet space (**??**)) which is elliptic (5.2.11.2). A solution of such a problem is a point $b \in B$ along with a quasi-holomorphic section $u : C_b \to W_b$.

Allowing domains with boundary in the parameterized context means that we allow $C \to B$ to be a submersion-with-boundary (**??**) (though $W \to C$ remains a submersion), and we impose boundary conditions taking the form of a submersion $K \to \partial C$ and an immersion $K \to W$ over C whose fibers $K_b \to W_b$ over points $b \in B$ are totally real (5.2.13) (or, in the quasi-holomorphic setting, elliptic (**??**)).

Parameterized problems in all the above senses pull back under maps $B' \to B$.

5.2.16 Example (Family of inhomogeneous terms)**.** Let C be a Riemann surface and X an almost complex manifold. We saw earlier (5.2.8) that almost complex structures on $X \times C$ for which the fiber inclusions $X = X \times c \subseteq X \times C$ and the projection $X \times C \to C$ are both almost complex are in natural bijection with sections $\gamma : C \times X \to TX \otimes \overline{T^*C}$, and that pseudo-holomorphicity of a section $(u, \mathbf{1}) : C \to X \times C$ with respect to such an almost complex structure amounts to the equation $(du)^{0,1} + \gamma(u) = 0$ for the map $u : C \to X$. Now fix a smooth manifold E and a section $\gamma : C \times X \times E \to TX \otimes \overline{T^*C}$. This gives rise to a pseudo-holomorphic section problem $C \times X \times E \to C \times E \to E$ to which a solution is a pair $(e \in E, u : C \to X)$ satisfying $(du)^{0,1} + \gamma(u, e) = 0$.

In fact, all that is really required to make sense of the various sorts of parameterized problems defined in (5.2.15) is a suitable notion of submersion (or submersion-with-boundary). Thus, the base B could in fact be a log smooth manifold (2.7), a derived smooth manifold (2.9), or an object of one of the 'hybrid categories' discussed in (2.10). It could also be any stack over these categories.

We will adopt the following definition of a point constraint for parameterized map problems. At first glance, it appears much less general than the class of point constraints considered earlier (5.2.10), but we will see that in fact it is not.

5.2.17 Definition (Parameterized point constraints)**.** A point constraint for a parameterized section problem $W \to C \to B$ is a map $f : A \to J^k_B(W/C)$ (recall the relative jet space (**??**)) whose composition $A \to C$ is a closed embedding and whose composition $A \to B$ is a proper local isomorphism (2.1.35). A solution to the constrained problem is a solution $(b, u : C_b \to W_b)$ of the unconstrained problem whose k-jet $J^k u : C_b \to J^k(W_b/C_b) = J^k_B(W/C)_b$ agrees with f under pullback to A_b.

5.2.18 Example. Consider a single point constraint in the sense of (5.2.10) for a section problem $W \to C$, namely a smooth manifold A with a map $A \to J^k(W/C)$. Such a constrained problem is 'equivalent' to the parameterized problem $W \times A \to C \times A \to A$ equipped with the single point constraint induced by the map $A \to J^k(W/C)$ regarded as a section of $J^k(W/C) \times A = J^k_A((W \times A)/(C \times A)) \to A$.

More generally, given a parameterized problem $W \to C \to B$ and a map $f : A \to J^k_B(W/C)$, we may wish to consider solutions $(b, u : C_b \to W_b)$ together with a point $a \in A$ such that the image of $J^k u$ contains $f(a)$. This is equivalent to the pullback $W \times_B A \to C \times_B A \to A$ equipped with the point constraint in the sense of (5.2.17) induced by f.

Log moduli problems

The theory of pseudo-holomorphic maps becomes most interesting when we allow domains and targets with cylindrical structure (2.7.15)(3.6.2) and when we allow them to break (degenerate) and glue as the base parameter $b \in B$ is varied. To describe such domains/targets and families thereof, we will use the language of log smooth manifolds developed in (2.7).

⋆ **5.2.19 Definition** (Parameterized log quasi-holomorphic section problem)**.** A quasi-holomorphic section problem over a log smooth manifold B is a pair of submersions (2.7.59) of log smooth manifolds $W \to C \to B$ where $C \to B$ is simply-broken (2.7.82) of relative dimension two, along with a vector bundle H/W and an affine linear map $\varphi : J^1_B(W/C) \to H$ which is elliptic (5.2.11.2) (where $J^1_B(W/C)$ denotes the space of sections of $T_{W/B} \to (W \to C)^* T_{C/B}$).

5.3 Moduli stacks

In the previous section (5.2), we introduced various sorts of pseudo-holomorphic map problems and solutions thereof (pseudo-holomorphic maps satisfying the relevant boundary conditions, point constraints, etc.). The goal of the present section is to formalize various notions (continuous, smooth, and otherwise) of *families* of solutions of such problems.

The *moduli stack* $\underline{\mathrm{Hol}}_B(C,W)$ associated to a given pseudo-holomorphic section problem $\wp = (W \to C \to B)$ associates to an object Z of the relevant geometric category (Top, Sm, LogSm, $\mathcal{D}\mathsf{Sm}$, etc.) the collection of families of solutions of $\wp$ parameterized by Z. Being a sheaf on a topological (∞-)site (2.8) such as Top, Sm, LogSm, $\mathcal{D}\mathsf{Sm}$, etc., the moduli stack may be regarded as a geometric object. To distinguish between the moduli stacks on different categories, we will say 'topological moduli stack' $\underline{\mathrm{Hol}}_B(C,W)_{\mathsf{Top}}$, 'smooth moduli stack' $\underline{\mathrm{Hol}}_B(C,W)_{\mathsf{Sm}}$, etc.

It will help to be familiar with mapping stacks (2.3.39)(2.3.52).

Smooth moduli stacks

The definition of the moduli stack on smooth manifolds Sm is straightforward.

⋆ **5.3.1 Definition** (Smooth moduli stack)**.** Fix a quasi-holomorphic section problem $(W \to C \to B, H, \varphi)$ (5.2.15). Recall that this means B is a smooth manifold, the map $C \to B$ is a submersion with two-dimensional fibers, $W \to C$ is a submersion, H/W is a vector bundle, and $\varphi : J^1_B(W/C) \to H$ is an affine linear map which is elliptic in the sense of (5.2.11.2); more generally, the map $C \to B$ can be a submersion-with-boundary, in which case suitable boundary conditions are imposed; we can also include point constraints (5.2.17).

The *smooth moduli stack* $\underline{\mathrm{Hol}}_B(C,W)$ associated to the problem $(W \to C \to B, H, \varphi)$ assigns to a smooth manifold Z the set of pairs (f,u) consisting of a smooth map $f : Z \to B$ and a smooth map $u : C \times_B Z \to W$ over C whose specialization $u_z : C_z \to W_z$ is quasi-holomorphic for every point $z \in Z$.

$$\begin{tikzcd} & W \arrow[d] \\ C \times_B Z \arrow[ur, "u"] \arrow[r] \arrow[d] & C \arrow[d] \\ Z \arrow[r, "f"] & B \end{tikzcd} \tag{5.3.1.1}$$

The map u is subject to whatever boundary conditions or point constraints exist in the input problem.

5.3.2 Example. Let C be a Riemann surface and X an almost complex manifold. A map $Z \to \underline{\mathrm{Hol}}(C,X)$ is a map $Z \times C \to X$ whose restriction to each fiber $z \times C$ is pseudo-holomorphic.

5.3.3 Exercise. Let $(W \to C \to B, H, \varphi)$ be a quasi-holomorphic section problem, and let $(W' \to C' \to B', H', \varphi')$ be its pullback under a map of smooth manifolds $B' \to B$. Define a tautological isomorphism $\underline{\mathrm{Hol}}_{B'}(C', W') = \underline{\mathrm{Hol}}_B(C, W) \times_B B'$.

The definition of the moduli stack $\underline{\mathrm{Hol}}_B(C, W)$ makes sense more generally for any smooth stack B.

Topological moduli stacks

We now define the moduli stack on topological spaces Top. Its definition depends on the 'hybrid category' of topological-smooth spaces TopSm (2.10).

⋆ **5.3.4 Definition** (Topological moduli stack)**.** Given a quasi-holomorphic section problem $(W \to C \to B, H, \varphi)$, a map $Z \to \underline{\mathrm{Hol}}_B(C, W)$ from a topological space Z is defined by replacing the category Sm in the definition of the moduli stack on Sm (5.3.1) with the category of topological-smooth spaces TopSm. That is, a map $Z \to \underline{\mathrm{Hol}}_B(C, W)$ is a diagram (5.3.1.1) in TopSm in which the specialization of the map u to the fiber over every point $z \in Z$ is quasi-holomorphic (and satisfies the relevant boundary conditions and point constraints, if any).

In this definition, the base B does not need to be a smooth manifold, rather it can be any topological-smooth space or every topological-smooth stack (in which case the maps $W \to C \to B$ must be submersive in the relevant sense (2.10.9)). The topological stack $\underline{\mathrm{Hol}}_B(C, W)$ is evidently unchanged by pulling back the moduli problem under $(\mathsf{Top} \to \mathsf{TopSm})_!(\mathsf{Top} \to \mathsf{TopSm})^*B \to B$ (indeed, formation of the moduli stack is compatible with pullback (2.3.54.2), and the operation $\times_B (\mathsf{Top} \to \mathsf{TopSm})_!(\mathsf{Top} \to \mathsf{TopSm})^*B$ is trivial over $\mathsf{Top} \subseteq \mathsf{TopSm}$). Thus for the purpose of defining the topological moduli stack, we lose no generality by restricting consideration to bases $B \in \mathsf{Shv}(\mathsf{Top}) \subseteq \mathsf{Shv}(\mathsf{TopSm})$. We are, in fact, usually interested in the case of smooth stacks $B \in \mathsf{Shv}(\mathsf{Sm})$, which thus for the purpose of defining the topological moduli stack can be replaced by their image $|B|_! = (\mathsf{Sm} \to \mathsf{Top})_! B \in \mathsf{Shv}(\mathsf{Top})$ (recall that $(\mathsf{Top} \hookrightarrow \mathsf{TopSm})^* = (\mathsf{TopSm} \xrightarrow{|\cdot|} \mathsf{Top})_!$ (**??**)).

We will use subscripts to distinguish the moduli stacks on different categories. This notation is, in particular, essential when discussing comparison maps between them.

⋆ **5.3.5 Definition** (Comparing smooth and topological moduli stacks)**.** A map $Z \to \underline{\mathrm{Hol}}_B(C, W)_{\mathsf{Sm}}$ determines, by forgetting structure, a map $|Z| \to \underline{\mathrm{Hol}}_B(C, W)_{\mathsf{Top}}$. This defines a tautological map $\underline{\mathrm{Hol}}_B(C, W)_{\mathsf{Sm}} \to (\mathsf{Sm} \to \mathsf{Top})^*\underline{\mathrm{Hol}}_B(C, W)_{\mathsf{Top}}$ for any smooth stack B. More geometrically significant is the associated (by adjunction) 'comparison map'

$$(\mathsf{Sm} \to \mathsf{Top})_!\underline{\mathrm{Hol}}_B(C, W)_{\mathsf{Sm}} \to \underline{\mathrm{Hol}}_B(C, W)_{\mathsf{Top}}. \tag{5.3.5.1}$$

Generally speaking, an 'open substack' of a moduli stack $\underline{\mathrm{Hol}}_B(C, W)$ refers to an open substack of the *topological* moduli stack $\underline{\mathrm{Hol}}_B(C, W)_{\mathsf{Top}}$. Such an open substack determines (by pulling back under the comparison map) corresponding open substacks of all other flavors of moduli stacks we consider. These other moduli stacks may have open substacks which do not arise like this (see (2.4.7) for example), but they are not of much relevance. The terms 'open covering' and 'locally' are to be understood accordingly. Open substacks (and open coverings) of $\underline{\mathrm{Hol}}_B(C,W)$ usually arise via pullback from open substacks/coverings of $\underline{\mathrm{Sec}}_B(C,W)$.

Fiber product presentations

The moduli stacks presented thus far admit evident fiber product presentations. Due to the 'diagrammatic' nature of these presentations, they are valid independent of which flavor of moduli stack is being considered.

⋆ **5.3.6 Definition** (Moduli stacks as fiber products)**.** Given a quasi-holomorphic section problem $(W \to C \to B, H, \varphi)$, there is a (quite tautological) pullback square presentation of the moduli stack $\underline{\mathrm{Hol}}_B(C,W)$ in terms of the stacks of smooth sections $\underline{\mathrm{Sec}}_B(C,W)$ and $\underline{\mathrm{Sec}}_B(C,H)$.

$$\begin{tikzcd}
\underline{\mathrm{Hol}}_B(C,W) \arrow[r] \arrow[d] & \underline{\mathrm{Sec}}_B(C,W) \arrow[d, "u\mapsto(u,0)"] \\
\underline{\mathrm{Sec}}_B(C,W) \arrow[r, "u\mapsto\varphi(du)"] & \underline{\mathrm{Sec}}_B(C,H)
\end{tikzcd} \tag{5.3.6.1}$$

In the presence of boundary conditions $(W,K) \to (C,\partial C)$, we consider the stacks of sections $\underline{\mathrm{Sec}}_B((C,\partial C),(W,K))$ which parameterize diagrams (5.2.13.2), meaning a map $Z \to \underline{\mathrm{Sec}}_B((C,\partial C),(W,K))$ is a diagram of the following shape.

$$\begin{tikzcd}
& & & K \arrow[r] \arrow[d] & W \arrow[d] \\
\partial C_Z \arrow[r] \arrow[urrr] \arrow[dr] & C_Z \arrow[urrr] \arrow[d] & & \partial C \arrow[r] \arrow[dr] & C \arrow[d] \\
& Z \arrow[rrr] & & & B
\end{tikzcd} \tag{5.3.6.2}$$

For $\underline{\mathrm{Sec}}_B(C,H)$ with boundary conditions, only the map to W is lifted to H.

Point constraints (5.2.17) may also be imposed via fiber product. Namely, for a point constraint $f : A \to J^k_B(W/C)$, we have the following fiber product presentation of the moduli stack $\underline{\mathrm{Hol}}_B(C,W)_f$ of solutions to the constrained problem.

$$\begin{tikzcd}
\underline{\mathrm{Hol}}_B(C,W)_f \arrow[r] \arrow[d] & B \arrow[d, "f"] \\
\underline{\mathrm{Hol}}_B(C,W) \arrow[r] & \underline{\mathrm{Sec}}_B(A, J^k_B(W/C))
\end{tikzcd} \tag{5.3.6.3}$$

The bottom map is the composition $\underline{\mathrm{Hol}}_B(C,W) \to \underline{\mathrm{Sec}}_B(C,W) \to \underline{\mathrm{Sec}}_B(C,J^k_B(W/C)) \to \underline{\mathrm{Sec}}_B(A,J^k_B(W/C))$ sending a map u to its k-jet restricted to $A \subseteq C$.

5.3.7 Corollary. *The map* $\underline{\mathrm{Hol}}_B(C,W) \to \underline{\mathrm{Sec}}_B(C,W)$ *is a closed embedding of topological stacks.*

Proof. It is a pullback (5.3.6.1) of 'zero' map $\underline{\mathrm{Sec}}_B(C,W) \to \underline{\mathrm{Sec}}_B(C,H)$, which is a closed embedding of topological stacks since $W \to H$ is a closed embedding and $C \to B$ is open (2.3.57). □

5.3.8 Corollary. *The map* $\underline{\mathrm{Hol}}_B(C,W) \to B$ *is separated (as a map of topological stacks) if* $W \to C$ *is separated.*

Proof. Since $\underline{\mathrm{Hol}}_B(C,W) \to \underline{\mathrm{Sec}}_B(C,W)$ is a closed embedding, it is separated, so it is enough to know that $\underline{\mathrm{Sec}}_B(C,W) \to B$ is separated, which is (**??**). □

Derived smooth moduli stacks

We now define the moduli stack on the ∞-category of derived smooth manifolds $\mathcal{D}\mathsf{Sm}$. In this context, vertical quasi-holomorphicity (i.e. vanishing of $\varphi(du)$) is not a fiberwise condition (a real valued function on a derived smooth manifold may be nonzero yet vanish at every point). In fact, it is not a condition at all, rather it is the *extra data* of a path between $\varphi(du)$ and zero in the space of sections over the total space of the family. For this reason, the most transparent definition of the derived smooth moduli stack of quasi-holomorphic maps is via the fiber product presentation (5.3.6) (in other words, vertical quasi-holomorphicity is expressed *diagrammatically*).

⋆ **5.3.9 Definition** (Derived smooth moduli stack)**.** The moduli stack $\underline{\mathrm{Hol}}_B(C,W)$ on the ∞-category of derived smooth manifolds is defined as the fiber product (5.3.6) of derived smooth stacks, where the stacks of sections $\underline{\mathrm{Sec}}_B(C,W)$ have their usual categorical meaning (2.3.52) (which is purely diagrammatic hence applies in any ∞-category). The bottom map $u \mapsto \varphi(du)$ in (5.3.6.1) is defined using the tangent functor on derived smooth manifolds (the tangent functor gives a map of derived smooth stacks $\underline{\mathrm{Sec}}_B(C,W) \to \underline{\mathrm{Sec}}_B(C,J^1(W/C))$ sending $u \mapsto du$). Point conditions are also imposed by fiber product against the relevant evaluation map(s), i.e. diagrammatically.

5.3.10 Definition (Comparing moduli functors over Sm, $\mathcal{D}\mathsf{Sm}$, Top)**.** There are tautological maps

$$\underline{\mathrm{Hol}}_B(C,W)_{\mathsf{Sm}} \to \underline{\mathrm{Hol}}_B(C,W)_{\mathcal{D}\mathsf{Sm}} \to \underline{\mathrm{Hol}}_B(C,W)_{\mathsf{Top}} \tag{5.3.10.1}$$

where the notion of a map from $X \in \mathsf{Shv}(\mathsf{C})$ to $Y \in \mathsf{Shv}(\mathsf{D})$ over a topological functor $f : \mathsf{C} \to \mathsf{D}$ is defined via the adjunction $(f_!, f^*)$ (2.8.36), namely it is a map $f_!X \to Y$ or equivalently a map $X \to f^*Y$. Indeed, the functors $\mathsf{Sm} \to \mathcal{D}\mathsf{Sm} \to \mathsf{Top}$ induce such comparison maps on stacks of sections $\underline{\mathrm{Sec}}$ (2.3.52) since they preserve pullbacks of $C \to B$, which induce the same on the stacks $\underline{\mathrm{Hol}}$ via their definition in terms of fiber products. Concretely, this just amounts to noting that the functors $\mathsf{Sm} \to \mathcal{D}\mathsf{Sm} \to \mathsf{Top}$ send families of quasi-holomorphic sections to families of quasi-holomorphic sections, since the notion of such a family is defined diagrammatically and these functors are compatible with the relevant tangent functors.

5.4 Tangent complexes

The (analytic) tangent complex $T_{\mathrm{an}}\underline{\mathrm{Hol}}_B(C, W)$ of a moduli stack of pseudo-holomorphic maps $\underline{\mathrm{Hol}}_B(C, W)$ measures the first order deformation theory of pseudo-holomorphic maps. It is obtained by 'linearizing' (differentiating) the pseudo-holomorphic map equation, yielding a family of elliptic operators over the moduli stack, whose pushforward is the tangent complex. The first goal of this section is to make this definition precise and to prove some basic properties about it. Given this definition, we may define (in various senses) the 'regular locus' of a moduli stack to be the locus where some (possibly relative) tangent complex vanishes in degrees > 0 (this is always an open substack (2.1.5)).

The procedure for obtaining the tangent complex may be described simply as applying the tangent functor T to the input moduli problem $\wp$ to obtain what we will call the 'tangent moduli problem' $T\wp$, whose moduli stack of solutions $\underline{\mathrm{Hol}}(T\wp)$ is what we have defined to be the analytic tangent complex $T_{\mathrm{an}}\underline{\mathrm{Hol}}(\wp)$ of $\underline{\mathrm{Hol}}(\wp)$. On the other hand, we may also apply (left Kan extension along) the tangent functor $T_!$ to the moduli stack $\underline{\mathrm{Hol}}(\wp)$, yielding what might be called the 'geometric' tangent bundle $T_!\underline{\mathrm{Hol}}(\wp)$. Now there is a tautological comparison map $T_!\underline{\mathrm{Hol}}(\wp) \to \underline{\mathrm{Hol}}(T\wp) = T_{\mathrm{an}}\underline{\mathrm{Hol}}(\wp)$. The second goal of this section is to note that this comparison map is an isomorphism on total spaces (hence is an isomorphism of perfect complexes on $\underline{\mathrm{Hol}}(\wp)$ once we prove the Derived Regularity Theorem (0.0.3)).

This discussion would apply equally well in any other non-linear elliptic Fredholm setting.

* *

To begin, we define the fiber of the tangent complex at a given pseudo-holomorphic map.

5.4.1 Definition (Analytic tangent space). Let $(W \to C, H, \varphi)$ be a quasi-holomorphic section problem, and let $u : C \to W$ be a point of its moduli stack $\underline{\mathrm{Hol}}(C, W)$. We consider the map

$$D_u : C^\infty(C, u^*T_{W/C}) \to C^\infty(C, u^*H) \tag{5.4.1.1}$$

measuring the first order variation in $\varphi(du)$ induced by first order variations of u (it is obtained formally by linearizing the triple $(W \to C, H, \varphi)$ around u to obtain $(u^*T_{W/C} \to C, u^*H, T\varphi(u, \cdot)))$. This D_u is a first order differential operator, whose symbol $\sigma(D_u) : T^*C \to \mathrm{Hom}(u^*T_{W/C}, u^*H)$ is the linear part of φ pulled back under u. It follows that D_u is elliptic by the hypothesis (suitably termed 'ellipticity') on φ (5.2.11.2).

We define the analytic tangent $T_u\underline{\mathrm{Hol}}(C, W)$ to be the operator D_u, regarded as a two-term complex supported in degrees [0 1]. If C is a compact Hausdorff smooth manifold, ellipticity of D_u means it has finite-dimensional cohomology (3.4.28)(3.4.30). More generally, D_u has finite-dimensional cohomology when C has asymptotically cylindrical ends (that is, is a compact Hausdorff marked depth one log smooth manifold) (3.6.2) and the ends of D_u are non-degenerate (3.6.13)(**??**)(**??**); note that non-degeneracy of the ends of D_u also means that its cohomology is unchanged by restricting its domain and codomain to functions which vanish on the ideal locus of C. There is an immediate generalization of this discussion to parameterized moduli problems $(W \to C \to B, H, \varphi)$. Given a point

$(b, u : C_b \to W_b) \in \underline{\mathrm{Hol}}_B(C, W)$, we may form the operator $D_u : u^*T_{W/C} \to u^*H$ over C_b as above (measuring first order variations in u, keeping b fixed). When $b \in B$ is a non-degenerate point This operator now defines the relative analytic tangent space $T_{(b,u)}(\underline{\mathrm{Hol}}_B(C, W)/B)$.

In the parameterized setting, we can also define a tangent complex which includes variations in the base directions.

5.4.2 Definition (Analytic tangent space with base directions)**.**

5.5 Elliptic bootstrapping

We now come to the first bit of analysis in our discussion of pseudo-holomorphic maps: *elliptic boostrapping*, which is a generalization of the linear elliptic estimates discussed in (3.4)–(3.7) to the present non-linear setting of pseudo-holomorphic (and more generally quasi-holomorphic) maps. Elliptic bootstrapping refers to estimates of the form

$$\|u\|_{s+1} \leq F_s(\|u\|_s) \tag{5.5.0.1}$$

for some functions F_s (depending on the geometry of the source and target), under the assumption that u satisfies some particular (possibly non-linear) elliptic equation. While for linear elliptic operators (with smooth coefficients) such estimates hold for all s and with F_s linear, in the present non-linear setting they only hold for sufficiently large s and with not necessarily linear F_s.

5.5.1 Exercise. Let $\Omega \subseteq \mathbb{R}^n$ be open and let $f : \Omega \to \mathbb{R}$ satisfy $\|f\|_{C^{k+1}} \leq M$. Show that for every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon, M) > 0$ such that if $\|f\|_{C^0} \leq \delta$ then $\|f\|_{C^k} \leq \varepsilon$. (Prove the case $k = 1$ directly and then use induction.)

5.6 Regularity

In this section, we prove the Regularity Theorem (0.0.2) (see (5.6.2) below) and its generalization the Log Regularity Theorem (**??**) (see (5.6.3)(**??**) below), which state the the regular loci in (log) smooth moduli spaces of pseudo-holomorphic maps are represented by (log) smooth manifolds. This is a non-linear generalization of the results about kernels of families of elliptic operators (3.4)–(3.8). The proof is in some sense analogous, though with quite a few additional technical complexities.

The reasoning here would apply equally well in any other non-linear elliptic Fredholm setting.

Outline

The proofs of the main results of this section (5.6.2)(5.6.3)(**??**) are quite heavy on notation, which can easily obscure the main ideas. We therefore find it helpful to begin by presenting the overall strategy in a somewhat abstract and idealized setting, free from the 'implementation details' of each individual case. This is then used as a template for our later arguments.

⋆ **5.6.1 Regularity Meta-Theorem.** *The regular locus* $\underline{\mathrm{Hol}}^{\mathrm{reg}}_{\mathsf{Sm}} \subseteq \underline{\mathrm{Hol}}_{\mathsf{Sm}}$ *inside the smooth moduli stack of solutions to a quasi-holomorphic sections problem is representable, and its comparison map* $(\mathsf{Sm} \to \mathsf{Top})_!\underline{\mathrm{Hol}}^{\mathrm{reg}}_{\mathsf{Sm}} \to \underline{\mathrm{Hol}}^{\mathrm{reg}}_{\mathsf{Top}}$ *is an isomorphism.*

Meta-Proof. We indicate in **bold** the key elements of the proof, which are in particular need of further elaboration in any particular implementation of this outline.

Our first step is to **fix local linear coordinates** for our moduli problem. That is, we phrase our problem (locally) as asking for solutions to $\mathbf{D}u = 0$ for some non-linear first order differential operator (**??**)

$$\mathbf{D} : C^\infty(M, E) \to C^\infty(M, F) \tag{5.6.1.1}$$

between spaces of sections of vector bundles E and F over a manifold M, so that the moduli stack $\underline{\mathrm{Hol}}$ is the fiber product $\underline{\mathrm{Sec}}(M, E) \times_{\underline{\mathrm{Sec}}(M,F)} *$ (over both Sm and Top).

Next, we fix a class of **linear projections**

$$\lambda : C^\infty(M, E) \to K \tag{5.6.1.2}$$

to finite-dimensional vector spaces K which separate points (for every non-zero element of $C^\infty(M, E)$, there is some λ in whose kernel it does not lie). Observe that every regular point $u \in \underline{\mathrm{Hol}}^{\mathrm{reg}}$, there exists a chosen linear projection λ which induces an isomorphism between the analytic tangent space $T_u\underline{\mathrm{Hol}}^{\mathrm{reg}}$ at u and K (this follows from separation of points). We will call such points λ-*regular*. Equivalently, $u \in \underline{\mathrm{Hol}}$ is λ-regular iff $T_u(\underline{\mathrm{Hol}}/K) = 0$; for this reason, λ-regular is also called *exactly regular relative* K and denoted $\underline{\mathrm{Hol}}^{\mathrm{xreg}/K}$.

With this preliminary setup concluded, our main goal is now to show that the map

$$\lambda|_{\underline{\mathrm{Hol}}} : \underline{\mathrm{Hol}} \to K \tag{5.6.1.3}$$

is a local isomorphism of topological and smooth stacks over the open locus $\underline{\mathrm{Hol}}^{\mathrm{xreg}/K} \subseteq \underline{\mathrm{Hol}}$ of λ-regular points.

To begin, we construct a local topological inverse to $\lambda|_{\underline{\mathrm{Hol}}}$ near a given λ-regular basepoint of $\underline{\mathrm{Hol}}$. We may assume wlog that this basepoint is the zero section $0 \in C^\infty(M, E)$ (this is permissible since the discussion so far has been invariant under fiberwise translation of E). Denote by

$$D : C^\infty(M, E) \to C^\infty(M, F) \tag{5.6.1.4}$$

the derivative of $\mathbf{D}$ at the basepoint $0 \in C^\infty(M, E)$, and note that λ-regularity of the basepoint means that $(D \oplus \lambda) : C^\infty(M, E) \to C^\infty(M, F) \oplus K$ is an isomorphism. Define the operator

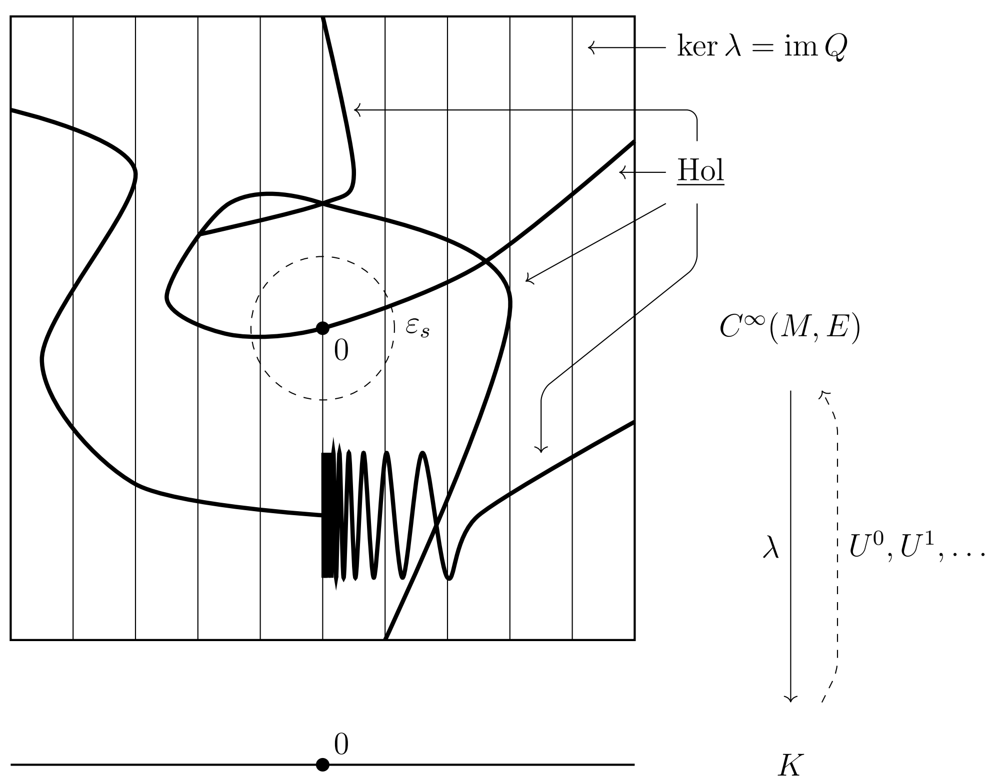
$$Q : C^\infty(M, F) \xrightarrow{\sim} \ker \lambda \subseteq C^\infty(M, E) \tag{5.6.1.5}$$

to be the first component of the inverse of $D \oplus \lambda$.

Now our local topological inverse $U^\infty : K \to \underline{\mathrm{Hol}}$ will be defined as the limit of a sequence of maps $U^0, U^1, U^2, \ldots : K \to C^\infty(M, E)$ defined by a 'Newton–Picard iteration'. Take $U^0 : K \to C^\infty(M, E)$ to be any (for example, linear) section of λ, and for $i > 0$ define

$$U^i = U^{i-1} - Q\mathbf{D}U^{i-1}. \tag{5.6.1.6}$$

Note that every U^i is a section of λ since $\operatorname{im} Q \subseteq \ker \lambda$.



(5.6.1.7)

The key to analyzing the behavior of U^i as $i \to \infty$ is the so-called **quadratic estimate** for $\xi, \zeta \in C^\infty(M, E)$ with respect to a particular choice of **Sobolev H^s-norm** on $C^\infty(M, E)$

and $C^\infty(M,F)$:

$$\|\mathbf{D}\xi - \mathbf{D}\zeta - D(\xi-\zeta)\|_{s-1} \leq \mathrm{const}_{N,s} \cdot (\|\xi\|_s + \|\zeta\|_s) \cdot \|\xi-\zeta\|_s \tag{5.6.1.8}$$

for $\|\xi\|_s, \|\zeta\|_s \leq N$ and $H^{s-1} \subseteq C^0$ (3.2.28) (it is convenient to introduce the notation $R(\xi,\zeta) = \mathbf{D}\xi - \mathbf{D}\zeta - D(\xi-\zeta)$ for the quantity whose norm is estimated here). Here $\|\cdot\|_s$ denotes the relevant Sobolev H^s-norm (3.2.14). The quadratic estimate (5.6.1.8) holds simply because D is the derivative of $\mathbf{D}$ at zero: the quantity $R(\xi,\zeta)$ is the result of applying to $(J^1\xi, J^1\zeta)$ (**??**) some non-linear function $A : (J^1E)^2 \to F$ which vanishes along the diagonal and to second order at the origin, which implies the quadratic estimate by (3.2.35).

Now let us return to the analysis of U^i as $i \to \infty$. When we write a Sobolev norm of U^i or an expression in it, we always mean the evaluation at some particular (unwritten) $k \in K$. We have $\mathbf{D}U^{i+1} = R(U^i - Q\mathbf{D}U^i, U^i)$ since $DQ = 1$, so the quadratic estimate (5.6.1.8) implies that

$$\|\mathbf{D}U^{i+1}\|_{s-1} \leq \mathrm{const}_{N,s} \cdot \|U^i\|_s \cdot \|\mathbf{D}U^i\|_{s-1} \tag{5.6.1.9}$$

for $\|U^i\|_s \leq N$ and $H^{s-1} \subseteq C^0$ (in addition to the quadratic estimate, we are also appealing to the elliptic estimate $\|Qu\|_s \leq \mathrm{const}_s \cdot \|u\|_{s-1}$ (3.4.24)(3.4.26) and the estimate $\|\mathbf{D}u\|_{s-1} \leq \mathrm{const}_{N,s} \cdot \|u\|_s$ for $\|u\|_s \leq N$ (3.2.32)). The estimate (5.6.1.9) implies that

$$\|\mathbf{D}U^{i+1}\|_{s-1} \leq \frac{1}{2} \cdot \|\mathbf{D}U^i\|_{s-1} \tag{5.6.1.10}$$

for $\|U^i\|_s \leq \varepsilon_s$ and $H^{s-1} \subseteq C^0$. Now $\|U^i\|_s \leq \|U^0\|_s + \|Q\|_{(s,s-1)}(\|\mathbf{D}U^0\|_{s-1} + \cdots + \|\mathbf{D}U^{i-1}\|_{s-1})$, and if $\|U^0\|_s, \ldots, \|U^{i-2}\|_s \leq \varepsilon_s$ so that the above estimate (5.6.1.10) applies, this is bounded by $\|U^0\|_s + 2\|Q\|_{(s,s-1)}\|\mathbf{D}U^0\|_{s-1} \leq \mathrm{const}_s \cdot \|U^0\|_s$. We conclude that

$$\|\mathbf{D}U^i\|_{s-1} \leq \qquad 2^{-i}\|\mathbf{D}U^0\|_{s-1} \tag{5.6.1.11}$$

$$\|U^{i+1} - U^i\|_s \leq \mathrm{const}_s \cdot 2^{-i}\|\mathbf{D}U^0\|_{s-1} \tag{5.6.1.12}$$

for all $i < \infty$, provided $\|U^0\|_s \leq \varepsilon_s$ and $H^{s-1} \subseteq C^0$ (for some $\varepsilon_s > 0$). Since $H^s(M,E)$ is complete, the limit $U^\infty = \lim_{i\to\infty} U^i$ exists as a continuous map $U^\infty : K \to H^s(M,E)$ (5.6.1.12) defined in a neighborhood of $0 \in K$ depending on s, and we have $\mathbf{D}U^\infty \equiv 0$ (5.6.1.11).

It is evident that $\lambda \circ U^\infty = 1_K$ over a neighborhood of $0 \in K$ (we have $\lambda \circ U^0 = 1_K$ by definition of U^0, so we have $\lambda \circ U^i = 1_K$ by induction on i since $\operatorname{im} Q \subseteq \ker\lambda$, which implies $\lambda \circ U^\infty = 1_K$ over the neighborhood of $0 \in K$ over which U^i converges in H^s, provided λ is H^s-continuous, which is the case for sufficiently large s).

Now we claim that $U^\infty \circ \lambda = 1_{\underline{\mathrm{Hol}}}$ over a neighborhood of the basepoint. Given the fact that $\lambda \circ U^\infty = 1_K$ over a neighborhood of $0 \in K$, it is enough to check that λ is injective on $\underline{\mathrm{Hol}}$. This follows from the quadratic estimate (5.6.1.8): if $\mathbf{D}\xi = \mathbf{D}\zeta = 0$, then $\|D(\xi-\zeta)\|_{s-1} \leq \mathrm{const}_{N,s} \cdot (\|\xi\|_s + \|\zeta\|_s) \cdot \|\xi-\zeta\|_s$, and if in addition $\xi - \zeta \in \ker\lambda$, then $\xi - \zeta = QD(\xi-\zeta)$, so we have $\|\xi-\zeta\|_s \subseteq \mathrm{const}_s \cdot (\|\xi\|_s + \|\zeta\|_s) \cdot \|\xi-\zeta\|_s$, which implies $\|\xi-\zeta\|_s = 0$ provided $\|\xi\|_s + \|\zeta\|_s$ is sufficiently small.

Now let us note that our continuous map $U^\infty : K \to H^s(M,E)$ (defined over a neighborhood of the origin $0 \in K$ depending on the choice of sufficiently large s) in fact lands (with effective bounds) inside $C^\infty(M,E) \subseteq H^s(M,E)$ (**??**) and hence is moreover continuous to $C^\infty(M,E)$ (5.5.1) over a neighborhood of $0 \in K$. The topological vector space $C^\infty(M,E)$ represents the topological stack $\underline{\mathrm{Sec}}(M,E)$ (**??**), so we have a morphism of topological stacks $U^\infty : K \to \underline{\mathrm{Hol}}(M,E)$ over a neighborhood of $0 \in K$.

We can now conclude that $\lambda : \underline{\mathrm{Hol}} \to K$ is a local isomorphism of topological stacks near the basepoint. Indeed, this follows from the fact that there exists a morphism of topological stacks $U^\infty : K \to \underline{\mathrm{Hol}}$ defined near $0 \in K$ and satisfying $\lambda \circ U^\infty = 1_K$ and $U^\infty \circ \lambda = 1_{\underline{\mathrm{Hol}}}$ over neighborhoods of $0 \in K$ and the basepoint in $\underline{\mathrm{Hol}}$, respectively.

We have now shown that our map of topological and smooth stacks $\lambda : \underline{\mathrm{Hol}}^{\mathrm{xreg}/K} \to K$ is a local isomorphism of topological stacks. It remains to show that it is also a local isomorphism of smooth stacks.

The local inverse of topological stacks $\lambda^{-1} : K \to \underline{\mathrm{Hol}}^{\mathrm{xreg}/K} \subseteq C^\infty(M,E)$ corresponds to a map $K \times M \to E$ which is continuous-smooth (2.10.1)(2.10.4) (that is, all its derivatives in the M direction exist and are continuous on $K \times M$). To show that λ is a local isomorphism of smooth stacks, it is equivalent to show that the local inverse of topological stacks λ^{-1} is in fact a morphism of smooth stacks, namely that the continuous-smooth map $K \times M \to E$ is in fact smooth.

To pass from continuous information to smooth information, we proceed one derivative at a time. That is, we shall show by induction on k that $\lambda : \underline{\mathrm{Hol}}^{\mathrm{xreg}/K} \to K$ is a local isomorphism of C^k-smooth stacks (stacks on the category of k times continuously differentiable manifolds); equivalently, that the local topological inverse of λ, regarded as a continuous-smooth map $K \times M \to E$, has continuous derivatives $D_K^\alpha D_M^\beta$ for $|\alpha| \le k$ and all β. The base case $k = 0$ follows from the fact that λ is a local isomorphism of topological stacks. The validity of the claim for all $k < \infty$ evidently implies the case $k = \infty$ (our desired conclusion). It thus suffices to treat the inductive step: we will show that if $\lambda : \underline{\mathrm{Hol}}^{\mathrm{xreg}/K} \to K$ is a local isomorphism of C^k-smooth stacks *for every* $\mathbf{D}$ (5.6.1.1) *and every* λ (5.6.1.2), then it is in fact a local isomorphism of C^{k+1}-smooth stacks.

The key to proving the inductive step is to apply the induction hypothesis to the **tangent moduli problem** (a form of which we have already met in (5.4)). Consider the derivative of the map $\mathbf{D}$ (5.6.1.1), which has the form

$$T\mathbf{D} : C^\infty(M, T_{E/M}) \to C^\infty(M, T_{F/M}) \tag{5.6.1.13}$$

where $T_{E/M}$ and $T_{F/M}$ denote the relative tangent bundles of $E \to M$ and $F \to M$ (note that $T_{E/M}$ and $T_{F/M}$ are vector bundles over M, by differentiating the vector bundle structure of $E \to M$ and $F \to M$). The *tangent moduli problem* asks for the zero set $T\underline{\mathrm{Hol}} = (T\mathbf{D})^{-1}(0) = \underline{\mathrm{Sec}}(M, T_{E/M}) \times_{\underline{\mathrm{Sec}}(M, T_{F/M})} *$ described by the operator $T\mathbf{D}$, which is itself a quasi-holomorphic section problem. The derivative of the linear projection λ (5.6.1.2) is a linear projection

$$T\lambda : C^\infty(M, T_{E/M}) \to TK, \tag{5.6.1.14}$$

so the induction hypothesis tells us that $T\lambda : (T\underline{\mathrm{Hol}})^{\mathrm{xreg}/TK} \to TK$ is a local isomorphism of C^k-smooth stacks.

Now we claim that the $T\lambda$-regular locus of $T\underline{\mathrm{Hol}}$ is precisely the inverse image of the λ-regular locus under the projection $T\underline{\mathrm{Hol}} \to \underline{\mathrm{Hol}}$. To see this, note that the tangent complex of $T\underline{\mathrm{Hol}}$ at a point $(u, \dot{u})$ is an extension of two copies of the tangent complex of $\underline{\mathrm{Hol}}$ at u (the map $T_u\underline{\mathrm{Hol}} \to T_{(u,\dot{u})}\underline{\mathrm{Hol}}$ corresponds to variations of $\dot{u}$ keeping u fixed, while the map $T_{(u,\dot{u})}\underline{\mathrm{Hol}} \to T_u\underline{\mathrm{Hol}}$ corresponds to remembering variations of u and forgetting variations of $\dot{u}$). This extension maps via $T\lambda$ to the canonical extension $0 \to K \to TK \to K \to 0$. Thus $T_{(u,\dot{u})}(T\underline{\mathrm{Hol}}/TK)$ is an extension of two copies of $T_u(\underline{\mathrm{Hol}}/K)$. It follows that if $T_u(\underline{\mathrm{Hol}}/K) = 0$ then $T_{(u,\dot{u})}(T\underline{\mathrm{Hol}}/TK) = 0$. Conversely, if $T_{(u,\dot{u})}(T\underline{\mathrm{Hol}}/TK) = 0$, then $T_u(\underline{\mathrm{Hol}}/K)$ is quasi-isomorphic to its shift, which, since it is bounded, implies it vanishes.

Now we would like to that $\lambda|_{\underline{\mathrm{Hol}}^{\mathrm{xreg}/K}}$ is a local isomorphism of C^{k+1}-smooth stacks given that $T\lambda|_{T\underline{\mathrm{Hol}}^{\mathrm{xreg}/TK}}$ is a local isomorphism of C^k-smooth stacks. That is, we want show that the local topological inverse to λ, regarded as a map $\lambda^{-1} : K \times M \to E$, has continuous derivatives $D_K^\alpha D_M^\beta$ for $|\alpha| \leq k+1$, provided that the local topological inverse to $T\lambda$, regarded as a map $(T\lambda)^{-1} : TK \times M \to E$, has continuous derivatives $D_K^\alpha D_M^\beta$ for $|\alpha| \leq k$. Evidently, it suffices to show that the derivative of λ^{-1} in the K direction exists and is given by $(T\lambda)^{-1}$; we call this the *tangent identity*:

$$T(\lambda|^{-1}_{\underline{\mathrm{Hol}}^{\mathrm{xreg}/K}}) = (T\lambda|_{T\underline{\mathrm{Hol}}^{\mathrm{xreg}/TK}})^{-1}. \tag{5.6.1.15}$$

This is a statement about the pointwise derivative of the function $\lambda^{-1} : K \times M \to E$. However, the equation (5.6.1.15) can also be interpreted as an assertion about the maps $\lambda^{-1} : K \to H^s(M, E)$ and $(T\lambda)^{-1} : TK \to H^s(M, T_{E/M})$, namely that the derivative of the former is the latter, pointwise on its domain in K, for some particular value of $s < \infty$. If $H^s \subseteq C^0$, then this implies the original form of the tangent identity about the maps $\lambda^{-1} : K \times M \to E$ and $(T\lambda)^{-1} : TK \times M \to E$. We will prove that the tangent identity holds in this stronger form, for all $s < \infty$.

To prove the tangent identity (5.6.1.15), it suffices (by translation invariance) to prove it at $0 \in K$. The tangent identity at $0 \in K$ is the assertion that

$$\|\lambda|^{-1}_{\underline{\mathrm{Hol}}}(k) - \lambda|^{-1}_{\ker D}(k)\|_s = o(|k|) \quad \text{as } k \to 0 \tag{5.6.1.16}$$

for all $s < \infty$. We may rewrite this in terms of $\xi = \lambda|^{-1}_{\underline{\mathrm{Hol}}}(k)$ as the assertion that

$$\|\xi - \alpha(\lambda(\xi))\|_s = o(|\lambda(\xi)|) \quad \text{as } \xi \to 0 \text{ with } \mathbf{D}(\xi) = 0, \tag{5.6.1.17}$$

where $\alpha : K \xrightarrow{\sim} \ker D \subseteq C^\infty(C, W)$ is the isomorphism which combines with Q to give the inverse of $D \oplus \lambda$. Now the left hand side is commensurate with the H^{s-1}-norm of $(D \oplus \lambda)(\xi - \alpha(\lambda(\xi))) = D(\xi)$ since $D \oplus \lambda$ and $Q \oplus \alpha$ are bounded $H^s \to H^{s-1}$ and $H^{s-1} \to H^s$. Since α is an isomorphism, the quantity $|\lambda(\xi)|$ is commensurate with the H^s-norm of $\alpha(\lambda(\xi)) = (1 - QD)\xi$. It is thus equivalent to show that

$$\|D(\xi)\|_{s-1} = o(\|(1 - QD)\xi\|_s) \quad \text{as } \xi \to 0 \text{ with } \mathbf{D}(\xi) = 0. \tag{5.6.1.18}$$

Now the quadratic estimate (5.6.1.8) gives that $\|D(\xi)\|_{s-1} \leq \mathrm{const}_s \|\xi\|_s^2$ for $\mathbf{D}(\xi) = 0$ and $\|\xi\|_s \leq 1$, which implies (5.6.1.18) (apply it to both occurences of $D\xi$, noting that $\|Q\|_{(s-1,s)} < \mathrm{const}_s$). $\square$

Smooth regularity

⋆ **5.6.2 Regularity Theorem** (stated earlier as (0.0.2)). *Let $(W \to C \to B, H, \varphi)$ be a quasi-holomorphic section problem over a smooth manifold B. Suppose $C \to B$ is proper. The regular locus $\underline{\mathrm{Hol}}_B(C,W)^{\mathrm{reg}}_{\mathsf{Sm}} \subseteq \underline{\mathrm{Hol}}_B(C,W)_{\mathsf{Sm}}$ is representable, and the comparison map $(\mathsf{Sm} \to \mathsf{Top})_! \underline{\mathrm{Hol}}_B(C,W)^{\mathrm{reg}}_{\mathsf{Sm}} \to \underline{\mathrm{Hol}}_B(C,W)^{\mathrm{reg}}_{\mathsf{Top}}$ is an isomorphism.*

Proof. We follow (5.6.1), which the reader should ensure they understand fully before proceeding here.

Let us first discuss how to **fix local linear coordinates** on our moduli problem (warning: this will involve a departure from the notation of the statement (5.6.2) above). We argue that any input moduli problem may be phrased (locally) as asking for the zero set of

$$\mathbf{D} : B \oplus C^\infty(C,W) \to C^\infty(C,H) \tag{5.6.2.1}$$
$$(b,u) \mapsto \varphi(b,du) \tag{5.6.2.2}$$

where B is a vector space and $W \to C$ and $H \to C$ are vector bundles, and $\varphi : B \times J^1(W/C) \to H$ is affine linear in the $J^1(W/C)$ coordinate. This is a purely topological result, which we proved in (2.4.19)(2.4.21) (modulo taking H to be pulled back from C; we leave it as an exercise for the reader to adapt the argument to prove this as well).

We consider **linear projections**

$$\lambda : B \oplus C^\infty(C,W) \to K \tag{5.6.2.3}$$

which on $C^\infty(C,W)$ are given by integration against a smooth section $C \to W^* \otimes \Omega_C \otimes K$. Such linear projections certainly separate points of $B \oplus C^\infty(C,W)$ and are H^s-continuous for all $s \in \mathbb{R}$. It thus suffices to show that the restriction

$$\lambda|_{\underline{\mathrm{Hol}}} : \underline{\mathrm{Hol}} \to K \tag{5.6.2.4}$$

is a local isomorphism of topological and smooth stacks, over the open locus $\underline{\mathrm{Hol}}^{\mathrm{xreg}/K} \subseteq \underline{\mathrm{Hol}}$ of λ-regular points (namely, where the relative tangent complex $T(\underline{\mathrm{Hol}}_B(C,W)/K)$ vanishes).

Now let us construct a local topological inverse to $\lambda|_{\underline{\mathrm{Hol}}}$ near a given λ-regular basepoint of $\underline{\mathrm{Hol}}$, which we may assume wlog is the zero element of $B \oplus C^\infty(C,W)$. Denote by $D : B \oplus C^\infty(C,W) \to C^\infty(C,H)$ the derivative of $\mathbf{D}$ at the basepoint (zero), so $D \oplus \lambda$ is an isomorphism since this point is λ-regular. Let $Q : C^\infty(C,H) \to \ker\lambda \subseteq B \oplus C^\infty(C,W)$ be the restriction of $(D \oplus \lambda)^{-1}$. Now our local topological inverse $U^\infty : K \to \underline{\mathrm{Hol}}_B(C,W)$ will be defined as the limit of a sequence $U^0, U^1, \ldots : K \to B \oplus C^\infty(C,W)$ defined by the iteration $U^i = U^{i-1} - Q\mathbf{D}U^{i-1}$.

As in (5.6.1), the key ingredient in analyzing the behavior of the sequence $U^0, U^1, \ldots$ is the **quadratic estimate** on $\mathbf{D}\xi - \mathbf{D}\zeta - D(\xi - \zeta)$ for $\xi, \zeta \in B \oplus C^\infty(C,W)$. Adopting the more precise notation $(b,\xi),(c,\zeta) \in B \oplus C^\infty(C,W)$, this desired estimate takes the form

$$\begin{aligned}\|\mathbf{D}(b,\xi) - \mathbf{D}(c,\zeta) - D(b-c,\xi-\zeta)\|_{s-1}& \\ \le \mathrm{const}_{N,s} \cdot (|b| + \|\xi\|_s + |c| + \|\zeta\|_s) &\cdot (|b-c| + \|\xi-\zeta\|_s),\end{aligned} \tag{5.6.2.5}$$

for $|b|, |c|, \|\xi\|_s, \|\zeta\|_s \leq N$ and $H^{s-1} \subseteq C^0$. Now the quadratic estimate (5.6.1.8) proven earlier applies to the quantity $\mathbf{D}(0,\xi) - \mathbf{D}(0,\zeta) - D(0,\xi-\zeta)$. It therefore remains to bound the difference

$$(\varphi_b - \varphi_0)(d\xi) - (\varphi_c - \varphi_0)(d\zeta) - \dot{\varphi}_{b-c}(0), \tag{5.6.2.6}$$

which we may write as $(\varphi_b - \varphi_0)(d\xi - d\zeta) + (\varphi_b - \varphi_c)(d\zeta - 0) + (\varphi_b - \varphi_c)(0) - \dot{\varphi}_{b-c}(0)$. The first two terms are both bounded as desired. The remainder $(\varphi_b - \varphi_c)(0) - \dot{\varphi}_{b-c}(0)$ is a smooth function in (b, c) which vanishes along the diagonal $b = c$ and to second order at $(0, 0)$, hence is also bounded as desired.

The rest of the outline (5.6.1) works as written. □

Log smooth regularity

⋆ **5.6.3 Log Regularity Theorem** (stated earlier as (**??**); case of non-degenerate ends). *Let $W \xrightarrow{\text{strict}} C \xrightarrow{\text{simply-broken}} B$ be a quasi-holomorphic section problem over a log smooth manifold B. Suppose $C \to B$ is proper and $W \to C \to B$ has non-degenerate ends. The regular locus $\underline{\mathrm{Hol}}_B(C,W)^{\mathrm{reg}}_{\mathsf{LogSm}} \subseteq \underline{\mathrm{Hol}}_B(C,W)_{\mathsf{LogSm}}$ is representable and the comparison map $(\mathsf{LogSm} \to \mathsf{LogTop})_! \underline{\mathrm{Hol}}_B(C,W)^{\mathrm{reg}}_{\mathsf{LogSm}} \to \underline{\mathrm{Hol}}_B(C,W)^{\mathrm{reg}}_{\mathsf{LogTop}}$ is an isomorphism.*

Proof. We follow (5.6.1)(5.6.2), which the reader should ensure they understand fully before proceeding here.

Let us first discuss how to **fix local linear coordinates** on our moduli problem (warning: this will involve a departure from the notation of the statement (5.6.3) above). A log moduli problem in linear coordinates shall mean one constructed as follows:

(5.6.3.1) Begin with a standard two-dimensional gluing family $C \to {}'\mathbb{R}^n_{\geq 0}$ associated to the data of the normalized zero fiber $\tilde{C}_0$, the involution of its ideal locus, and a choice of ideal collar (2.7.87) (so n indexes the components of the nodal locus of C_0). Let $W \to C$ and $H \to C$ be standard gluing family vector bundles associated to the data of their restriction to the normalized zero fiber and a pullback isomorphism over its ideal collar (2.7.87). Now fix a log smooth manifold M, a map $M \to {}'\mathbb{R}^n_{\geq 0}$, a vector space V, and a map $\varphi : J^1_{V\times M}(W/C) \to H$. This describes a quasi-holomorphic section problem over the base $B = V \times M$, whose moduli stack of solutions we denote by $\underline{\mathrm{Hol}}_B(C,W)$ (implicitly pulling back $W \to C \to {}'\mathbb{R}^n_{\geq 0}$ to B).

Note that the standard gluing family over ${}'\mathbb{R}^n_{\geq 0}$ need only be defined over a neighborhood of the image of $M \to {}'\mathbb{R}^n_{\geq 0}$. Such a moduli problem is usually regarded as a germ near $0 \in V$ times a basepoint $m \in M$ (in which case the gluing family $C \to {}'\mathbb{R}^n_{\geq 0}$ need only be defined over a neighborhood of the image of m in ${}'\mathbb{R}^n_{\geq 0}$).

We will be interested in the locus of points in $\underline{\mathrm{Hol}}_B(C,W)$ which are *regular relative* M, meaning the relative tangent complex $T(\underline{\mathrm{Hol}}_B(C,W)/M)$ is supported in degree zero (this is the significance of the splitting $B = V \times M$). The regular relative M locus is denoted $\underline{\mathrm{Hol}}_B(C,W)^{\mathrm{reg}/M} \subseteq \underline{\mathrm{Hol}}_B(C,W)$.

The moduli stack associated to a moduli problem (5.6.3.1) is the zero set of the map

$$\mathbf{D} : V \times \underline{\mathrm{Sec}}_M(C,W) \to \underline{\mathrm{Sec}}_M(C,H) \tag{5.6.3.2}$$

over M, that is $\underline{\mathrm{Hol}}_B(C,W) = (V \times \underline{\mathrm{Sec}}_M(C,W)) \times_{\underline{\mathrm{Sec}}_M(C,H)} M$.

It is a purely topological result that every quasi-holomorphic section problem over a log smooth manifold has an open cover by linear models (2.7.92) (as before, we leave it as an exercise for the reader to deal with H); note that for this result, there is no need to split $B = V\times M$ (that is, we can just take $V = 0$). Now we claim that, moreover, the regular locus (inside any quasi-holomorphic section problem over a log smooth manifold) is covered by the regular relative M loci $\underline{\mathrm{Hol}}_B(C,W)^{\mathrm{reg}/M}$ of linear models (5.6.3.1) (it is here that the splitting $B = V\times M$ is crucial). To see this, given a point of the regular locus, choose a splitting $(B,b) = (V,0)\times(M,m)$ near its image $b \in B$, where V is a vector space and $m \in M$ is its own local stratum. Now in the open cover by linear models (2.7.92), we may take the classifying map of the standard gluing family $B \to {}'\mathbb{R}^n_{\geq 0}$ to factor through the projection $B \to M$. Since $m \in M$ is its own stratum, a point lying over it is regular iff it is regular relative M (**??**).

It therefore suffices to show, for linear model moduli problems (5.6.3.1), that the regular relative M locus $\underline{\mathrm{Hol}}_B(C,W)^{\mathrm{reg}/M}_{\mathsf{LogSm}}$ is representable and that its $(\mathsf{LogSm} \to \mathsf{LogTop})_!$ comparison map is an isomorphism.

To show this, we consider **linear projections** on log moduli problems (5.6.3.1)

$$\lambda : V \oplus \underline{\mathrm{Sec}}_{'\mathbb{R}^n_{\geq 0}}(C,W) \to K \tag{5.6.3.3}$$

which are given on $\underline{\mathrm{Sec}}_{'\mathbb{R}^n_{\geq 0}}(C,W)$ by integration against a smooth section $C \to W^* \otimes \Omega_{C/'\mathbb{R}^n_{\geq 0}} \otimes K$ which is supported away from the nodes and pulled back from the zero fiber $\tilde{C}_0$ under the defining trivialization (away from the nodes) of the gluing family $C \to {}'\mathbb{R}^n_{\geq 0}$. We will be interested in the locus of λ-regular points, also known as 'exactly regular relative $K\times M$', namely where $T(\underline{\mathrm{Hol}}_B(C,W)/(K\times M)) = 0$. Since our class of linear projections λ separates points, the union of these loci is the regular locus relative M.

It now suffices to show that the restriction

$$\lambda|_{\underline{\mathrm{Hol}}} : \underline{\mathrm{Hol}}_B(C,W) \to K\times M \tag{5.6.3.4}$$

is a local isomorphism of log topological and log smooth stacks over the λ-regular locus $\underline{\mathrm{Hol}}^{\mathrm{xreg}/K\times M} \subseteq \underline{\mathrm{Hol}}$. This is a local assertion near any particular λ-regular point $(b = (v,m), u : C_m \to W_m) \in \underline{\mathrm{Hol}}_B(C,W)^{\mathrm{xreg}/K\times M}$. We claim that we may assume wlog that u is the zero section (recall $W \to C$ is a vector bundle (5.6.3.1)). To see this, it suffices to apply a translation automorphism of $W_B \to C_B$ to our moduli problem (meaning, concretely, to pull back the map φ under such an automorphism) specializing to u at $b \in B$. To see that $u : C_b \to W_b$ extends to a log smooth section $C_B \to W_B$ (or $C \to W$), note that this is a local problem on C_b (partition of unity and properness of $C \to {}'\mathbb{R}^n_{\geq 0}$) and appeal to the basic extension result for real-valued functions on strata of log smooth manifolds (2.7.51).

Now let us construct a local inverse to $\lambda : \underline{\mathrm{Hol}}_B(C,W) \to K\times M$ (as a map of log topological stacks) near a given λ-regular basepoint $(b = (v,m), u : C_m \to W_m) \in \underline{\mathrm{Hol}}_B(C,W)^{\mathrm{xreg}/K\times M}$, assuming wlog that $v = 0$ and $u = 0$. Denote by

$$D : V \oplus \underline{\mathrm{Sec}}_M(C,W) \to \underline{\mathrm{Sec}}_M(C,H) \tag{5.6.3.5}$$

the family of derivatives of $\mathbf{D}$ at the zero sections $(b' = (0, m'), 0 : C_{m'} \to W_{m'})$ (so this is our basepoint when $m' = m$); this is a family of first order elliptic operators on the family $C_M \to M$. Denoting by $D_m : V \oplus C^\infty(C_m, W_m) \to C^\infty(C_m, H_m)$ the specialization of D to the fiber over $m \in M$, note that $D_m \oplus \lambda$ is an isomorphism since our basepoint is assumed to be λ-regular. It follows that $D_{m'} \oplus \lambda$ is invertible for all $m' \in M$ in a neighborhood of m and that its family of inverses $(D \oplus \lambda)^{-1} : \underline{\mathrm{Sec}}_M(C, H) \oplus K \to V \oplus \underline{\mathrm{Sec}}_M(C, W)$ is an isomorphism of log topological stacks (**??**). We denote by

$$Q : \underline{\mathrm{Sec}}_M(C, H) \to \ker \lambda \subseteq V \oplus \underline{\mathrm{Sec}}_M(C, W) \tag{5.6.3.6}$$

the restriction of $(D \oplus \lambda)^{-1}$. Now our local topological inverse $U^\infty : K \times M \to \underline{\mathrm{Hol}}_B(C, W)$ will be defined as the limit of the sequence $U^0, U^1, \ldots : K \times M \to V \oplus \underline{\mathrm{Sec}}_M(C, W)$ defined by the iteration $U^i = U^{i-1} - Q\mathbf{D}U^{i-1}$, with initial condition $U^0 : K \times M \to V \oplus \underline{\mathrm{Sec}}_M(C, W)$ any (e.g. linear) section of λ (e.g. the restriction of $(D \oplus \lambda)^{-1}$). $\square$

5.7 Derived Regularity

In this section, we prove the Derived Regularity Theorem (0.0.3) (see (5.7.10) below), which states that every quasi-holomorphic section problem is 'derived regular' in the following sense.

5.7.1 Definition (Derived Regular)**.** A quasi-holomorphic section problem $\wp = (W \to C \to B)$ over a derived smooth stack B is called *derived regular* when the morphism $\underline{\mathrm{Hol}}(\wp)_{\mathcal{D}\mathsf{Sm}} \to B$ is representable and the comparison map $(\mathcal{D}\mathsf{Sm} \to \mathsf{Top})_!\underline{\mathrm{Hol}}(\wp)_{\mathcal{D}\mathsf{Sm}} \to \underline{\mathrm{Hol}}(\wp)_{\mathsf{Top}}$ is an isomorphism.

5.7.2 Exercise (Locality of derived regularity)**.** Show that derived regularity is a *local property* in the sense that if $\wp$ is derived regular over each of a collection of open substacks $U_i \subseteq \underline{\mathrm{Hol}}(\wp)_{\mathsf{Top}}$, then it is derived regular over their union $\bigcup_i U_i \subseteq \underline{\mathrm{Hol}}(\wp)_{\mathsf{Top}}$.

In brief, the proof we are about to give of the Derived Regularity Theorem proceeds as follows. Recall the Regularity Theorem (**??**), which asserts that $\underline{\mathrm{Hol}}(\wp)^{\mathrm{reg}}_{\mathsf{Sm}}$ is representable and that $(\mathsf{Sm} \to \mathsf{Top})_!\underline{\mathrm{Hol}}(\wp)^{\mathrm{reg}}_{\mathsf{Sm}} \to \underline{\mathrm{Hol}}(\wp)^{\mathrm{reg}}_{\mathsf{Top}}$ is an isomorphism. The Derived Regularity Theorem is stronger in two respects: it concerns the entire moduli stack rather than just the regular locus, and it concerns the derived smooth moduli stack rather than just the smooth moduli stack. The first difference is easily dealt with: since derived regularity is preserved under pullback, a standard thickening argument (5.7.6) shows that it is enough to prove the Derived Regularity Theorem over the regular locus. To prove the Derived Regularity Theorem over the regular locus (which is the main difficulty), it suffices (given the Regularity Theorem) to show that the comparison map $(\mathsf{Sm} \to \mathcal{D}\mathsf{Sm})_!\underline{\mathrm{Hol}}(\wp)^{\mathrm{reg}}_{\mathsf{Sm}} \to \underline{\mathrm{Hol}}(\wp)^{\mathrm{reg}}_{\mathcal{D}\mathsf{Sm}}$ is an isomorphism (5.7.9). The analogous comparison map for the stack $\underline{\mathrm{Sec}}_B(C, W)$ of all sections is an isomorphism by (2.9.41)(2.9.43), and the moduli stack $\underline{\mathrm{Hol}}_B(C, W)$ is a fiber product of these. Now left Kan extension $(\mathsf{Sm} \to \mathcal{D}\mathsf{Sm})_!$ does not preserve all pullbacks, but it does preserve submersive pullbacks, and the Regularity Theorem implies that the relevant pullback is submersive over $\underline{\mathrm{Hol}}_B(C, W)^{\mathrm{reg}}$, so we are done.

It is remarkable that this argument reveals the Derived Regularity Theorem to be a *formal (yet nontrivial) consequence* of the Regularity Theorem! At no point in the argument do we need to contemplate the meaning of, or do any *hard analysis* (such as invoking Sobolev spaces or elliptic regularity) with, a family of quasi-holomorphic sections parameterized by a derived smooth manifold. This was quite a welcome surprise to the present author.

Initial reductions

We begin our treatment of the Derived Regularity Theorem with some initial reductions.

5.7.3 Lemma (Pullback and descent for derived regularity)**.** *Let $\wp$ be a quasi-holomorphic section problem over a derived smooth stack $B \in \mathsf{Shv}(\mathcal{D}\mathsf{Sm})$.*

(5.7.3.1) *If $\wp$ is derived regular, then so is its pullback $\wp' = \wp \times_B B'$ under any map of derived smooth stacks $B' \to B$.*

(5.7.3.2) *If the pullback $\wp'$ is derived regular for every map $B' \to B$ from a derived smooth manifold B', then $\wp$ is derived regular.*

Proof. It follows directly from the definition that $\underline{\mathrm{Hol}}(\wp') = \underline{\mathrm{Hol}}(\wp) \times_B B'$.

$$\begin{tikzcd} \underline{\mathrm{Hol}}(\wp') \arrow[r] \arrow[d] & \underline{\mathrm{Hol}}(\wp) \arrow[d] \\ B' \arrow[r] & B \end{tikzcd} \tag{5.7.3.3}$$

Thus representability of $\underline{\mathrm{Hol}}(\wp) \to B$ implies representability of the pullback $\underline{\mathrm{Hol}}(\wp') \to B'$, and representability of the pullback $\underline{\mathrm{Hol}}(\wp') \to B'$ for all maps from derived smooth manifolds $B' \to B$ implies representability of $\underline{\mathrm{Hol}}(\wp) \to B$.

Now let us compare the comparison maps (between derived smooth and topological moduli stacks) associated to $\wp$ and its pullback $\wp'$. Let us abbreviate $|\cdot|_! = (\mathcal{D}\mathsf{Sm} \to \mathsf{Top})_!$, and recall that $\underline{\mathrm{Hol}}(\wp)_{\mathsf{Top}} = \underline{\mathrm{Hol}}(\wp \times_B |B|_!)$, so the comparison map for $\wp$ takes the form $|\underline{\mathrm{Hol}}(\wp)|_! \to \underline{\mathrm{Hol}}(\wp \times_B |B|_!)$. Now the comparison maps for $\wp$ and $\wp'$ fit into a commuting diagram of the following shape.

$$\begin{tikzcd} {|\underline{\mathrm{Hol}}(\wp')|_!} \arrow[r] \arrow[d] & {|\underline{\mathrm{Hol}}(\wp)|_!} \arrow[d] \\ \underline{\mathrm{Hol}}(\wp' \times_{B'} |B'|_!) \arrow[r] \arrow[d] & \underline{\mathrm{Hol}}(\wp \times_B |B|_!) \arrow[d] \\ {|B'|_!} \arrow[r] & {|B|_!} \end{tikzcd} \tag{5.7.3.4}$$

The bottom square is a fiber square since the formation of $\underline{\mathrm{Hol}}$ is compatible with pullback (by inspection). The composite square is a pullback provided $\underline{\mathrm{Hol}}(\wp) \to B$ is representable, since left Kan extension $(\mathcal{D}\mathsf{Sm} \to \mathsf{Top})_!$ preserves pullbacks of representable morphisms (2.8.49). It thus follows from cancellation (1.1.57) that the top square is a pullback (when $\underline{\mathrm{Hol}}(\wp) \to B$ is representable).

Now if $\underline{\mathrm{Hol}}(\wp) \to B$ is representable and the comparison map for $\wp$ is an isomorphism, it follows that the comparison map for $\wp'$ is an isomorphism (since the top square in (5.7.3.4) above is a pullback). Conversely, suppose $\underline{\mathrm{Hol}}(\wp') \to B'$ is representable and the comparison map of $\wp'$ is an isomorphism, for all maps from derived smooth manifolds $B' \to B$. We already saw that this means $\underline{\mathrm{Hol}}(\wp) \to B$ is representable, so the top square in (5.7.3.4) is a pullback for all $B' \to B$. To check that the comparison map for $\wp$ is an isomorphism, it suffices to check that it pulls back to an isomorphism under any map from a topological space $Z \to |B|_!$ (1.1.127). By the definition of left Kan extension $|\cdot|_! = (\mathcal{D}\mathsf{Sm} \to \mathsf{Top})_! : \mathsf{Shv}(\mathcal{D}\mathsf{Sm}) \to \mathsf{Shv}(\mathsf{Top})$, such a map locally factors through the map $|B'|_! \to |B|_!$ associated to some map from a derived smooth manifold $B' \to B$. Now the comparison map of $\wp$ pulls back to an isomorphism under any such map by assumption. $\square$

5.7.4 Lemma (Reduction to smooth bases)**.** *If every quasi-holomorphic section problem over a smooth manifold is derived regular, then every quasi-holomorphic section problem over a derived smooth manifold is derived regular.*

Proof. Let $W \to C \to B$ be a quasi-holomorphic section problem over a derived smooth manifold B. The construction of an open cover of $\underline{\mathrm{Hol}}_B(C, W)$ by linear models (**??**)(**??**) (5.6.3.1) given earlier in the case B is a smooth manifold applies without change in the case B is a derived smooth manifold. Since derived regularity is local on $\underline{\mathrm{Hol}}_B(C, W)$, we may assume wlog that our quasi-holomorphic section problem $W \to C \to B$ is a linear model.

Since derived regularity is preserved under pullback (5.7.3.1), it suffices to show that every linear model quasi-holomorphic section problem over a derived smooth manifold B is, locally on B, a pullback of a quasi-holomorphic section problem over a smooth manifold. In fact, something much stronger is true, namely that the stack of linear model quasi-holomorphic section problems over derived smooth manifolds B is left Kan extended from smooth manifolds (**??**). □

5.7.5 Exercise (Derived regularity and point conditions)**.** Fix a morphism $A \to S$ in $(\mathsf{Shv}(\mathcal{D}\mathsf{Sm}) \downarrow \mathsf{Shv}(\mathsf{Top}))$ (meaning $A = (A_{\mathcal{D}\mathsf{Sm}}, A_{\mathsf{Top}})$ consists of a pair of derived smooth and topological stacks together with a comparison map $(\mathcal{D}\mathsf{Sm} \to \mathsf{Top})_! A_{\mathcal{D}\mathsf{Sm}} \to A_{\mathsf{Top}}$, etc.). Consider a pullback diagram of the following shape.

$$\begin{array}{ccc} \underline{\mathrm{Hol}}_B(C, W)' & \longrightarrow & A \\ \big\downarrow & & \big\downarrow \\ \underline{\mathrm{Hol}}_B(C, W) & \longrightarrow & S \end{array} \tag{5.7.5.1}$$

Recall that left Kan extension $\mathsf{Shv}(\mathcal{D}\mathsf{Sm}) \to \mathsf{Shv}(\mathsf{Top})$ preserves pullbacks of representable morphisms (2.8.49), and conclude that if the comparison maps of A and S are isomorphisms and $A_{\mathcal{D}\mathsf{Sm}} \to S_{\mathcal{D}\mathsf{Sm}}$ (hence also $A_{\mathsf{Top}} \to S_{\mathsf{Top}}$) is representable, then derived regularity of $\underline{\mathrm{Hol}}_B(C, W)$ implies the same for $\underline{\mathrm{Hol}}_B(C, W)'$.

Reduction to the regular locus

The following (trivial) result is quite useful for proving results about moduli stacks of pseudo-holomorphic maps/sections.

⋆ **5.7.6 Proposition** (Reduction to the regular locus)**.** *A condition on solutions to quasi-holomorphic section problems over log smooth manifolds which is preserved under pullback and holds over the regular locus holds everywhere.*

Proof. It suffices (in fact, is equivalent) to show that for any quasi-holomorphic moduli problem $\wp$ over a log smooth manifold B, we have

$$\underline{\mathrm{Hol}}(\wp) = \bigcup_{\wp = \tilde{\wp} \times_{\tilde{B}} B} \underline{\mathrm{Hol}}(\tilde{\wp})^{\mathrm{reg}} \times_{\tilde{B}} B \tag{5.7.6.1}$$

where the union (of open substacks of $\underline{\mathrm{Hol}}(\wp)$) is over all maps of log smooth manifolds $B \to \tilde{B}$, all quasi-holomorphic moduli problems $\tilde{\wp}$ over $\tilde{B}$, and all isomorphisms $\wp = \tilde{\wp} \times_{\tilde{B}} B$.

Let $\wp = (W \to C \to B, H, \varphi)$, and fix a point $(b \in B, u : C_b \to W_b) \in \underline{\mathrm{Hol}}(\wp)$ which we would like to show lies in the union (5.7.6.1) above. Let $\tilde{B} = B \times \mathbb{R}^k \supseteq B \times 0 = B$. Define $\tilde{\wp}/\tilde{B}$ as the pullback of $\wp$ under the projection $\tilde{B} \to B$, except that instead of taking

$$\tilde{\varphi} : J^1_{\tilde{B}}(\tilde{W}/\tilde{C}) = J^1_B(W/C) \times \mathbb{R}^k \to H \tag{5.7.6.2}$$

to simply equal φ, we add to it some linear map $\alpha : \mathbb{R}^k \to H$ over W (evidently $\tilde{\wp} \times_{\tilde{B}} B = \wp$). The point $(b, u) \in \underline{\mathrm{Hol}}(\wp) = \underline{\mathrm{Hol}}(\tilde{\wp}) \times_{\tilde{B}} B$ will lie in $\underline{\mathrm{Hol}}(\tilde{\wp})^{\mathrm{reg}} \times_{\tilde{B}} B$ iff the composition

$$\mathbb{R}^k \xrightarrow{\alpha} C^\infty(W, H) \xrightarrow{u_b^*} C^\infty(C_b, u^*H) \twoheadrightarrow T^1_u\underline{\mathrm{Hol}}(C_b, W_b) \twoheadrightarrow T^1_{(u,b)}\underline{\mathrm{Hol}}_B(C, W) \tag{5.7.6.3}$$

is surjective. In fact, we can choose α so that *a fortiori* the composition $\mathbb{R}^k \to T^1_u\underline{\mathrm{Hol}}(C_b, W_b)$ is surjective, since $C^\infty(C_b, u^*H) \twoheadrightarrow T^1_u\underline{\mathrm{Hol}}(C_b, W_b)$ is the cokernel of an elliptic operator on C_b (5.4), hence is finite-dimensional since C_b is compact (3.4)(3.6)(3.7). □

Derived regularity over the regular locus

We now come to the main point in the proof of the Derived Regularity Theorem, namely the proof that the comparison map $(\mathsf{Sm} \to \mathcal{D}\mathsf{Sm})_!\underline{\mathrm{Hol}}(\wp)^{\mathrm{reg}}_{\mathsf{Sm}} \to \underline{\mathrm{Hol}}(\wp)^{\mathrm{reg}}_{\mathcal{D}\mathsf{Sm}}$ is an isomorphism for any quasi-holomorphic section problem $\wp$ over a smooth manifold B (5.7.9). In other words, families of regular quasi-holomorphic sections parameterized by derived smooth manifolds are completely classified by such families over smooth manifolds. It is quite surprising that this turns out to be a formal consequence of the fact that $\underline{\mathrm{Hol}}(\wp')^{\mathrm{reg}}_{\mathsf{Sm}}$ is representable (**??**) for all moduli problems $\wp'$ over smooth manifolds. The key inputs are the fact that $\mathsf{Sm} \to \mathcal{D}\mathsf{Sm}$ preserves submersive pullbacks, hence so does $(\mathsf{Sm} \to \mathcal{D}\mathsf{Sm})_! : \mathsf{Shv}(\mathsf{Sm}) \to \mathsf{Shv}(\mathcal{D}\mathsf{Sm})$ (2.8.49), and the fact that the comparison map for stacks of all sections $(\mathsf{Sm} \to \mathcal{D}\mathsf{Sm})_!\underline{\mathrm{Sec}}_B(C, W)_{\mathsf{Sm}} \to \underline{\mathrm{Sec}}_B(C, W)_{\mathcal{D}\mathsf{Sm}}$ is an isomorphism (2.9.43).

To make the argument, we need a technical fact, namely we need to realize $\underline{\mathrm{Hol}}_B(C, W)$ as a fiber of a map of stacks of smooth sections (note that the most apparent fiber product presentation of $\underline{\mathrm{Hol}}_B(C, W)$ (5.3.6) is not of this form).

5.7.7 Lemma. *Let $(W \to C \to B, H, \varphi)$ be quasi-holomorphic section problem. If H/W is the pullback of H_0/C, then there is a fiber diagram*

$$\begin{tikzcd} \underline{\mathrm{Hol}}_B(C, W) \arrow[r] \arrow[d] & \underline{\mathrm{Sec}}_B(C, W) \arrow[d] \\ B \arrow[r] & \underline{\mathrm{Sec}}_B(C, H_0) \end{tikzcd} \tag{5.7.7.1}$$

for all flavors of moduli stacks.

Proof. We have $H = W \times_C H_0$ as a fiber product in Sm (and also in $\mathcal{D}\mathsf{Sm}$ since $\mathsf{Sm} \to \mathcal{D}\mathsf{Sm}$ preserves transverse fiber products (2.9.3.2)(2.9.22)). We thus have $\underline{\mathrm{Sec}}_B(C, H) = \underline{\mathrm{Sec}}_B(C, W) \times_B \underline{\mathrm{Sec}}_B(C, H_0)$ (this is a purely categorical consequence of $H = W \times_C H_0$). Now $\underline{\mathrm{Hol}}_B(C, W) \to \underline{\mathrm{Sec}}_B(C, W)$ is by definition a pullback of the zero section $\underline{\mathrm{Sec}}_B(C, W) \to \underline{\mathrm{Sec}}_B(C, H) = \underline{\mathrm{Sec}}_B(C, W) \times_B \underline{\mathrm{Sec}}_B(C, H_0)$, which is a pullback of $B \to \underline{\mathrm{Sec}}_B(C, H_0)$. □

The presentation (5.7.7) is 'better' than the fiber product presentation (5.3.6) in that one of the factors is B. We need to improve it further to make this factor as small as possible.

5.7.8 Lemma. *In the setup of (5.7.7), if $H_0 \to C \to B$ is the pullback of $H_0' \to C_0' \to B'$ under a map $B \to B'$, then there is a fiber diagram*

$$\begin{tikzcd} \underline{\mathrm{Hol}}_B(C,W) \arrow[r] \arrow[d] & \underline{\mathrm{Sec}}_B(C,W) \arrow[d] \\ B' \arrow[r] & \underline{\mathrm{Sec}}_{B'}(C',H_0') \end{tikzcd} \tag{5.7.8.1}$$

for all flavors of moduli stacks.

Proof. We have (by cancellation (1.1.57)) a pair of fiber squares.

$$\begin{tikzcd} B \arrow[r] \arrow[d] & \underline{\mathrm{Sec}}_B(C,H_0) \arrow[r] \arrow[d] & B \arrow[d] \\ B' \arrow[r] & \underline{\mathrm{Sec}}_{B'}(C',H_0') \arrow[r] & B' \end{tikzcd} \tag{5.7.8.2}$$

Stacking the left square with the square (5.7.7) gives the desired result. □

5.7.9 Proposition. *For any quasi-holomorphic section problem $\wp$ over a smooth manifold, the comparison map $(\mathsf{Sm} \to \mathcal{D}\mathsf{Sm})_!\underline{\mathrm{Hol}}(\wp)^{\mathrm{reg}}_{\mathsf{Sm}} \to \underline{\mathrm{Hol}}(\wp)^{\mathrm{reg}}_{\mathcal{D}\mathsf{Sm}}$ is an isomorphism.*

Proof. The desired assertion is local, so we may fix a basepoint of $\underline{\mathrm{Hol}}(\wp)^{\mathrm{reg}}$ and prove it just in a neighborhood of this point. There exists a local linear model in which this point is regular relative X_P (5.6.3.1). We are thus reduced to considering the comparison map for $\underline{\mathrm{Hol}}_B(C_B,W_B)^{\mathrm{reg}/X_P}$ for a linear model.

By (5.7.8) we have a fiber product presentation of the following form.

$$\begin{tikzcd} \underline{\mathrm{Hol}}_B(C_B,W_B) \arrow[r] \arrow[d] & \underline{\mathrm{Sec}}_B(C_B,W_B) \arrow[d] \\ X_P \arrow[r] & \underline{\mathrm{Sec}}_{X_P}(C,H) \end{tikzcd} \tag{5.7.9.1}$$

We now consider the $(\mathsf{Sm} \to \mathcal{D}\mathsf{Sm})_!$ comparison cube of this fiber square. The comparison maps for the parameterized section functors $\underline{\mathrm{Sec}}$ on the right are isomorphisms (2.9.43), as is the comparison map for X_P. Thus to show that the comparison map for $\underline{\mathrm{Hol}}_B(C_B,W_B)^{\mathrm{reg}/X_P}$ is an isomorphism, it suffices to show that $(\mathsf{Sm} \to \mathcal{D}\mathsf{Sm})_!$(5.7.9.1) is a fiber square (over the regular locus relative X_P).

To show that $(\mathsf{Sm} \to \mathcal{D}\mathsf{Sm})_!$(5.7.9.1) is a fiber square over the regular locus relative X_P, recall that $\mathsf{Sm} \to \mathcal{D}\mathsf{Sm}$ preserves submersive pullbacks (2.9.22), hence so does left Kan extension $(\mathsf{Sm} \to \mathcal{D}\mathsf{Sm})_!$ (2.8.49). It thus suffices to show that the right vertical map

$\underline{\mathrm{Sec}}_B(C_B, W_B) \to \underline{\mathrm{Sec}}_{X_P}(C, H)$ is submersive over the relative regular locus. For a map $Z \to \underline{\mathrm{Sec}}_{X_P}(C, H)$ from a log smooth manifold Z, the pullback

$$\begin{tikzcd} \underline{\mathrm{Hol}}_{B\times_{X_P}Z}(C_{B\times_{X_P}Z}, W_{B\times_{X_P}Z}) \arrow[r] \arrow[d] & \underline{\mathrm{Sec}}_B(C_B, W_B) \arrow[d] \\ Z \arrow[r] & \underline{\mathrm{Sec}}_{X_P}(C, H) \end{tikzcd} \tag{5.7.9.2}$$

is itself a moduli stack of quasi-holomorphic sections over the parameter space $B \times_{X_P} Z = V \times Z$, hence its relative regular locus is submersive over Z by the Regularity Theorem (**??**). Thus $\underline{\mathrm{Sec}}_B(C_B, W_B) \to \underline{\mathrm{Sec}}_{X_P}(C, H)$ is submersive over the relative regular locus, as desired. □

Conclusion

We may now conclude with the proof of the Derived Regularity Theorem (0.0.3).

5.7.10 Derived Regularity Theorem (stated earlier as (0.0.3))**.** *Every quasi-holomorphic section problem over a derived smooth stack is derived regular.*

Proof. By our initial reductions (5.7.3.2)(5.7.4), it suffices to consider quasi-holomorphic section problems over smooth manifolds. Derived regularity is a local property on $\underline{\mathrm{Hol}}(\wp)$, so we may consider the maximal open subset $\underline{\mathrm{Hol}}(\wp)^{\mathrm{dreg}} \subseteq \underline{\mathrm{Hol}}(\wp)$ which is derived regular. Derived regularity is preserved under pullback (5.7.3.1), so by reduction to the regular locus (5.7.6), to show that $\underline{\mathrm{Hol}}(\wp)^{\mathrm{dreg}} = \underline{\mathrm{Hol}}(\wp)$ it suffices to show that $\underline{\mathrm{Hol}}(\wp)^{\mathrm{dreg}}$ contains $\underline{\mathrm{Hol}}(\wp)^{\mathrm{reg}}$, which follows from the Regularity Theorem (**??**) and the fact that the comparison map $(\mathsf{Sm} \to \mathfrak{D}\mathsf{Sm})_! \underline{\mathrm{Hol}}_B(C, W)^{\mathrm{reg}}_{\mathsf{Sm}} \to \underline{\mathrm{Hol}}_B(C, W)^{\mathrm{reg}}_{\mathfrak{D}\mathsf{Sm}}$ is an isomorphism (5.7.9). □

5.8 Stability

5.9 A priori estimates

We now provide a treatment of the standard *a priori* estimates on pseudo-holomorphic maps.

It is difficult to trace the origin of the results in this section. Many appear in some form in Gromov [35], where they were considered too trivial to require anything more than a very brief justification. Subsequent work of many authors has supplied various different ways turning Gromov's brief hints into complete proofs.

⋆ **5.9.1 Definition** (Bound on geometry). Let (X, g) be a Riemannian manifold. A *bound on the geometry* of (X, g) at a point $p \in X$ is a sequence of real numbers $\varepsilon > 0$ and $M_0, M_1, \ldots < \infty$ such that there exists a smooth map $\Phi : (B(1), 0) \to (X, p)$ with $\Phi^* g \geq \varepsilon \cdot g_{\mathrm{std}}$ and $\|\Phi^* g\|_{C^k} \leq M_k$ for all k. A bound on the geometry of (X, g) (resp. over a subset $A \subseteq X$) is a sequence $\varepsilon > 0$ and $M_0, M_1, \ldots < \infty$ which bounds the geometry of X at every point (resp. of A). A bound on the geometry *and injectivity radius* means that in addition Φ is required to be injective (beware that in standard terminology, a 'bound on the geometry' is usually taken to mean what we have decided to call a 'bound on the geometry and injectivity radius').

A bound on the geometry of (X, g, τ) for some additional structure τ (e.g. a symplectic form, almost complex structure, or any combination thereof) means that $\|\Phi^* \tau\|_{C^k} \leq M_k$ as well. When the data τ itself determines a Riemannian metric g_τ (e.g. a tame pair (J, ω) determining the metric $\omega(v, Jw) + \omega(w, Jv)$), we may simply say (X, τ) has bounded geometry to mean that (X, g_τ, τ) has bounded geometry.

We say that a constant 'depends on the geometry of X (resp. over a $A \subseteq X$)' to mean that said constant may be bounded in terms of a bound on the geometry of X (resp. over A).

⋆ **5.9.2 Definition** (Energy). Let (X, g) be a Riemannian manifold, and let C be a Riemann surface. The *energy* of a map $u : C \to (X, g)$ is the integral

$$E(u) = \int_C \tfrac{1}{2}|du|^2 \tag{5.9.2.1}$$

where the integrand $\frac{1}{2}|du|^2$ is by definition $\frac{1}{2}(|u_x|^2 + |u_y|^2)\, dx\, dy = g(u_z, u_{\bar{z}})\, i\, dz\, d\bar{z}$ in local holomorphic coordinates.

5.9.3 Exercise. Show that if $u : C \to (X, J)$ is pseudo-holomorphic and J is compatible with ω, then $u^*\omega = \frac{1}{2}|du|^2$, so we have $E(u) = \int_C u^*\omega$.

* *

⋆ **5.9.4 Proposition** (Gradient bounds imply C^∞ bounds). *Let $u : D^2 \to (X, J, g)$ be a pseudo-holomorphic map. If $\sup |du| \leq M$, then*

$$|D^k u(0)| \leq \mathrm{const} \cdot E(u)^{1/2} \tag{5.9.4.1}$$

for some const $< \infty$ *depending on $k < \infty$, $M < \infty$, and the geometry of (X, J, g) over the image of u.*

Proof. Choose local linear coordinates on the target X. Write the pseudo-holomorphic map equation in local coordinates as $u_x + J(u)u_y = 0$. Applying $\frac{d}{dx} - \frac{d}{dy}J(u)$ to this equation yields the higher order equation

$$u_{xx} + u_{yy} = \dot{J}(u, u_y)u_x - \dot{J}(u, u_x)u_y, \tag{5.9.4.2}$$

which has the virtue that its leading order terms have constant coefficients.

We now bound the $W^{k,2}$-norm of u (over any compact subset of $(D^2)^\circ$) using two successive bootstrapping arguments based on (5.9.4.2). The first bounds $\|u\|_{k,2}$ by some (unspecified) function of M. The second bounds $\|u\|_{k,2}$ by $E(u)^{1/2}$ times some (unspecified) function of M (which is enough by Sobolev embedding (3.2.28)).

For the first bootstrap, we note that L^2-norm of the right side of (5.9.4.2) is bounded in terms of M, so by elliptic regularity (3.4) we have a bound on the $W^{2,2}$-norm of u in terms of M. This implies a $W^{1,2}$-bound on the right side of (5.9.4.2) (inspect its derivative) in terms of M, hence by elliptic regularity we have a bound on the $W^{3,2}$-norm of u in terms of M. Now we claim that for $k \geq 3$, a bound on the $W^{k,2}$-norm of u implies a bound on the $W^{k+1,2}$-norm of u. Indeed, the right side of (5.9.4.2) is a smooth function vanishing at zero applied to (u, Du), hence since $W^{k-1,2} \subseteq C^0$ for $k \geq 3$ (3.2.28), the $W^{k-1,2}$-norm of the right side is bounded in terms of the $W^{k-1,2}$-norm of (u, Du) (3.2.32), thus in terms of the $W^{k,2}$-norm of u. By induction, the $W^{k,2}$-norm of u is bounded in terms of M for all $k < \infty$.

Now we do the second bootstrap. We know from the first bootstrap that the derivatives of u are bounded in terms of M. In particular, the factors $\dot{J}(u, u_x)$ and $\dot{J}(u, u_y)$ are bounded in C^∞ in terms of M. It follows (3.3.7) that the $W^{k-1,2}$-norm of the right side of (5.9.4.2) is bounded linearly in terms of the $W^{k,2}$-norm of u. Applying elliptic regularity (3.4), we conclude that the $W^{k+1,2}$-norm of u is bounded linearly in terms of the $W^{k,2}$-norm of u. In the base case $k = 1$, the $W^{k-1,2}$-norm of the right side is (by inspection) bounded linearly in terms of $E(u)^{1/2}$. By induction, the $W^{k,2}$-norm of u is bounded linearly in terms of $E(u)^{1/2}$ for all $k < \infty$. □

5.9.5 Exercise. Use a rescaling argument to deduce from (5.9.4) that $|D^k u(p)| \leq \text{const} \cdot d(p, \partial D^2)^{-(k-1)} \cdot E(u)^{1/2}$ under the same hypotheses.

⋆ **5.9.6 Hofer's Lemma** ([43, Lemma 3.3])**.** *Let (X, d) be a complete metric space, let $f : X \to \mathbb{R}_{\geq 0}$ be locally bounded, and let $M < \infty$. For every $p_0 \in X$, there exists $p \in X$ with $f(p) \geq f(p_0)$ and $d(p, p_0) \leq 2M \cdot f(p_0)^{-1}$ such that $d(x, p) \leq M \cdot f(p)^{-1} \implies f(x) \leq 2f(p)$.*

Proof. If p_0 does not satisfy the desired property, then there exists a violation point p_1, i.e. $d(p_0, p_1) \leq M \cdot f(p_0)^{-1}$ and $f(p_1) \geq 2f(p_0)$. If p_1 does not satisfy the desired property, there is a subsequent violation point p_2. We have $f(p_i) \geq 2^i f(p_0)$, hence $d(p_i, p_{i+1}) \leq 2^{-i}M \cdot f(p_0)^{-1}$, so $d(p_0, p_i) \leq 2M \cdot f(p_0)^{-1}$. This process $p_0, p_1, \ldots$ will eventually terminate at a suitable point p, since otherwise it would converge (since X is complete) to a point p_∞ near which f is not locally bounded. □

⋆ **5.9.7 Proposition** (Small energy bounds imply gradient bounds). *Let $u : D^2 \to (X, J, g)$ be a pseudo-holomorphic map. If $\int_{D^2} |du|^2 < \varepsilon$ then*

$$|du(0)| \leq 5 \tag{5.9.7.1}$$

for some $\varepsilon > 0$ depending on the geometry of (X, J, g) over the image of u.

Proof. If $|du(0)| \geq 5$, then we can use Hofer's Lemma (5.9.6) to find a point $p \in D^2$ at distance at most $\frac{2}{5}$ from the origin such that $|du| \leq 2|du(p)|$ over the disk of radius $|du(p)|^{-1}$ around p (which is entirely contained in D^2 since $\frac{2}{5} + \frac{1}{5} < 1$). Now consider the map $\tilde{u} : D^2 \to (X, J, g)$ obtained from u by rescaling the disk of radius $|du(p)|^{-1}$ around p to the disk D^2. We have $\sup |d\tilde{u}| \leq 2$ by construction, hence we have C^∞ bounds (5.9.4) on $\tilde{u}$ over D^2. We also have $|d\tilde{u}(0)| = 1$ by construction, which combined with C^∞ bounds on $\tilde{u}$ implies a lower bound on the energy of $\tilde{u}$, hence also on the energy of u. Now take $\varepsilon > 0$ to be smaller than this lower bound. □

⋆ **5.9.8 Proposition** (Energy bound and C^0-bound imply removable singularity). *A pseudo-holomorphic map $u : D^2 \setminus 0 \to (X, J)$ extends smoothly to D^2 iff its image is relatively compact in X and it has finite energy.*

Proof. Consider the coordinates $z = e^{-s-it}$ on $D^2 \setminus 0 = [0, \infty) \times S^1$, and let $E(s)$ denote the energy of u over $[s, \infty) \times S^1$. Finiteness of the energy implies $E(s) \to 0$ as $s \to \infty$, and small energy decreases exponentially in long cylinders (**??**), so we have $E(s) \leq A \cdot e^{-2s}$ for some $A < \infty$. Since sufficiently small energy bounds imply gradient bounds (5.9.7), we have

$$|du(s, t)| \leq \mathrm{const} \cdot E(s-1)^{1/2} \leq B \cdot e^{-s} \tag{5.9.8.1}$$

for some $B < \infty$ and $s \geq 1$. This implies that $|du|$ is bounded on $D^2 \setminus 0$ and hence that u extends continuously to D^2. Note that the factor in the exponent in (5.9.8.1) is sharp and that we need it to be: the bound $|du(s,t)| \leq B \cdot e^{-\delta s}$ is equivalent to the bound $|du| \leq B \cdot r^{\delta - 1}$ on $D^2 \setminus 0$, which for $\delta > 1$ is false and for $\delta < 1$ does not imply the desired boundedness of $|du|$ on $D^2 \setminus 0$.

Now we would like to use the fact that gradient bounds imply C^∞ bounds (5.9.4) to conclude that u is smooth on D^2. That result assumed u to be smooth, whereas here u is merely continuous on D^2 and smooth on $D^2 \setminus 0$, with $|du|$ bounded. So, let us check that the proof of (5.9.4) goes through in this setting. We first check that the key equation (5.9.4.2)

$$u_{xx} + u_{yy} = \dot{J}(u, u_y)u_x - \dot{J}(u, u_x)u_y \tag{5.9.8.2}$$

holds as an equality of distributions on D^2. □

5.9.9 Gromov–Schwarz Lemma (Exact C^0-bounds imply gradient bounds [35, 1.3.A]).
⋆ *Let $(X, J, d\lambda)$ be tame. For any pseudo-holomorphic map $u : D^2 \to X$, we have*

$$|du(0)| \leq \mathrm{const}, \tag{5.9.9.1}$$

for some constant depending on a bound on the geometry of (X, J, λ).

Proof. Assume $|du(0)| \geq 1$. Consider $\tilde{u} : D^2 \to X$ obtained by rescaling D^2 to the disk of radius $|du(0)|^{-1}$ around zero and composing with u. Now $|d\tilde{u}(0)| = 1$, so since small energy bounds imply gradient bounds (5.9.7), we obtain a constant lower bound on the energy of u over the disk of radius $|du(0)|^{-1}$ centered at zero.

Consider cylindrical coordinates $z = e^{-s-it}$ on $[0, \infty) \times S^1 = D^2 \setminus 0 \subseteq D^2$, and let $E(s)$ be the energy of u over $[s, \infty) \times S^1$. We just saw that $E(\log |du(0)|) \geq \varepsilon$ for some $\varepsilon > 0$ depending on the geometry of (X, λ, J).

We now establish a differential inequality for E which, together with the lower bound $E(\log |du(0)|) \geq \varepsilon$ and finiteness of $E(0)$, gives an upper bound on $\log |du(0)|$. We have

$$-2E'(s) = \int_{s \times S^1} |du|^2 \geq \frac{1}{2\pi} \left(\int_{s \times S^1} |du| \right)^2 \geq \frac{1}{2\pi \|\lambda\|} \left(\int_{s \times S^1} u^* \lambda \right)^2 = \frac{E(s)^2}{2\pi \|\lambda\|}. \qquad (5.9.9.2)$$

This differential inequality blows up in finite time (5.9.10), which gives the desired upper bound on $\log |du(0)|$. ☐

5.9.10 Exercise. Use the fact that $\sum_n n^{-2} < \infty$ to show that for every $\varepsilon > 0$ there exists $N = N(\varepsilon) < \infty$ such that there exists no smooth function $f : [0, N] \to \mathbb{R}$ satisfying $f(0) \geq \varepsilon$ and $f'(x) \geq f(x)^2$.

Bibliography

[1] *Théorie des topos et cohomologie étale des schémas. Tome 1: Théorie des topos*. Lecture Notes in Mathematics, Vol. 269. Springer-Verlag, Berlin-New York, 1972. Séminaire de Géométrie Algébrique du Bois-Marie 1963–1964 (SGA 4), Dirigé par M. Artin, A. Grothendieck, et J. L. Verdier. Avec la collaboration de N. Bourbaki, P. Deligne et B. Saint-Donat.

[2] Dan Abramovich and Qile Chen. Stable logarithmic maps to Deligne-Faltings pairs II. *Asian J. Math.*, 18(3):465–488, 2014.

[3] Dan Abramovich, Qile Chen, Mark Gross, and Bernd Siebert. Decomposition of degenerate Gromov-Witten invariants. *Compos. Math.*, 156(10):2020–2075, 2020.

[4] Dan Abramovich, Qile Chen, Mark Gross, and Bernd Siebert. Punctured logarithmic maps. *arxiv:2009.07720v3*, pages 1–125, 2024.

[5] Robert A. Adams. *Sobolev spaces*. Academic Press [A subsidiary of Harcourt Brace Jovanovich, Publishers], New York-London, 1975. Pure and Applied Mathematics, Vol. 65.

[6] Lars V. Ahlfors. *Complex analysis. An introduction to the theory of analytic functions of one complex variable*. McGraw-Hill Book Co., Inc., New York-Toronto-London, 1953.

[7] Lars V. Ahlfors. *Complex analysis*. International Series in Pure and Applied Mathematics. McGraw-Hill Book Co., New York, third edition, 1978. An introduction to the theory of analytic functions of one complex variable.

[8] Jarod Alper, Jack Hall, and David Rydh. A Luna étale slice theorem for algebraic stacks. *Ann. of Math. (2)*, 191(3):675–738, 2020.

[9] M. Artin. Versal deformations and algebraic stacks. *Invent. Math.*, 27:165–189, 1974.

[10] M. F. Atiyah and R. Bott. A Lefschetz fixed point formula for elliptic complexes. I. *Ann. of Math. (2)*, 86:374–407, 1967.

[11] M. F. Atiyah, N. J. Hitchin, and I. M. Singer. Self-duality in four-dimensional Riemannian geometry. *Proc. Roy. Soc. London Ser. A*, 362(1711):425–461, 1978.

[12] K. Behrend. Introduction to algebraic stacks. In *Moduli spaces*, volume 411 of *London Math. Soc. Lecture Note Ser.*, pages 1–131. Cambridge Univ. Press, Cambridge, 2014.

[13] J. M. Boardman and R. M. Vogt. *Homotopy invariant algebraic structures on topological spaces*. Lecture Notes in Mathematics, Vol. 347. Springer-Verlag, Berlin-New York, 1973.

[14] A. K. Bousfield and D. M. Kan. *Homotopy limits, completions and localizations*, volume Vol. 304 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin-New York, 1972.

[15] Qile Chen. Stable logarithmic maps to Deligne-Faltings pairs I. *Ann. of Math. (2)*, 180(2):455–521, 2014.

[16] Jean-Marc Cordier. Sur la notion de diagramme homotopiquement cohérent. *Cahiers Topologie Géom. Différentielle*, 23(1):93–112, 1982. Third Colloquium on Categories, Part VI (Amiens, 1980).

[17] Marius Crainic and Ivan Struchiner. On the linearization theorem for proper Lie groupoids. *Ann. Sci. Éc. Norm. Supér. (4)*, 46(5):723–746, 2013.

[18] Matias del Hoyo and Rui Loja Fernandes. Riemannian metrics on Lie groupoids. *J. Reine Angew. Math.*, 735:143–173, 2018.

[19] P. Deligne and D. Mumford. The irreducibility of the space of curves of given genus. *Inst. Hautes Études Sci. Publ. Math.*, (36):75–109, 1969.

[20] Jean Dieudonné. Une généralisation des espaces compacts. *J. Math. Pures Appl. (9)*, 23:65–76, 1944.

[21] Albrecht Dold. Homology of symmetric products and other functors of complexes. *Ann. of Math. (2)*, 68:54–80, 1958.

[22] Albrecht Dold. Partitions of unity in the theory of fibrations. *Ann. of Math. (2)*, 78:223–255, 1963.

[23] Eduardo J. Dubuc. C^∞-schemes. *Amer. J. Math.*, 103(4):683–690, 1981.

[24] Yuri V. Egorov and Bert-Wolfgang Schulze. *Pseudo-differential operators, singularities, applications*, volume 93 of *Operator Theory: Advances and Applications*. Birkhäuser Verlag, Basel, 1997.

[25] Charles Ehresmann. Sur les espaces fibrés différentiables. *C. R. Acad. Sci. Paris*, 224:1611–1612, 1947.

[26] Charles Ehresmann. Les connexions infinitésimales dans un espace fibré différentiable. In *Colloque de topologie (espaces fibrés), Bruxelles, 1950*, pages 29–55. Georges Thone, Liège; Masson et Cie., Paris, 1951.

[27] Charles Ehresmann. Catégories topologiques et catégories différentiables. In *Colloque Géom. Diff. Globale (Bruxelles, 1958)*, pages 137–150. Centre Belge Rech. Math., Louvain, 1959.

[28] Samuel Eilenberg and J. A. Zilber. Semi-simplicial complexes and singular homology. *Ann. of Math. (2)*, 51:499–513, 1950.

[29] Y. Eliashberg, A. Givental, and H. Hofer. Introduction to symplectic field theory. pages 560–673. 2000. GAFA 2000 (Tel Aviv, 1999).

[30] Kenji Fukaya, Yong-Geun Oh, Hiroshi Ohta, and Kaoru Ono. *Lagrangian intersection Floer theory: anomaly and obstruction. Part I*, volume 46 of *AMS/IP Studies in Advanced Mathematics*. American Mathematical Society, Providence, RI; International Press, Somerville, MA, 2009.

[31] Kenji Fukaya, Yong-Geun Oh, Hiroshi Ohta, and Kaoru Ono. *Lagrangian intersection Floer theory: anomaly and obstruction. Part II*, volume 46 of *AMS/IP Studies in Advanced Mathematics*. American Mathematical Society, Providence, RI; International Press, Somerville, MA, 2009.

[32] Kenji Fukaya and Kaoru Ono. Arnold conjecture and Gromov-Witten invariant. *Topology*, 38(5):933–1048, 1999.

[33] Peter Gabriel and Friedrich Ulmer. *Lokal präsentierbare Kategorien*. Lecture Notes in Mathematics, Vol. 221. Springer-Verlag, Berlin-New York, 1971.

[34] Jean Giraud. *Cohomologie non abélienne*. Die Grundlehren der mathematischen Wissenschaften, Band 179. Springer-Verlag, Berlin-New York, 1971.

[35] M. Gromov. Pseudo holomorphic curves in symplectic manifolds. *Invent. Math.*, 82(2):307–347, 1985.

[36] Mark Gross and Bernd Siebert. Logarithmic Gromov-Witten invariants. *J. Amer. Math. Soc.*, 26(2):451–510, 2013.

[37] André Haefliger. Groupoïdes d'holonomie et classifiants. Number 116, pages 70–97. 1984. Transversal structure of foliations (Toulouse, 1982).

[38] André Haefliger. Orbi-espaces. In *Sur les groupes hyperboliques d'après Mikhael Gromov (Bern, 1988)*, volume 83 of *Progr. Math.*, pages 203–213. Birkhäuser Boston, Boston, MA, 1990.

[39] André Haefliger. Complexes of groups and orbihedra. In *Group theory from a geometrical viewpoint (Trieste, 1990)*, pages 504–540. World Sci. Publ., River Edge, NJ, 1991.

[40] J. Heinloth. Notes on differentiable stacks. In *Mathematisches Institut, Georg-August-Universität Göttingen: Seminars Winter Term 2004/2005*, pages 1–32. Universitätsdrucke Göttingen, Göttingen, 2005.

[41] V. A. Hinich and V. V. Schechtman. On homotopy limit of homotopy algebras. In *K-theory, arithmetic and geometry (Moscow, 1984–1986)*, volume 1289 of *Lecture Notes in Math.*, pages 240–264. Springer, Berlin, 1987.

[42] H. Hofer. A general Fredholm theory and applications. In *Current developments in mathematics, 2004*, pages 1–71. Int. Press, Somerville, MA, 2006.

[43] H. Hofer and C. Viterbo. The Weinstein conjecture in the presence of holomorphic spheres. *Comm. Pure Appl. Math.*, 45(5):583–622, 1992.

[44] Lars Hörmander. *The analysis of linear partial differential operators. I*, volume 256 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Math-*

ematical Sciences]. Springer-Verlag, Berlin, 1983. Distribution theory and Fourier analysis.

[45] Luc Illusie, Kazuya Kato, and Chikara Nakayama. Quasi-unipotent logarithmic Riemann-Hilbert correspondences. *J. Math. Sci. Univ. Tokyo*, 12(1):1–66, 2005.

[46] Niles Johnson and Donald Yau. *2-dimensional categories*. Oxford University Press, Oxford, 2021.

[47] A. Joyal. Quasi-categories and Kan complexes. *J. Pure Appl. Algebra*, 175(1-3):207–222, 2002. Special volume celebrating the 70th birthday of Professor Max Kelly.

[48] André Joyal. Notes on quasi-categories. 2004, 2007, 2008.

[49] André Joyal. The Theory of Quasi-Categories and its Applications. 2008.

[50] Dominic Joyce. An introduction to d-manifolds and derived differential geometry. In *Moduli spaces*, volume 411 of *London Math. Soc. Lecture Note Ser.*, pages 230–281. Cambridge Univ. Press, Cambridge, 2014.

[51] Dominic Joyce. A new definition of Kuranishi space. *arxiv:math/1409.6908v3*, pages 1–193, 2015.

[52] Dominic Joyce. Manifolds with analytic corners. *arXiv:1605.05913*, 2016.

[53] Dominic Joyce. Algebraic geometry over C^∞-rings. *Mem. Amer. Math. Soc.*, 260(1256):v+139, 2019.

[54] Dominic Joyce. Kuranishi spaces as a 2-category. In *Virtual fundamental cycles in symplectic topology*, volume 237 of *Math. Surveys Monogr.*, pages 253–298. Amer. Math. Soc., Providence, RI, 2019.

[55] Daniel M. Kan. Abstract homotopy. III. *Proc. Nat. Acad. Sci. U.S.A.*, 42:419–421, 1956.

[56] Daniel M. Kan. On c. s. s. complexes. *Amer. J. Math.*, 79:449–476, 1957.

[57] Daniel M. Kan. On c.s.s. categories. *Bol. Soc. Mat. Mexicana*, 2:82–94, 1957.

[58] Daniel M. Kan. Functors involving c.s.s. complexes. *Trans. Amer. Math. Soc.*, 87:330–346, 1958.

[59] Kazuya Kato. Logarithmic structures of Fontaine-Illusie. In *Algebraic analysis, geometry, and number theory (Baltimore, MD, 1988)*, pages 191–224. Johns Hopkins Univ. Press, Baltimore, MD, 1989.

[60] Yoshiki Kinoshita. Nobuo Yoneda (1930–1996). *Math. Japon.*, 47(1):155, 1998.

[61] W. Kondrachov. Sur certaines propriétés des fonctions dans l'espace. *C. R. (Doklady) Acad. Sci. URSS (N.S.)*, 48:535–538, 1945.

[62] Maxim Kontsevich. Enumeration of rational curves via torus actions. In *The moduli space of curves (Texel Island, 1994)*, volume 129 of *Progr. Math.*, pages 335–368. Birkhäuser Boston, Boston, MA, 1995.

[63] M. Kuranishi. New proof for the existence of locally complete families of complex structures. In *Proc. Conf. Complex Analysis (Minneapolis, 1964)*, pages 142–154. Springer, Berlin, 1965.

[64] Markus Land. *Introduction to infinity-categories*. Compact Textbooks in Mathematics. Birkhäuser/Springer, Cham, [2021] ©2021.

[65] Gérard Laumon and Laurent Moret-Bailly. *Champs algébriques*, volume 39 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, 2000.

[66] H. Blaine Lawson, Jr. and Marie-Louise Michelsohn. *Spin geometry*, volume 38 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 1989.

[67] Tom Leinster. *Basic category theory*, volume 143 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2014.

[68] Jean Leray. L'anneau d'homologie d'une représentation. *C. R. Acad. Sci. Paris*, 222:1366–1368, 1946.

[69] Jun Li. Stable morphisms to singular schemes and relative stable morphisms. *J. Differential Geom.*, 57(3):509–578, 2001.

[70] Jun Li. A degeneration formula of GW-invariants. *J. Differential Geom.*, 60(2):199–293, 2002.

[71] Robert B. Lockhart and Robert C. McOwen. Elliptic differential operators on noncompact manifolds. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 12(3):409–447, 1985.

[72] Jacob Lurie. *Derived algebraic geometry*. ProQuest LLC, Ann Arbor, MI, 2004. Thesis (Ph.D.)–Massachusetts Institute of Technology.

[73] Jacob Lurie. *Derived Algebraic Geometry V: Structured Spaces*. 2009.

[74] Jacob Lurie. *Higher topos theory*, volume 170 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2009.

[75] Jacob Lurie. *Higher algebra*. Book manuscript, 2017.

[76] Jacob Lurie. Kerodon. `https://kerodon.net`, 2024.

[77] Saunders Mac Lane. *Categories for the working mathematician*, volume 5 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1998.

[78] Saunders MacLane. The Yoneda lemma. *Math. Japon.*, 47(1):156, 1998.

[79] Richard B. Melrose. Pseudodifferential operators, corners and singular limits. In *Proceedings of the International Congress of Mathematicians, Vol. I, II (Kyoto, 1990)*, pages 217–234. Math. Soc. Japan, Tokyo, 1991.

[80] Richard B. Melrose. Calculus of conormal distributions on manifolds with corners. *Internat. Math. Res. Notices*, (3):51–61, 1992.

[81] Richard B. Melrose. *The Atiyah-Patodi-Singer index theorem*, volume 4 of *Research Notes in Mathematics*. A K Peters, Ltd., Wellesley, MA, 1993.

[82] Haynes Miller. Leray in Oflag XVIIA: the origins of sheaf theory, sheaf cohomology, and spectral sequences. Number 84, suppl., pages 17–34. 2000. Jean Leray (1906–1998).

[83] Ieke Moerdijk and Gonzalo E. Reyes. *Models for smooth infinitesimal analysis*. Springer-Verlag, New York, 1991.

[84] J. C. Moore. Homotopie des complexes monoïdaux, I. *Séminaire Henri Cartan*, 7(2), 1954-1955. talk:18.

[85] Jürgen Moser. A rapidly convergent iteration method and non-linear partial differential equations. I. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (3)*, 20:265–315, 1966.

[86] James R. Munkres. *Topology*. Prentice Hall, Inc., Upper Saddle River, NJ, 2000. Second edition of [MR0464128].

[87] Jun-iti Nagata. On a necessary and sufficient condition of metrizability. *J. Inst. Polytech. Osaka City Univ. Ser. A*, 1:93–100, 1950.

[88] Chikara Nakayama and Arthur Ogus. Relative rounding in toric and logarithmic geometry. *Geom. Topol.*, 14(4):2189–2241, 2010.

[89] Behrang Noohi. Foundations of topological stacks I. *arXiv:math/0503247*, 2005.

[90] John Pardon. Representability in non-linear elliptic Fredholm analysis. *Submitted to the Proceedings of the International Congress of Basic Science.*, 2023.

[91] Brett Parker. Exploded manifolds. *Adv. Math.*, 229(6):3256–3319, 2012.

[92] The Univalent Foundations Program. *Homotopy type theory—univalent foundations of mathematics*. The Univalent Foundations Program, Princeton, NJ; Institute for Advanced Study (IAS), Princeton, NJ, 2013.

[93] Michael Reed and Barry Simon. *Methods of modern mathematical physics. I.* Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York, second edition, 1980. Functional analysis.

[94] C. L. Reedy. Homotopy theory of model categories. 1974.

[95] F. Rellich. Ein Satz über mittlere Konvergenz. *Nachr. Ges. Wiss. Göttingen, Math.-Phys. Kl.*, 1930:30–35, 1930.

[96] Emily Riehl and Dominic Verity. *Elements of ∞-category theory*, volume 194 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2022.

[97] J. Rosický. On homotopy varieties. *Adv. Math.*, 214(2):525–550, 2007.

[98] I. Satake. On a generalization of the notion of manifold. *Proc. Nat. Acad. Sci. U.S.A.*, 42:359–363, 1956.

[99] Jean-Pierre Serre. *Algèbre locale. Multiplicités*, volume 11 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin-New York, 1965. Cours au Collège de France, 1957–1958, rédigé par Pierre Gabriel, Seconde édition, 1965.

[100] Yu. Smirnov. A necessary and sufficient condition for metrizability of a topological space. *Doklady Akad. Nauk SSSR (N.S.)*, 77:197–200, 1951.

[101] David I. Spivak. *Quasi-smooth derived manifolds*. ProQuest LLC, Ann Arbor, MI, 2007. Thesis (Ph.D.)–University of California, Berkeley.

[102] David I. Spivak. Derived smooth manifolds. *Duke Math. J.*, 153(1):55–128, 2010.

[103] Pelle Steffens. Representability of Elliptic Moduli Problems in Derived C^∞-Geometry. *arxiv:2404.07931v2*, pages 1–82, 2024.

[104] William P. Thurston. *Three-dimensional geometry and topology. Vol. 1*, volume 35 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 1997. Edited by Silvio Levy.

[105] Bertrand Toën and Gabriele Vezzosi. Homotopical algebraic geometry. I. Topos theory. *Adv. Math.*, 193(2):257–372, 2005.

[106] Bertrand Toën and Gabriele Vezzosi. Homotopical algebraic geometry. II. Geometric stacks and applications. *Mem. Amer. Math. Soc.*, 193(902):x+224, 2008.

[107] Alan Weinstein. Linearization problems for Lie algebroids and Lie groupoids. volume 52, pages 93–102. 2000. Conference Moshé Flato 1999 (Dijon).

[108] Alan Weinstein. Linearization of regular proper groupoids. *J. Inst. Math. Jussieu*, 1(3):493–511, 2002.

[109] R. O. Wells, Jr. *Differential analysis on complex manifolds*. Prentice-Hall Series in Modern Analysis. Prentice-Hall, Inc., Englewood Cliffs, N.J., 1973.

[110] J. H. C. Whitehead. Combinatorial homotopy. I. *Bull. Amer. Math. Soc.*, 55:213–245, 1949.

[111] E. Zermelo. Beweis, daßjede Menge wohlgeordnet werden kann. *Math. Ann.*, 59(4):514–516, 1904.

[112] Nguyen Tien Zung. Proper groupoids and momentum maps: linearization, affinity, and convexity. *Ann. Sci. École Norm. Sup. (4)*, 39(5):841–869, 2006.

Index