Derived moduli spaces of pseudo-holomorphic curves

John Pardon

April 2024

Disclaimer: This a small excerpt of unfinished work-in-progress. Its release in such incomplete form is unconventional, but is made due to popular request. Comments and corrections are always welcome.

Contents

Contents **2**

Preface **4**

1 Category theory **9**
- 1.1 Categories . . . 10
- 1.2 2-categories . . . 29
- 1.3 Simplicial objects . . . 31
- 1.4 Simplicial homotopy theory . . . 38
- 1.5 ∞-categories . . . 43

2 Topology **66**
- 2.1 Properties of morphisms . . . 67
- 2.2 Sheaves . . . 72
- 2.3 Topological stacks . . . 75
- 2.4 Mapping stacks . . . 80
- 2.5 Stability . . . 83
- 2.6 Smooth manifolds . . . 85
- 2.7 Smooth stacks . . . 95
- 2.8 Log smooth manifolds . . . 103
- 2.9 Topological sites . . . 128
- 2.10 Derived smooth manifolds . . . 146
- 2.11 Derived smooth stacks . . . 161
- 2.12 Hybrid categories . . . 162

3 Analysis **165**
- 3.1 Function spaces . . . 166
- 3.2 Differential operators . . . 175
- 3.3 Elliptic operators . . . 178
- 3.4 Rough coefficients . . . 185
- 3.5 Ellipticity in cylindrical ends . . . 186
- 3.6 Elliptic boundary conditions . . . 190
- 3.7 Families of elliptic operators . . . 191

4 Riemann surfaces **195**

5 Pseudo-holomorphic maps **196**
5.1 Symplectic and almost complex structures . . . 197
5.2 Pseudo-holomorphicity . . . 199
5.3 Moduli stacks . . . 204
5.4 Tangent complexes . . . 208
5.5 Elliptic bootstrapping . . . 210
5.6 Regularity . . . 211
5.7 Derived Regularity . . . 218
5.8 A priori estimates . . . 224

Bibliography **227**

Index **234**

Preface

This text is intended as a logical starting point for the theory of moduli spaces of pseudo-holomorphic curves, as founded by Gromov [26] and subsequently developed by Floer, Hofer, Eliashberg, Fukaya, Kontsevich, Seidel, Abouzaid, and many others. A distinguishing feature of our treament of this subject is its generality: we formulate the main foundations of the theory in a way which is logically sufficient for applications. In fact, much of what we do is applicable well beyond the setting of pseudo-holomorphic curves, to any non-linear elliptic Fredholm problem. A summary of this work appeared in [72].

We assume minimal prerequisites and thus include a substantial amount of advanced graduate level background. Great effort is made to include all relevant foundational material, much in the form of exercises, so that our treatment may qualify as self-contained. An artifact of the resulting length is that the interesting material is spread a bit thin. The reader is therefore advised not to read the text linearly, but rather to seek out their specific topics of interest, and to refer to the other parts as they are cross-referenced.

The writing of this text has been something of an 'architecture problem'. Once the correct blueprint for the logical structure has been prescribed, the details fall into place with little resistance. While developing a blueprint with simple and clear logical structure (both globally and locally) is ultimately good for the subject, it also minimizes the apparent depth of the final result, especially in comparison to the amount of work leading up to it.

Despite their analytic nature, the main results of this work rely fundamentally on the framework of ∞-categories.

* *

To introduce the topics we cover in more detail, we begin with some background. A map $u : C \to X$ from a Riemann surface C to an almost complex manifold X (i.e. a manifold equipped with an endomorphism $J : TX \to TX$ squaring to $-\mathbf{1}$) is called *pseudo-holomorphic* when its differential $du : TC \to TX$ is $\mathbb{C}$-linear. The equation $J \circ du = du \circ j$ asserting $\mathbb{C}$-linearity of du is a non-linear elliptic Fredholm partial differential equation. Though we focus our attention on this particular equation and its variants, the vast majority of the framework we develop applies equally to any other non-linear elliptic Fredholm partial differential equation.

The primary objects of study in this text are the *moduli spaces* of solutions to the pseudo-holomorphic map equation. Our main goal is to *describe precisely what sort of mathematical objects these moduli spaces are.*

To explain the answer this question, it is helpful to begin in the linear setting. Fix a linear elliptic operator $L : E \to F$ acting on sections of vector bundles E and F over a compact manifold M. The 'space of solutions' to $Lu = 0$ is most immediately the finite-dimensional vector space $\ker L \in \mathsf{Vect}_{\mathbb{R}}$. However, for many purposes, it is better to instead consider the two-term complex $[L] = [C^\infty(M, E) \xrightarrow{L} C^\infty(M, F)]$ regarded as an object of the ∞-category $\mathsf{K}^{\geq 0}(\mathsf{Vect}_{\mathbb{R}})$ of complexes of vector spaces supported in non-negative cohomological degrees (in which it is isomorphic to $[\ker L \xrightarrow{0} \operatorname{coker} L]$). For example, if L_t is a family of operators parameterized by a smooth manifold T, then $\ker L_t$ is not generally a smooth vector on T, while $[L_t]$ is, locally on the parameter space T, equivalent to a two-term complex of smooth vector bundles. It is very reasonable to regard the two-term complex $[L]$ as equally deserving of the descriptor 'space of solutions to $Lu = 0$'. Indeed, while $\ker L$ is the fiber product $C^\infty(M, E) \times_{C^\infty(M,F)} 0$ in the category $\mathsf{Vect}_{\mathbb{R}}$, the two-term complex $[L]$ is the same fiber product taken in the ∞-category $\mathsf{K}^{\geq 0}(\mathsf{Vect}_{\mathbb{R}})$.

Now let us move to the non-linear setting. The moduli space $\mathcal{M}$ of solutions to a non-linear elliptic partial differential equation on a compact manifold may be identified locally with the zero set $f^{-1}(0)$ of a smooth map $f : \mathbb{R}^n \to \mathbb{R}^m$. This is a classical fact going back to Kuranishi [49] and Atiyah–Hitchin-Singer [6], and such charts are often called *Kuranishi charts*. It is desirable to regard $\mathcal{M}$ not just as a topological space, but to also remember its Kuranishi charts and the relations among them; this generalizes the passage from $\ker L$ to $[L]$ in the linear setting. One very direct way to do this is to simply equip $\mathcal{M}$ with an *atlas* of Kuranishi charts, as first appeared in work of Fukaya–Ono [24] and developed further by Fukaya–Oh–Ohta–Ono [22, 23] and others. It is natural to ask whether $\mathcal{M}$ is naturally an object of a non-linear analogue of the ∞-category $\mathsf{K}^{\geq 0}(\mathsf{Vect}_{\mathbb{R}})$.

	linear	non-linear
category	$\mathsf{Vect}_{\mathbb{R}}$	smooth manifolds Sm
∞-category	$\mathsf{K}^{\geq 0}(\mathsf{Vect}_{\mathbb{R}})$	derived smooth manifolds Der

The relevant non-linear analogue of the ∞-category $\mathsf{K}^{\geq 0}(\mathsf{Vect}_{\mathbb{R}})$ is the ∞-category of *derived smooth manifolds*, which we denote by Der and study in (2.10).

The ∞-category of derived smooth manifolds was introduced by Spivak [84], and it may also be called the ∞-category of locally finitely presented C^∞-schemes. It can be regarded as a special case of the rather general theory of *derived geometry* introduced by Lurie [56] and Toën–Vezzosi [87, 88]. We will adopt the perspective that the ∞-category of derived smooth manifolds Der obtained from the category Sm by *formally adjoining finite ∞-limits modulo preserving finite transverse ∞-limits*. It can be shown that a derived fiber product (i.e. fiber product in Der) of smooth manifolds *remembers its fiber product presentation locally, modulo transverse fiber products of smooth manifolds*. The connection between multiplicities of non-transverse intersections and derived geometry was suggested long ago by the Serre intersection formula [81, V.C.1], and this has been a key motivation for the development of derived geometry since its inception.

We may now state a refined version of our main goal: we seek to *construct moduli spaces of pseudo-holomorphic curves as derived smooth manifolds*.

* *

To explain the construction of moduli spaces of pseudo-holomorphic curves as derived smooth manifolds, we must begin by recalling Grothendieck's technique of constructing moduli spaces by representing functors.

To specify an object $\mathcal{M}$ of a category C, it is equivalent to specify the functor $\operatorname{Hom}_{\mathsf{C}}(-, \mathcal{M}) : \mathsf{C}^{\mathsf{op}} \to \mathsf{Set}$ associating to every object $Z \in \mathsf{C}$ the set of maps $Z \to \mathcal{M}$. More precisely, given a functor $F : \mathsf{C}^{\mathsf{op}} \to \mathsf{Set}$, an object $\mathcal{M} \in \mathsf{C}$ together with an element $\xi \in F(\mathcal{M})$ is said to *represent* F when the map $\operatorname{Hom}_{\mathsf{C}}(Z, \mathcal{M}) \to F(Z)$ given by $f \mapsto f^*\xi$ is an isomorphism for every $Z \in \mathsf{C}$ (that is, pulling back ξ defines an isomorphism of functors $\operatorname{Hom}_{\mathsf{C}}(-, \mathcal{M}) \to F(-)$). When such a representing pair $(\mathcal{M}, \xi)$ exists, we say that F is *representable*. It is straightforward to check that any two representing pairs $(\mathcal{M}, \xi)$ and $(\mathcal{M}', \xi')$ are uniquely isomorphic. The property of a pair $(\mathcal{M}, \xi)$ representing a particular functor is often also called satisfying a particular *universal property*. Bored experts may at this point take note that so far this discussion does not require any version of the Yoneda Lemma (nor is equivalent to it in any way).

The vague idea that a moduli space $\mathcal{M}$ 'parameterizes all objects of some type $\mathcal{O}$' naturally lends itself to a precise formulation in terms of representable functors. Indeed, consider the *moduli functor* F sending a space Z to the set of all families of objects of type $\mathcal{O}$ over Z (the terms 'space' and 'family' are placeholders for whatever the relevant sort of mathematical items may be). To represent F now means to find a space $\mathcal{M}$ and a family $\mathcal{U} \to \mathcal{M}$ of objects of type $\mathcal{O}$ which is 'universal' in the sense that every family of objects of type $\mathcal{O}$ over a space Z is the pullback of $\mathcal{U} \to \mathcal{M}$ under a unique map $Z \to \mathcal{M}$. As remarked above, such a pair $(\mathcal{M}, \mathcal{U} \to \mathcal{M})$ is unique up to unique isomorphism if it exists, and in this case $\mathcal{M}$ is called the 'moduli space' and $\mathcal{U} \to \mathcal{M}$ the 'universal family'. (One important caveat about this discussion is that it often needs a higher categorical context, that is we should replace the category of sets Set with the 2-category of groupoids Grpd or the ∞-category of spaces Spc.) The formalism of moduli functors may seem trivial and tautological at first glance, and it is perhaps for this reason that moduli spaces were studied for quite some time before the introduction of moduli functors.

Despite the apparent triviality of the formalism of moduli functors, it turns out to be extraordinarily useful from a technical standpoint, for a few different reasons.

First of all, the moduli functor $\operatorname{Hom}(-, \mathcal{M})$ is usually much easier to describe than the moduli space $\mathcal{M}$ itself. Indeed, the moduli functor simply consists of sets and maps between them, while the moduli space is an object of some category (e.g. smooth manifolds) which may be rather complicated to describe directly (e.g. a set, a topology on that set, and a collection of charts with smooth transition functions). The notion of a 'family of objects of type $\mathcal{O}$ parameterized by Z' is usually quite transparent, while turning 'the set of all objects of type $\mathcal{O}$' into an object of some category (e.g. describing a topology on this set) is virtually guaranteed to be quite a bit more complicated. For this reason, the moduli functor is often unquestionably canonical, while the same cannot be said for (an explicit construction of) the moduli space. Crucially, representability is a *property* (rather than *extra structure*), so whatever arbitrary choices may go into proving that a functor is representable necessarily do not affect the resulting representing object, which is automatically (and trivially) identified with the result of any other construction of a representing object.

Second, if C is a category of 'geometric objects', then one can regard the category of functors ('presheaves') $\mathsf{P}(\mathsf{C}) = \mathsf{Fun}(\mathsf{C}^{\mathsf{op}}, \mathsf{Set})$ itself as a category of geometric objects containing C (it is here that we need the Yoneda Lemma, which in particular says that $\mathsf{C} \subseteq \mathsf{P}(\mathsf{C})$). This makes it possible to reason geometrically with moduli functors, similarly to how we might reason with moduli spaces, without proving (or perhaps before we prove) they are representable. In fact, many moduli functors simply *are not* representable by objects of our 'original' geometric category C, but instead satisfy a weaker condition which nevertheless makes them reasonable geometric objects (e.g. the moduli functor of closed Riemann surfaces is not a smooth manifold, rather a smooth orbifold). When regarding a (moduli) functor (object of $\mathsf{P}(\mathsf{C})$) as a geometric object in this way, we may also call it a *(moduli) stack*. Crucially, representability is a *local property* of a stack.

The fact that representability is a local property is of decisive importance, particularly so for our application to moduli spaces of pseudo-holomorphic curves. Let us explain why. As we have already noted above, the local structure of moduli spaces $\mathcal{M}$ of pseudo-holomorphic curves (or, more generally, solutions to any non-linear elliptic Fredholm problem) has been well understood since [49, 6]: we have $\mathcal{M} = f^{-1}(0)$ (locally) for smooth maps $f : \mathbb{R}^n \to \mathbb{R}^m$. However, such local charts and the data relating them are non-unique (this is inevitable given the higher homotopical nature of the ∞-category of derived smooth manifolds), and this is the root cause of the worst technical complications in all prior work on the subject. The fact that representability is a local property gives a decisive solution to this problem: concretely, it tells us that the data relating local charts exists and is unique for formal reasons (provided we construct these local charts to represent a canonical moduli functor), and so there is no need to construct it explicitly! The use of moduli functors thus resolves one of the main difficulties in the subject.

* *

We can now state our main results. In a word, we define moduli functors associated to a general class of pseudo-holomorphic moduli problems, and we show they are representable by derived smooth manifolds. This answers a conjecture of Joyce [40, §5.3].

0.0.1 Derived Regularity Theorem. *Let $W \to C \to B$ be a pseudo-holomorphic section problem over a derived smooth stack B. The morphism of derived smooth stacks $\underline{\mathrm{Hol}}_B(C, W) \to B$ is representable, and the comparison map $|\underline{\mathrm{Hol}}_B(C, W)|_! \to \underline{\mathrm{Hol}}_{|B|_!}(C_{|B|_!}, W_{|B|_!})$ is an isomorphism of topological stacks.*

The proof of this result is given in (5.7). An independent proof has been announced by Steffens [85]. We should emphasize that this is not a theorem about pseudo-holomorphic curves, rather it is a theorem about non-linear elliptic Fredholm problems. The proof, properly understood, applies in significantly greater generality than what we have stated here.

Acknowledgements

The presentation in this text has benefitted in innumerable ways from many conversations over many years with many different people. With advance apologies for any omissions, let me acknowledge in particular communication with Mohammed Abouzaid, Shaoyun Bai, Dustin Clausen, Yasha Eliashberg, Tobias Ekholm, Helmut Hofer, Daniel Litt, Sergey Nersisyan, Stefan Schwede, Paul Seidel, Mohan Swaminathan. I also thank my collaborators Sheel Ganatra and Vivek Shende from whom I learned much about how to write mathematics. A conversation with Yongbin Ruan provided meaningful motivation for writing these notes.

Many references were taken from the nLab `https://ncatlab.org/`.

Any errors or omissions remain the responsibility of the author.

Chapter 1

Category theory

In any mathematical discussion, it is helpful to have available multiple different perspectives on the same situation, as it often happens that something which is opaque from one perspective turns out to be clear from another. Category theory provides such an additional perspective in virtually any mathematical setting. It has an uncanny ability to reveal large parts of mathematical arguments to be 'purely formal', thus clarifying where the true content really lies and eliminating redundant arguments. It is easy to reach the mistaken conclusion that this means all of category theory is trivial! On the contrary, its utility in crafting efficient arguments makes it indispensable in many settings.

1.1 Categories

The reader may refer to Leinster [53] for a first introduction to category theory and to MacLane [60] for a comprehensive treatment.

⋆ **1.1.1 Definition.** A *category* C consists of the following data:

(1.1.1.1) For every pair of objects $X, Y \in \mathsf{C}$, a set $\operatorname{Hom}(X, Y)$, whose elements are called the *morphisms* $X \to Y$ in C.

(1.1.1.2) A set C, whose elements are called the *objects* of C.

(1.1.1.3) For every triple of objects $X, Y, Z \in \mathsf{C}$, a map

$$\operatorname{Hom}(X, Y) \times \operatorname{Hom}(Y, Z) \to \operatorname{Hom}(X, Z)$$

called *composition*, such that for every composable triple of morphisms a, b, c, the two compositions $(ab)c$ and $a(bc)$ are equal (composition is *associative*).

(1.1.1.4) For every object $X \in \mathsf{C}$, an element $\mathbf{1}_X \in \operatorname{Hom}(X, X)$ called the *identity morphism* such that composition with $\mathbf{1}_X$ defines the identity map $\operatorname{Hom}(X, Y) \to \operatorname{Hom}(X, Y)$ and $\operatorname{Hom}(Z, X) \to \operatorname{Hom}(Z, X)$ for all $Y, Z \in \mathsf{C}$.

The set of morphisms $\operatorname{Hom}(X, Y)$ may also be denoted $\operatorname{Hom}_{\mathsf{C}}(X, Y)$ or $\mathsf{C}(X, Y)$.

⋆ **1.1.2 Example** (Categories of sets, groups, and topological spaces)**.** The following are categories:

(1.1.2.1) Set, the category of sets: an object is a set, and a morphism is a map of sets.

(1.1.2.2) Grp, the category of groups: an object is a group, and a morphism is a group homomorphism.

(1.1.2.3) Top, the category of topological spaces: an object is a topological space, and a morphism is a continuous map.

Except not quite: a category needs a *set* of objects (1.1.1.2), and there is no ‘set of all sets’, ‘set of all groups’, or ‘set of all topological spaces’. So, we should really say that we get a category of sets, groups, or topological spaces by choosing a set and, for each element of that set, a set, group, or topological space. The notation Set, Grp, Top is thus somewhat abusive, since it hides these choices. This is, in fact, an advantage, as we will see shortly that such choices are to a large extent irrelevant (see the ‘principle of equivalence’ (1.1.32)(1.1.33)(1.1.34) below).

1.1.3 Example. To each poset S, we can associate a category whose objects are the elements of S and in which

$$\operatorname{Hom}(s, t) = \begin{cases} * & s \leq t \\ \varnothing & \text{else} \end{cases} \tag{1.1.3.1}$$

1.1.4 Exercise (Identity morphisms are a property)**.** Show that in a category, the identity morphisms (1.1.1.4) are uniquely determined by the rest of the data (1.1.1.2)–(1.1.1.3) provided they exist.

1.1.5 Exercise (Isomorphisms and inverses)**.** A morphism $X \to Y$ in a category is called an *isomorphism* iff there exists a morphism $Y \to X$ such that the compositions $X \to Y \to X$ and $Y \to X \to Y$ are the identity morphisms $\mathbf{1}_X$ and $\mathbf{1}_Y$. Show that a given morphism $X \to Y$ as at most one such 'inverse' morphism $Y \to X$.

1.1.6 Example (Cardinal)**.** A *cardinal* is an isomorphism class of objects in the category Set.

1.1.7 Example (Groups up to finite index)**.** Consider the category whose objects are groups and whose morphisms $G \to H$ are pairs (G', f) where $G' \leq G$ is a finite index subgroup and $f : G' \to H$ is a group homomorphism, modulo the equivalence relation that $(G', f) \sim (G'', g)$ iff there exists a finite index subgroup $G''' \leq G' \cap G''$ such that $f|_{G'''} = g|_{G'''}$. In this category, all finite groups are isomorphic to the trivial group.

1.1.8 Exercise (Germs of topological spaces)**.** The category of germs of topological spaces is defined as follows. Its objects are pairs (X, x) where X is a topological space and $x \in X$ is a point. Its morphisms $(X, x) \to (Y, y)$ are pairs (U, f) where $U \subseteq X$ is an open set containing x and $f : U \to Y$ is a continuous map with $f(x) = y$, modulo the equivalence relation that $(U, f) \sim (U', f')$ iff there exists an open set $A \subseteq U \cap U'$ containing x for which $f|_A = f'|_A$. The composition of $(U, f) : (X, x) \to (Y, y)$ and $(V, g) : (Y, y) \to (Z, z)$ is given by $(f^{-1}(V), g \circ f|_{f^{-1}(V)})$. Show that a morphism $(X, x) \to (Y, y)$ in this category is an isomorphism iff can be realized as a pair (U, f) for which f is an open embedding.

1.1.9 Definition (Groupoid)**.** A *groupoid* is a category in which every morphism is an isomorphism.

1.1.10 Example (Fundamental groupoid)**.** Let X be a topological space. Its *fundamental groupoid* $\pi_1(X)$ is the category whose objects are points x of X and whose morphisms $x \to y$ are paths from x to y modulo homotopy rel endpoints, with composition being given by concatenation (which is indeed associative on homotopy classes). The automorphism group of a point $x \in X$ in $\pi_1(X)$ is the fundamental group $\pi_1(X, x)$ of X based at x.

1.1.11 Example (Core)**.** For any category C, we can consider the category $\mathsf{C}_\simeq$ with the same objects and whose morphisms are the isomorphisms in C; thus $\mathsf{C}_\simeq$ is a groupoid. For example, $\mathsf{Set}_\simeq$ consists of sets and bijections of sets.

1.1.12 Definition (Full subcategory)**.** For a category C, the *full subcategory* spanned by a set of objects of C is the category whose objects are this set and whose morphisms are the same as in C.

1.1.13 Example. The category of abelian groups Ab is a full subcategory of the category of groups Grp.

1.1.14 Definition (Opposite category)**.** For a category C, its *opposite* C^{op} has the same objects, but morphisms are reversed: $\mathsf{C}^{\mathsf{op}}(X, Y) = \mathsf{C}(Y, X)$.

Every notion for categories has a *dual* notion obtained by applying the original notion to the opposite; this is usually indicated linguistically with the prefix 'co-'.

⋆ **1.1.15 Definition** (Functor)**.** A *functor* $F : \mathsf{C} \to \mathsf{D}$ between categories consists of the following data:

(1.1.15.1) For every object $X \in \mathsf{C}$, an object $F(X) \in \mathsf{D}$.

(1.1.15.2) For every pair of objects $X, Y \in \mathsf{C}$, a map $F : \mathrm{Hom}(X, Y) \to \mathrm{Hom}(F(X), F(Y))$ such that $F(\mathbf{1}_X) = \mathbf{1}_{F(X)}$ and such that composing and applying F in either order define the same map

$$\mathrm{Hom}(X, Y) \times \mathrm{Hom}(Y, Z) \to \mathrm{Hom}(F(X), F(Z)).$$

1.1.16 Example (Free and forgetful functors)**.** Associating to a group or topological space its underlying set defines 'forgetful' functors $\mathsf{Grp} \to \mathsf{Set}$ and $\mathsf{Top} \to \mathsf{Set}$. Associating to a set the free group on that set or the discrete topology on that set defines functors $\mathsf{Set} \to \mathsf{Grp}$ and $\mathsf{Set} \to \mathsf{Top}$.

As in (1.1.2), there is a caveat. To define categories Set and Grp, we should first choose a set of sets and a set of groups. Then to define, say, the 'free group' functor $\mathsf{Set} \to \mathsf{Grp}$, we should choose, for each $S \in \mathsf{Set}$, a group $G \in \mathsf{Grp}$ and an identification of G with the free group generated by S. If such a group $G \in \mathsf{Grp}$ exists for every $S \in \mathsf{Set}$, we can then define the desired functor $\mathsf{Set} \to \mathsf{Grp}$. Fortunately, this sort of discussion can (and should) be systematically avoided (see the 'principle of equivalence' (1.1.32)(1.1.33)(1.1.34) below).

1.1.17 Example (Homology and homotopy groups)**.** Homology groups are a sequence of functors $H_n : \mathsf{Top} \to \mathsf{Ab}$ from topological spaces to abelian groups for integers $n \geq 0$. The homotopy groups π_n are functors $\mathsf{Top}_* \to \mathsf{Ab}$ for $n \geq 2$ and $\pi_1 : \mathsf{Top}_* \to \mathsf{Grp}$ and $\pi_0 : \mathsf{Top}_* \to \mathsf{Set}_*$, where Top_* denotes the category of pointed topological spaces and Set_* that of pointed sets (and, in both cases, pointed maps). The functors H_n and π_n are homotopy invariant, meaning they factor through the functors $\mathsf{Top} \to h\mathsf{Top}$ and $\mathsf{Top}_* \to h\mathsf{Top}_*$, where the h indicates morphisms are now homotopy classes of (pointed) maps.

1.1.18 Example (Functors on fundamental groupoids)**.** A map of topological spaces $X \to Y$ induces a functor on fundamental groupoids $\pi_1(X) \to \pi_1(Y)$. A functor $\pi_1(X) \to \mathsf{C}$ is known as a *local system* on X valued in C.

1.1.19 Example (Hom functor)**.** Sending $(X, Y) \mapsto \mathrm{Hom}(X, Y)$ is a functor $\mathsf{C}^{\mathsf{op}} \times \mathsf{C} \to \mathsf{Set}$ for any category C.

1.1.20 Definition (Fully faithful)**.** A functor F is called *fully faithful* when its constituent maps $\mathrm{Hom}(X, Y) \to \mathrm{Hom}(F(X), F(Y))$ are bijections of sets. A fully faithful functor is also called an *embedding* or an *inclusion*, and full faithfulness is often indicated with the hooked arrow $\hookrightarrow$.

1.1.21 Definition (Essential image)**.** The *essential image* of a functor $F : \mathsf{C} \to \mathsf{D}$ is the full subcategory $\mathsf{im}(F) \subseteq \mathsf{D}$ spanned by those objects $Y \in \mathsf{D}$ which are isomorphic to $F(X)$ for some $X \in \mathsf{C}$. When every object of D lies in $\mathsf{im}(F)$, we say that F is *essentially surjective.*

⋆ **1.1.22 Definition** (Natural transformation)**.** A *natural transformation* $F \to G$ between functors $F, G : \mathsf{C} \to \mathsf{D}$ consists of:

(1.1.22.1) For every object $X \in \mathsf{C}$, a morphism $F(X) \to G(X)$ such that for every morphism $X \to Y$, the two compositions $F(X) \to G(X) \to G(Y)$ and $F(X) \to F(Y) \to G(Y)$ agree.

Given categories C and D, there is a category $\mathsf{Fun}(\mathsf{C}, \mathsf{D})$ whose objects are functors $\mathsf{C} \to \mathsf{D}$ and whose morphisms are natural transformations.

1.1.23 Example (Homology of local systems)**.** Local systems on X valued in C form a category $\mathsf{Fun}(\pi_1(X), \mathsf{C})$. Homology with twisted coefficients is a sequence of functors $H_n : \mathsf{Fun}(\pi_1(X), \mathsf{Ab}) \to \mathsf{Ab}$ for $n \geq 0$.

1.1.24 Example. For every group G, there is a groupoid BG with a single object whose automorphism group is G. A functor $BG \to BH$ is a group homomorphism $\phi : G \to H$. A natural isomorphism of (functors associated to) group homomorphisms $\phi \to \phi'$ is an element $h \in H$ conjugating ϕ to ϕ', namely satisfying $h\phi(g) = \phi'(g)h$.

1.1.25 Example. If D is a groupoid, then the functor category $\mathsf{Fun}(\mathsf{C}, \mathsf{D})$ is a groupoid.

A naive notion of 'isomorphism' between categories is that of a compatible bijection between objects and morphisms. The following weaker notion turns out to be much more meaningful (see the 'principle of equivalence' (1.1.32) below):

⋆ **1.1.26 Definition** (Equivalence of categories)**.** A functor $\mathsf{C} \to \mathsf{D}$ is called an *equivalence* iff there exists a functor $\mathsf{D} \to \mathsf{C}$ such that the compositions $\mathsf{C} \to \mathsf{D} \to \mathsf{C}$ and $\mathsf{D} \to \mathsf{C} \to \mathsf{D}$ are naturally isomorphic (in $\mathsf{Fun}(\mathsf{C}, \mathsf{C})$ and $\mathsf{Fun}(\mathsf{D}, \mathsf{D})$, respectively) to the identity functors $\mathbf{1}_{\mathsf{C}}$ and $\mathbf{1}_{\mathsf{D}}$.

An equivalence of categories often expresses the fact that two different definitions of some type of mathematical object (vector spaces, smooth manifolds, etc.) are equivalent.

1.1.27 Exercise. Show that a functor is an equivalence iff it is fully faithful and essentially surjective.

1.1.28 Example. Let C be a category with set of objects S. Given any map of sets $S' \to S$, we can form a new category C' with set of objects S' and with a fully faithful functor $\mathsf{C}' \to \mathsf{C}$ acting as $S' \to S$ on objects. If $S' \to S$ is surjective, then $\mathsf{C}' \to \mathsf{C}$ is an equivalence of categories.

1.1.29 Example. Fix a field k, and let Vect_k denote the category of vector spaces and linear maps over k. Now consider a category in which an object is a vector space over k with a chosen basis. There are two reasonable notions of a morphism between two such objects:

(1.1.29.1) A linear map over k.

(1.1.29.2) A linear map over k sending basis elements to basis elements.

In the first case, the resulting category is equivalent to Vect_k, via the functor forgetting the basis. In the second case, the resulting category is equivalent to Set, via the functor remembering just the basis. This illustrates how the information in a category is carried by the morphisms, not the objects.

1.1.30 Example. Given a set S, we can regard S as a groupoid in which $\operatorname{Hom}(x, x) = *$ and $\operatorname{Hom}(x, y) = \varnothing$ for $x \neq y$. A groupoid is called *discrete* when it is equivalent to (the groupoid associated to) a set. A groupoid is discrete iff the automorphism group of every object is trivial.

1.1.31 Exercise (Posets as categories)**.** Show that a category is equivalent to the category associated to a poset (1.1.3) iff for every ordered pair of objects x, y, there is at most one morphism $x \to y$. Show that for any two such categories C and D, the groupoid $\mathsf{Fun}(\mathsf{C}, \mathsf{D})_{\simeq}$ is discrete. Let Po' denote the category whose objects are categories in which there is at most one morphism for each ordered pair of objects, and whose morphisms are functors up to natural isomorphism (which, in view of the previous sentence, is unique if it exists). Let Po denote the category of posets and weakly order preserving maps ($s \leq t$ implies $f(s) \leq f(t)$). Show that the natural functor $\mathsf{Po} \to \mathsf{Po}'$ is an equivalence of categories. This equivalence justifies using the term 'poset' for an object of Po'.

The following is a fundamental principle of category theory.

$\star$ **1.1.32 Remark** (Principle of equivalence)**.** Equivalence of categories (1.1.26) plays the role that isomorphism plays for most other mathematical objects one is used to dealing with. The reason for this difference is that most common mathematical objects (sets, groups, rings, modules, fields, vector spaces, topological spaces, manifolds, sheaves, schemes, cohomology theories, functors, etc.) form categories, whereas categories form a *2-category* (see (1.1.35)).

The *principle of equivalence* declares a statement involving categories to be 'meaningful' iff it is invariant under equivalence. For example, the cardinality of the set of isomorphism classes of objects in a category is invariant under equivalence, hence is a meaningful (albeit very crude) invariant to attach to a category. The cardinality of the set of objects in a category is not invariant under equivalence, hence is not a meaningful invariant of a category. A somewhat more subtle observation is that the principle of equivalence allows us to identify the notions of 'full subcategory' and 'fully faithful functor'.

Intuitively speaking, a statement about categories is invariant under equivalence provided it makes no reference to the notion of 'equality' of objects (and instead says things about morphisms between objects). Virtually any statement about categories which is invariant under equivalence is obviously so, to the extent that there is usually no need to state it explicitly. In particular, a construction involving categories will be invariant under equivalence whenever it is appropriately acted on by functors (and natural isomorphisms between them) of the categories in question (i.e. it should be 2-functorial on the 2-category of categories Cat (1.1.35)). It follows, for example, that basic constructions such as formation of functor categories respect the principle of equivalence by sending equivalences to equivalences.

The importance of the principle of equivalence stems from the fact that most 'categories' of interest, such as Set, Grp, Top (1.1.2), are, at best, only well defined up to (canonical) equivalence (1.1.33)–(1.1.34), and so specializing a statement about categories to one of these is only meaningful when that statement is invariant under equivalence.

To develop the foundations of category theory in standard mathematical language does require some (minimal) breaking of the principle of equivalence. Indeed, the very definition of a category (1.1.1) involves a *set of objects*, in which there is necessarily a notion of equality. Proofs of statements about categories typically involve quantifying or inducting over sets of objects. This is unavoidable (though see Voevodsky [74]) but benign.

⋆ **1.1.33 Remark** (Small vs large categories)**.** A category in the sense of (1.1.1) is often called a *small category*, the adjective 'small' indicating that there is a *set* of objects and a *set* of morphisms between any pair of objects. As we have seen in (1.1.2), many, or perhaps most, 'categories' of interest are not small. There is thus a certain amount of dissonance between the foundations of the theory of categories in the sense of (1.1.1) and the scope of the intended applications of this theory.

A *large category* has a 'notion of object', a 'notion of morphism between objects', a 'notion of equality of between morphisms', and a 'notion of associative composition of morphisms'; one similarly has a notion of functor between large categories. We do not regard this sentence as a precise mathematical definition. Rather, the notion of a large category is a meta-mathematical framework into which typical categories of interest such as Set, Grp, Top (1.1.2) fall.

A large category is called *essentially small* when it is equivalent to a small category. Equivalently, a large category is essentially small when there is a set of objects representing every isomorphism class and the collection of morphisms between any pair of objects is a set. For example, the large categories Set, Grp, Top are not essentially small, although their full subcategories Set_κ, Grp_κ, Top_κ of sets, abelian groups, and topological spaces of cardinality less than a given cardinal κ are essentially small.

Given an essentially small category C, a *small model* of C is a small category C_0 together with an equivalence $\mathsf{C}_0 \to \mathsf{C}$. Small models always exist (by definition of essentially small), and they are moreover unique up to canonical equivalence (1.1.34). It follows that any result for small categories which adheres to the principle of equivalence remains valid for essentially small categories.

Applying category theory to large categories which are not essentially small requires either realizing that the underlying arguments go through without any smallness assumptions (that is to say, they are meta-mathematical) or working with appropriately chosen essentially small subcategories. Set-theoretic complications rarely arise as long as one avoids arguments which are obviously 'circular'.

1.1.34 Remark (Uniqueness of small models)**.** We explain uniqueness of small models in the case of Top_κ, but the reasoning applies to any essentially small category.

(1.1.34.1) Given a set S along with, for every $s \in S$, a topological space X_s, such that every topological space of cardinality $< \kappa$ is isomorphic to some X_s, we obtain a small category Top_κ^S (whose set of objects is S and in which a morphism $s \to s'$ is a continuous map $X_s \to X_{s'}$). Such sets S exist: for example, fix a set U of cardinality $\geq \kappa$, and let S consist of all subsets of U of cardinality $< \kappa$ equipped with a topology.

(1.1.34.2) Given any two S and S' as above, a choice of function $f : S \to S'$ along with isomorphisms $X_s \xrightarrow{\sim} X_{f(s)}$ defines an equivalence of categories $\mathsf{Top}_\kappa^S \to \mathsf{Top}_\kappa^{S'}$; this recipe is moreover compatible with composition of functors. Such functions f exist by the axiom of choice.

(1.1.34.3) Given any two $f, g : S \to S'$ as above, there is a canonical natural isomorphism between the two induced functors $\mathsf{Top}_\kappa^S \to \mathsf{Top}_\kappa^{S'}$, namely that defined by the isomorphisms $X_{f(s)} \xleftarrow{\sim} X_s \xrightarrow{\sim} X_{g(s)}$. This construction is also compatible with composition.

1.1.35 Example (Categories of categories)**.** There are at least three diffferent answers to the question of what is the *category of categories*, related by functors

$$\mathsf{Cat}_{\mathsf{strict}} \to \mathsf{Cat} \to \mathsf{hCat}. \tag{1.1.35.1}$$

At one extreme is the category $\mathsf{Cat}_{\mathsf{strict}}$, whose objects are (small) categories and whose morphisms are functors. The category $\mathsf{Cat}_{\mathsf{strict}}$ does not see natural transformations between functors. Because of this, an equivalence of categories need not be an isomorphism in $\mathsf{Cat}_{\mathsf{strict}}$ (rather isomorphism in $\mathsf{Cat}_{\mathsf{strict}}$ is the notion of a 'naive isomorphism of categories' mentioned above (1.1.26)). Regarding categories as objects of $\mathsf{Cat}_{\mathsf{strict}}$ thus violates the principle of equivalence, which means $\mathsf{Cat}_{\mathsf{strict}}$ is mostly useless for doing any category theory.

At another extreme is the category hCat whose objects are (small) categories and whose morphisms are natural isomorphism classes of functors. It is promising to note that a functor is an isomorphism in hCat iff it is an equivalence of categories. Unfortunately, it turns out that hCat is a poor input to most other categorical constructions, notably limits and colimits.

The objects of Cat are again (small) categories, and $\mathrm{Hom}_{\mathsf{Cat}}(\mathsf{C}, \mathsf{D}) = \mathsf{Fun}(\mathsf{C}, \mathsf{D})_{\simeq}$. As $\mathsf{Fun}(\mathsf{C}, \mathsf{D})_{\simeq}$ is not a set but rather a groupoid, Cat is not a category but rather a *2-category* as we will explain in more detail (1.2.4) once we have in hand the language of 2-categories. It is this 2-category Cat which is really the true category of categories.

* *

⋆ **1.1.36 Definition** (Monomorphism and epimorphism)**.** A morphism $X \to Y$ is called a *monomorphism* (or *monic*) iff the induced map $\mathrm{Hom}(Z, X) \to \mathrm{Hom}(Z, Y)$ is injective for all Z. Dually, $X \to Y$ is an *epimorphism* (or *epic*) when $\mathrm{Hom}(Y, Z) \to \mathrm{Hom}(X, Z)$ is injective for all Z.

1.1.37 Exercise. Show that a morphism of sets is monic iff it is injective, and is epic iff it is surjective. Show that a morphism of commutative rings is monic iff it is injective. Show that surjections and localizations of commutative rings are epimorphisms.

1.1.38 Exercise. Given a pair of morphisms $X \xrightarrow{f} Y \xrightarrow{g} X$ composing to the identity $\mathbf{1}_X$, we say that the morphism g is a *retraction* of f and that f is a *section* of g; we also say that the object X is a *retract* of Y. A morphism admitting a retraction (resp. section) is called a *split monomorphism* (resp. *split epimorphism*); these notions are dual. Show that a split monomorphism (resp. split epimorphism) is a monomorphism (resp. epimorphism). Show that a morphism which is both a split monomorphism and a split epimorphism is an isomorphism.

* *

1.1.39 Definition (Property of objects)**.** A *property* of objects in a category C is a set $\mathcal{P}$ of isomorphism classes in C. An object 'has $\mathcal{P}$' or 'is $\mathcal{P}$' when its isomorphism class is in $\mathcal{P}$.

1.1.40 Definition (Arrow category)**.** The morphisms in a category C are themselves the objects of a category, namely the *arrow category* $(\mathsf{C} \downarrow \mathsf{C}) = \mathsf{Fun}(\Delta^1, \mathsf{C})$ where $\Delta^1 = (\bullet \to \bullet)$ denotes the category with two objects and a single non-identity morphism from one to the other. A morphism is said to be a retract of another when it is true for the corresponding objects of $\mathsf{Fun}(\Delta^1, \mathsf{C})$. A *property of morphisms* in C is a property of objects in $\mathsf{Fun}(\Delta^1, \mathsf{C})$.

1.1.41 Example. In any category, properties of morphisms include being an isomorphism, being a monomorphism, or being an epimorphism.

1.1.42 Example. Consider the category whose objects are finite subsets $S \subseteq \mathbb{Z}$ and whose morphisms are arbitrary maps $S \to T$. The property of a map $f : S \to T$ satisfying $f(s) \leq f(s')$ for $s \leq s'$ not a property of morphisms, because it is not invariant under isomorphisms in the category. This category is equivalent to the category of finite sets, a context in which asking for a morphism to be weakly increasing evidently has no meaning.

1.1.43 Definition (Property closed under composition)**.** A property of morphisms $\mathcal{P}$ is said to be *closed under composition* iff every isomorphism has $\mathcal{P}$ and the composition of any two $\mathcal{P}$-morphisms has $\mathcal{P}$.

1.1.44 Example. Isomorphisms, monomorphisms, and epimorphisms (in any category) are closed under composition.

1.1.45 Definition (2-out-of-3 property)**.** A property of morphisms $\mathcal{P}$ is said to have the *2-out-of-3 property* when any two out of f, g, $g \circ f$ having $\mathcal{P}$ implies that the third does too.

1.1.46 Example. In the category of abelian groups, the property of having finite kernel and finite cokernel satisfies the 2-out-of-3 property.

1.1.47 Definition (Preservation, reflection, and lifting of properties)**.** Let $\mathcal{P}$ be a property of objects in categories C and D, and let $F : \mathsf{C} \to \mathsf{D}$ be a functor. We say F *preserves* $\mathcal{P}$-objects when $c \in \mathcal{P}$ implies $F(c) \in \mathcal{P}$ for every morphism $c \in \mathsf{C}$. We say F *reflects* $\mathcal{P}$-objects when $F(c) \in \mathcal{P}$ implies $c \in \mathcal{P}$ for every object $c \in \mathsf{C}$. We say F *lifts* $\mathcal{P}$-objects when every $d \in \mathcal{P}$ is isomorphic to $F(c)$ for some $c \in \mathcal{P}$.

1.1.48 Example. The forgetful functor $\mathsf{Grp} \to \mathsf{Set}$ reflects isomorphisms (a group homomorphism is an isomorphism iff it is a bijection of sets). The forgetful functor $\mathsf{Top} \to \mathsf{Set}$ does not reflect isomorphisms (a continuous bijection of topological spaces need not have a continuous inverse).

⋆ **1.1.49 Definition** (Final and initial objects)**.** A *final object* in a category C is an object X such that $\mathrm{Hom}(Z, X) = *$ for every $Z \in \mathsf{C}$. Final objects are *unique up to unique isomorphism*: if X and X' are both final objects, then there is a unique isomorphism $X \to X'$; because of this, we may speak of *the* final object of C (in accordance with the principle of equivalence (1.1.32)). Dually, an object X is *initial* when $\mathrm{Hom}(X, Z) = *$ for every Z. An object which is both final and initial is called a *zero object*. A category which has a zero object is called *pointed*.

1.1.50 Example. The initial objects of Set and Top are the empty set/space $\varnothing$. The final objects of Set and Top are the one-point set/space $*$. The category Grp is pointed: the trivial group $\mathbf{1}$ is a zero object (both initial and final).

⋆ **1.1.51 Definition** (Diagram)**.** A *diagram shape* J consists of a set of 0-cells (vertices), a set of 1-cells (arrows between vertices), and a set of 2-cells, disks with boundary of the form

$$\begin{tikzcd}[column sep=small, row sep=small] & a_1 \arrow[r] & \cdots \arrow[r] & a_n \arrow[dr] & \\ x \arrow[ur] \arrow[dr] & & & & y \\ & b_1 \arrow[r] & \cdots \arrow[r] & b_m \arrow[ur] & \end{tikzcd} \tag{1.1.51.1}$$

for some integers $n, m \geq 0$. A *diagram* of shape J in a category C is a map $D : J \to \mathsf{C}$ associating to each 0-cell an object, to each 1-cell a morphism, such that for each 2-cell (1.1.51.1), composition along the two paths from x to y yields the same morphism $x \to y$. Diagrams form a category $\mathsf{Fun}(J, \mathsf{C})$ in which a morphism $D \to D'$ associates to each 0-cell $j \in J$ a morphism $D(j) \to D'(j)$ such that for each 1-cell $j \to j'$, the two compositions $D(j) \to D'(j) \to D'(j')$ and $D(j) \to D(j') \to D'(j')$ coincide.

There is an evident similarity between a diagram shape and a category, and between a diagram and a functor; in fact, this is more than just a similarity. We can regard a category as a diagram shape by taking its objects to be the 0-cells, its morphisms to be the 1-cells, and adding a triangular 2-cell

$$\begin{tikzcd}[column sep=small, row sep=small] & b \arrow[dr] & \\ a \arrow[ur] \arrow[rr] & & c \end{tikzcd} \tag{1.1.51.2}$$

for each pair of morphisms $a \to b \to c$ composing to a morphism $a \to c$. In the other direction, a diagram shape determines a category whose objects are the 0-cells and whose morphisms are directed paths (formal compositions) of 1-cells, modulo the relation that the formal compositions of the two maximal paths bounding a 2-cell (1.1.51.1) are the same. A diagram $J \to \mathsf{C}$ is then exactly the same as a functor to C from the category associated to J.

We emphasize that a diagram $J \to \mathsf{C}$ consists of *specified data* for each 0-cell and 1-cell, satisfying a *property* for each 2-cell. The assertion that a given diagram 'commutes' is simply the assertion that certain evident 2-cells (usually all possible 2-cells) are present; often this assertion is implicit in writing the diagram (diagrams commute unless the contrary is explicitly specified).

1.1.52 Exercise (Cancellation for fiber products)**.** Fix a diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[d] \\ D \arrow[r] & E \arrow[r] & F \end{tikzcd} \tag{1.1.52.1}$$

in which the right square (involving B, C, E, F) is a fiber square. Consider the induced maps

$$\begin{tikzcd} A \arrow[r] \arrow[rrr, bend left] & B \times_E D \arrow[r, "\sim"] & C \times_F E \times_E D \arrow[r, equal] & C \times_F D \end{tikzcd} \tag{1.1.52.2}$$

and conclude that the composite square (involving A, C, D, F) is a fiber square iff the left square (involving A, B, D, E) is a fiber square.

1.1.53 Definition (Pullback and pushout of a morphism)**.** Let $f : X \to Y$ be a morphism. A pullback of f is a morphism $f' : X' \to Y'$ fitting into a pullback square:

$$\begin{tikzcd} X' \arrow[r] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r] & Y \end{tikzcd} \tag{1.1.53.1}$$

Dually, a pushout of f is a morphism f' fitting into a pushout square:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\ X' \arrow[r, "f'"] & Y' \end{tikzcd} \tag{1.1.53.2}$$

1.1.54 Definition (Property preserved under pullback)**.** A property of morphisms $\mathcal{P}$ is said to be *preserved under pullback* when the following implication holds:

(1.1.54.1) For every $\mathcal{P}$-morphism $X \to Y$ and every morphism $Z \to Y$, the pullback $X \times_Y Z \to Z$ exists and has $\mathcal{P}$.

More generally, we say $\mathcal{P}$ is *preserved under* $\mathcal{Q}$*-pullback* ($\mathcal{Q}$ another property of morphisms) when the implication (1.1.54.1) holds provided $Z \to Y$ has $\mathcal{Q}$.

1.1.55 Exercise. Show that isomorphisms, monomorphisms, and split epimorphisms are preserved under pullback.

1.1.56 Exercise. Suppose $\mathcal{P}$ is a property of morphisms which is preserved under pullback and closed under composition. Show that $\mathcal{P}$ is preserved under fiber product, in the sense that for $\mathcal{P}$-morphisms $X \to Y$ and $X' \to Y'$ and any morphisms $Y \to Z \leftarrow Y'$, if $Y \times_Z Y'$ exists then so does $X \times_Z X'$ and the morphism $X \times_Z X' \to Y \times_Z Y'$ has $\mathcal{P}$.

⋆ **1.1.57 Definition** (Relative diagonal)**.** For any morphism $X \to Y$ in a category, the diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & X \arrow[d] \\ X \arrow[r] & Y \end{tikzcd} \tag{1.1.57.1}$$

induces a morphism $X \to X \times_Y X$ called the *(relative) diagonal* of $X \to Y$. The *nth diagonal* is the nth iterate of this construction: the zeroth diagonal of $X \to Y$ is $X \to Y$ itself, the first diagonal is $X \to X \times_Y X$, the second diagonal is $X \to X \times_{X \times_Y X} X$, etc.

1.1.58 Exercise. Show that the diagonal of any map of sets is injective, and that the diagonal of an injective map of sets is an isomorphism. Show that in any category, the diagonal of any morphism (if it exists) is a monomorphism, and that the diagonal of any monomorphism exists and is an isomorphism.

1.1.59 Definition (Properties of the diagonal)**.** Let $\mathcal{P}$ be any property of morphisms in a category which has all fiber products. A morphism is said to have property $\mathcal{P}_\Delta$ when its relative diagonal has property $\mathcal{P}$.

1.1.60 Exercise (The diagonal of a pullback is a pullback of the diagonal)**.** Use cancellation for fiber products (1.1.52) to show that if the left square below is a fiber square, then so are the right two squares.

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & Y' \arrow[d] \\ X \arrow[r] & Y \end{tikzcd} \quad \Longrightarrow \quad \begin{tikzcd} X' \arrow[r] \arrow[d] & X' \times_{Y'} X' \arrow[r] \arrow[d] & Y' \arrow[d] \\ X \arrow[r] & X \times_Y X \arrow[r] & Y \end{tikzcd} \tag{1.1.60.1}$$

Conclude that if $\mathcal{P}$ is preserved under pullback then so is $\mathcal{P}_\Delta$.

1.1.61 Exercise (The diagonal of a composition is a composition of pullbacks of diagonals)**.** Show that if $X \to Y \to Z$ are morphisms, then $X \times_Y X \to X \times_Z X$ is a pullback of $Y \to Y \times_Z Y$. Conclude that if $\mathcal{P}$ is preserved under pullback and closed under composition then so is $\mathcal{P}_\Delta$.

1.1.62 Lemma (Cancellation)**.** *Let $\mathcal{P}$ be a property of morphisms preserved under pullback and closed under composition. If the composition $X \to Y \to Z$ has $\mathcal{P}$ and $Y \to Z$ has $\mathcal{P}_\Delta$, then $X \to Y$ has $\mathcal{P}$.*

Proof. Factor $X \to Y$ into $X \to X \times_Z Y \to Y$. The map $X \to X \times_Z Y$ is a pullback of $Y \to Y \times_Z Y$ so has $\mathcal{P}$. The map $X \times_Z Y \to Y$ is a pullback of $X \to Z$ so has $\mathcal{P}$. □

1.1.63 Exercise. If $X \to Y$ and $Y \to Z$ are maps of sets whose composition $X \to Z$ is injective, then the first map $X \to Y$ is also injective. Prove this using the abstract cancellation property (1.1.62).

1.1.64 Definition (Twisted arrow category)**.** Let C be a category, and recall the arrow category $(\mathsf{C} \downarrow \mathsf{C})$ (1.1.40), whose objects are morphisms $c \to d$ in C and whose morphisms $(c \to d) \to (c' \to d')$ are commutative squares of the following shape.

$$\begin{tikzcd} c \arrow[r] \arrow[d] & c' \arrow[d] \\ d \arrow[r] & d' \end{tikzcd} \tag{1.1.64.1}$$

The *twisted arrow category* $(\mathsf{C}^{\mathsf{op}} \downarrow \mathsf{C})$ has the same objects, but a morphism $(c \to d) \to (c' \to d')$ is a commutative square of the following shape.

$$\begin{tikzcd} c \arrow[d] & c' \arrow[l] \arrow[d] \\ d \arrow[r] & d' \end{tikzcd} \tag{1.1.64.2}$$

Note the direction of the top arrow.

1.1.65 Definition (End and coend)**.** Let $F : \mathsf{C}^{\mathsf{op}} \times \mathsf{C} \to \mathsf{D}$ be a functor. Its *end* is the limit of its pullback to the twisted arrow category $(\mathsf{C}^{\mathsf{op}} \downarrow \mathsf{C})$.

$$\lim_{c^{\mathsf{op}} \to c} F(c^{\mathsf{op}}, c) = \mathrm{Eq}\left(\prod_c F(c,c) \overset{\prod_f (\mathbf{1}_{\mathsf{C}^{\mathsf{op}}} \times f)}{\underset{\prod_f (f \times \mathbf{1}_{\mathsf{C}})}{\rightrightarrows}} \prod_{f : c_1 \to c_2} F(c_1, c_2) \right) \tag{1.1.65.1}$$

Dually, the *coend* of a functor $F : \mathsf{C} \times \mathsf{C}^{\mathsf{op}} \to \mathsf{D}$ is the colimit of its pullback to $(\mathsf{C} \downarrow \mathsf{C}^{\mathsf{op}})$.

$$\operatorname*{colim}_{c \to c^{\mathsf{op}}} F(c, c^{\mathsf{op}}) = \mathrm{Coeq}\left(\coprod_{f : c_1 \to c_2} F(c_1, c_2) \overset{\coprod_f (\mathbf{1}_{\mathsf{C}} \times f)}{\underset{\coprod_f (f \times \mathbf{1}_{\mathsf{C}^{\mathsf{op}}})}{\rightrightarrows}} \coprod_c F(c,c) \right) \tag{1.1.65.2}$$

1.1.66 Exercise. Let $F, G : \mathsf{C} \to \mathsf{E}$ be two functors, which together determine a functor of the following shape.

$$\mathsf{C}^{\mathsf{op}} \times \mathsf{C} \to \mathsf{Set} \tag{1.1.66.1}$$

$$(c^{\mathsf{op}}, c) \mapsto \mathrm{Hom}_{\mathsf{E}}(F(c^{\mathsf{op}}), G(c)) \tag{1.1.66.2}$$

Show that the set of natural transformations $F \to G$ is naturally identified with end of this functor.

$$\mathrm{Hom}_{\mathsf{Fun}(\mathsf{C},\mathsf{E})}(F, G) = \lim_{c^{\mathsf{op}} \to c} \mathrm{Hom}_{\mathsf{E}}(F(c^{\mathsf{op}}), G(c)) \tag{1.1.66.3}$$

⋆ **1.1.67 Exercise** (Final and initial functors)**.** Show that for a functor $F : \mathsf{C} \to \mathsf{D}$, the following are equivalent:

(1.1.67.1) For every diagram $p : \mathsf{D} \to \mathsf{A}$ which has a limit, the pullback diagram $F^*p : \mathsf{C} \to \mathsf{A}$ also has a limit and the map $\lim_{\mathsf{D}} p \to \lim_{\mathsf{C}} F^*p$ is an isomorphism.

(1.1.67.2) For every $d \in \mathsf{D}$, the colimit $\operatorname{colim}_{\mathsf{C}_{/d}} *$ in Set is $*$.

A functor satisfying these properties is called *initial*. Show that a functor $d : * \to \mathsf{D}$ is initial iff d is an initial object of D. Although initial functors generalize initial objects, their use is somewhat different. The dual notion of initial is called *final* (F is final iff F^{op} is initial). Formulate precisely the duals of both properties above.

1.1.68 Lemma. *Every left adjoint functor is initial.*

Proof. A functor $F : \mathsf{C} \to \mathsf{D}$ has a right adjoint iff the category $\mathsf{C}_{/d}$ has a final object for every $d \in \mathsf{D}$. The colimit of the constant diagram $*$ over any category with a final object is $*$. □

⋆ **1.1.69 Definition** (Preservation of colimits)**.** A functor $F : \mathsf{C} \to \mathsf{D}$ is said to *preserve* a colimit diagram in C when it is sent to a colimit diagram in D by F. For example, we can ask that a functor preserve pushouts, initial objects, finite coproducts, all coproducts, finite colimits, filtered colimits, sifted colimits, simplicial realizations, all colimits, etc. A functor which preserves all colimits is called *cocontinuous*.

1.1.70 Exercise. Show that the forgetful functor $\mathsf{Ab} \to \mathsf{Set}$ preserves limits but not colimits.

* *

⋆ **1.1.71 Definition** (Presheaf)**.** A *presheaf* on a category C is simply a functor $\mathsf{C}^{\mathsf{op}} \to \mathsf{Set}$. The category of presheaves on C is denoted $\mathsf{P}(\mathsf{C}) = \mathsf{Fun}(\mathsf{C}^{\mathsf{op}}, \mathsf{Set})$.

⋆ **1.1.72 Definition** (Representable)**.** A presheaf $F \in \mathsf{P}(\mathsf{C})$ is called *representable* when it is isomorphic to a presheaf of the form $\mathrm{Hom}(-, X)$ for some $X \in \mathsf{C}$.

1.1.73 Exercise (Idempotent completion)**.** Let C be a category. An endomorphism π of an object $X \in \mathsf{C}$ is called *idempotent* when $\pi^2 = \pi$. Given a retraction $Y \to X \to Y$, the composition $X \to Y \to X$ is idempotent. Show that this idempotent, call it π, determines the retraction uniquely up to unique isomorphism by showing that $\mathrm{Hom}(Z, Y) = \mathrm{Hom}(Z, X)\pi \subseteq \mathrm{Hom}(Z, X)$ is the set of maps $Z \to X$ which factor as πf for some $f : Z \to X$ (thus the pair (X, π) determines the Yoneda functor of Y). We say that an idempotent π *splits* when it comes from a retraction (that is, when the functor $\mathrm{Hom}(-, X)\pi$ is representable). Split idempotents are preserved by any functor. A category is called *idempotent complete* (or *closed under retracts*) when every idempotent splits.

Given a category C, its *idempotent completion* $\Pi\mathsf{C}$ is defined as follows. An object of $\Pi\mathsf{C}$ is a pair (X, π) where $X \in \mathsf{C}$ is an object and $\pi \in \mathrm{Hom}_{\mathsf{C}}(X, X)$ is idempotent. Morphisms in $\Pi\mathsf{C}$ are given by

$$\mathrm{Hom}_{\Pi\mathsf{C}}((X, \pi), (X', \pi')) = \pi\mathrm{Hom}_{\mathsf{C}}(X, X')\pi' \subseteq \mathrm{Hom}_{\mathsf{C}}(X, X'), \tag{1.1.73.1}$$

namely the subset of $\mathrm{Hom}(X, X')$ consisting of morphisms which admit a factorization $\pi' f \pi$ (equivalently those morphisms g satisfying $g = \pi' g \pi$). There is an evident fully faithful embedding $\mathsf{C} \hookrightarrow \Pi\mathsf{C}$ given by $X \mapsto (X, \mathbf{1}_X)$. The maps $\pi : X \to (X, \pi)$ and $\pi : (X, \pi) \to X$ express (X, π) as a retract of X in the category $\Pi\mathsf{C}$. Show that $\Pi\mathsf{C}$ is idempotent complete and that for any idempotent complete category D, the restriction functor $\mathsf{Fun}(\Pi\mathsf{C}, \mathsf{D}) \to \mathsf{Fun}(\mathsf{C}, \mathsf{D})$ is an equivalence of categories.

1.1.74 Example. The idempotent completion of the category of free R-modules is the category of projective R-modules.

* *

⋆ **1.1.75 Definition** (Reflective subcategory)**.** A *reflective subcategory* is a full subcategory $i : \mathsf{A}_0 \subseteq \mathsf{A}$ whose inclusion functor i has a left adjoint r, called the *reflector*.

1.1.76 Example. The category of abelian groups Ab is a reflective subcategory of the category of all groups Grp. The reflector $\mathsf{Grp} \to \mathsf{Ab}$ is the abelianization functor $G \mapsto G/[G, G]$.

1.1.77 Example. Let hSpc denote the category of CW-complexes and homotopy classes of maps. Discrete spaces $\mathsf{Set} \subseteq \mathsf{hSpc}$ form a reflective subcategory, with reflector the π_0 functor $\mathsf{hSpc} \to \mathsf{Set}$.

1.1.78 Exercise. Show that if $\mathsf{A} \subseteq \mathsf{B}$ and $\mathsf{B} \subseteq \mathsf{C}$ are reflective, then $\mathsf{A} \subseteq \mathsf{C}$ is reflective and $r_{\mathsf{AC}} = r_{\mathsf{AB}} r_{\mathsf{BC}}$.

⋆ **1.1.79 Exercise** (Limits and colimits in a reflective subcategory)**.** Let $i : \mathsf{A}_0 \to \mathsf{A}$ be the inclusion of a reflective subcategory with left adjoint r. Show that $\mathsf{A}_0 \subseteq \mathsf{A}$ is closed under all limits (i.e. a limit of objects of A_0 which exists in A must in fact lie in A_0). Show that if a diagram in A_0 has a colimit in A, then it has a colimit in A_0, namely the image of the colimit in A under the reflector r.

1.1.80 Exercise (Cocontinuity on a reflective subcategory)**.** Let $\mathsf{A}_0 \subseteq \mathsf{A}$ be a reflective subcategory, and let E be a cocomplete category. Show that a functor $\mathsf{A} \to \mathsf{E}$ sending reflections to isomorphisms is cocontinuous iff its restriction to A_0 is cocontinuous.

1.1.81 Exercise. Let $\mathsf{C}_0 \subseteq \mathsf{C}$ be a reflective subcategory. Let $\mathcal{P}$ be a property of objects in C which is satisfied by all objects of C_0. Show that if the reflector $\mathsf{C} \to \mathsf{C}_0$ reflects isomorphisms when restricted to the full subcategory $\mathsf{C}_{\mathcal{P}} \subseteq \mathsf{C}$ of objects satisfying $\mathcal{P}$, then conversely all objects satisfying $\mathcal{P}$ lie in C_0.

1.1.82 Definition (Local object)**.** Let C be a category, and let Λ be a set of morphisms in C. An object $X \in \mathsf{C}$ is called *(right)* Λ*-local* when the functor $\operatorname{Hom}(-, X)$ sends morphisms in Λ to isomorphisms.

1.1.83 Lemma. *Let $\mathsf{C}_0 \subseteq \mathsf{C}$ be a reflective subcategory. An object $X \in \mathsf{C}$ lies in C_0 iff the functor* $\operatorname{Hom}(-, X)$ *sends all reflections $Y \to rY$ to isomorphisms.*

Proof. Suppose $\operatorname{Hom}(-, X)$ sends reflections to isomorphisms, and let us show that $X \in \mathsf{C}_0$ (the other direction is trivial). Apply the hypothesis on X to the reflection $\ell_X : X \to rX$ to see that $\operatorname{Hom}(rX, X) \xrightarrow{\circ \ell_X} \operatorname{Hom}(X, X)$ is an isomorphism. Lifting the identity map $\mathbf{1}_X$ produces a map $s : rX \to X$ for which the composition $X \xrightarrow{\ell_X} rX \xrightarrow{s} X$ is the identity. To show that the other composition $rX \xrightarrow{s} X \xrightarrow{\ell_X} rX$ is the identity, it suffices to show it is an isomorphism. Consider the commuting square obtained by applying the functor r to the morphism s.

$$\begin{tikzcd} rX \arrow[r, "s"] \arrow[d, "\ell_{rX}"'] & X \arrow[d, "\ell_X"] \\ rrX \arrow[r, "rs"] & rX \end{tikzcd} \tag{1.1.83.1}$$

The morphism ℓ_{rX} is an isomorphism, so it suffices to show that rs is an isomorphism. Now r sends ℓ_X to an isomorphism, so it must also send its retraction s to an isomorphism. □

1.1.84 Definition (Passing a functor to reflective subcategories)**.** Let $\mathsf{A}_0 \subseteq \mathsf{A}$ and $\mathsf{B}_0 \subseteq \mathsf{B}$ be reflective subcategories. A functor $f : \mathsf{A} \to \mathsf{B}$ induces a functor $f_0 = rfi : \mathsf{A}_0 \to \mathsf{B}_0$. For functors $f : \mathsf{A} \to \mathsf{B}$ and $g : \mathsf{B} \to \mathsf{C}$, there is a canonical natural transformation $(gf)_0 = rgfi \xrightarrow{rg\eta fi} rgirfi = g_0 f_0$. For a third functor $h : \mathsf{C} \to \mathsf{D}$, the diagram of canonical natural transformations

$$\begin{tikzcd} (hgf)_0 \arrow[r] \arrow[d] & h_0(gf)_0 \arrow[d] \\ (hg)_0 f_0 \arrow[r] & h_0 g_0 f_0 \end{tikzcd} \tag{1.1.84.1}$$

commutes.

1.1.85 Exercise. Let $f : \mathsf{A} \to \mathsf{B}$ be a functor, and let $\mathsf{A}_0 \subseteq \mathsf{A}$ and $\mathsf{B}_0 \subseteq \mathsf{B}$ be reflective subcategories with reflectors r_A and r_B. Show that f preserves reflections (i.e. sends reflections to reflections) iff $f(\mathsf{A}_0) \subseteq \mathsf{B}_0$ and $r_\mathsf{B} f$ sends reflections to isomorphisms.

1.1.86 Exercise. Let $f_! : \mathsf{A} \rightleftarrows \mathsf{B} : f^*$ be adjoint $(f_!, f^*)$, and let $\mathsf{A}_0 \subseteq \mathsf{A}$ and $\mathsf{B}_0 \subseteq \mathsf{B}$ be reflective subcategories with reflectors r_A and r_B. Use (1.1.83) to show that $r_\mathsf{B} f_!$ sends reflections to isomorphisms iff $f^*(\mathsf{B}_0) \subseteq \mathsf{A}_0$.

1.1.87 Exercise (Passing an adjunction to reflective subcategories)**.** Let $f_! : \mathsf{A} \rightleftarrows \mathsf{B} : f^*$ be adjoint $(f_!, f^*)$. Let $i_\mathsf{A} : \mathsf{A}_0 \subseteq \mathsf{A}$ and $i_\mathsf{B} : \mathsf{B}_0 \subseteq \mathsf{B}$ be reflective subcategories with reflectors r_A and r_B. Show that if $f^*(\mathsf{B}_0) \subseteq \mathsf{A}_0$ then there is an adjunction $(r_\mathsf{B} f_!, f^*)$ of functors $r_\mathsf{B} f_! : \mathsf{A}_0 \rightleftarrows \mathsf{B}_0 : f^*$. More precisely, show that such an adjunction is given by the identifications

$$\operatorname{Hom}(r_\mathsf{B} f_! X, Y) \overset{(r_\mathsf{B}, i_\mathsf{B})}{=\!=\!=} \operatorname{Hom}(f_! X, Y) \overset{(f_!, f^*)}{=\!=\!=} \operatorname{Hom}(X, f^* Y) \tag{1.1.87.1}$$

for $X \in \mathsf{A}_0$ and $Y \in \mathsf{B}_0$, corresponding to the unit and counit maps

$$\mathbf{1} \xrightarrow{\eta} f^* f_! \xrightarrow{f^* \eta_B f_!} f^* r_\mathsf{B} f_! \quad : \mathsf{A}_0 \to \mathsf{A}_0 \tag{1.1.87.2}$$

$$r_\mathsf{B} f_! f^* \xrightarrow{r_\mathsf{B} \varepsilon} r_\mathsf{B} \underset{\sim}{\xleftarrow{\eta_\mathsf{B}}} \mathbf{1} \quad : \mathsf{B}_0 \to \mathsf{B}_0 \tag{1.1.87.3}$$

where $\eta : \mathbf{1} \to f^* f_!$ and $\varepsilon : f_! f^* \to \mathbf{1}$ are the unit and counit maps of the adjunction $(f_!, f^*)$ and $\eta_\mathsf{B} : \mathbf{1} \to r_\mathsf{B}$ is the unit of the reflection r_B.

1.1.88 Lemma. *In the setup of* (1.1.87)*, if $f_!$ is fully faithful and $r_\mathsf{A} f^*$ sends reflections to isomorphisms, then $r_\mathsf{B} f_!$ is fully faithful.*

Proof. It is equivalent (**??**) to show that the unit map $\mathbf{1} \to f^* r_\mathsf{B} f_!$ (1.1.87.2) is an isomorphism. We are given that the unit map $\eta : \mathbf{1} \to f^* f_!$ is an isomorphism (since $f_!$ is fully faithful), so it suffices to show that the map

$$f^* f_! \xrightarrow{f^* \eta_B f_!} f^* r_\mathsf{B} f_! \quad : \mathsf{A}_0 \to \mathsf{A}_0 \tag{1.1.88.1}$$

is an isomorphism. By hypothesis, the map

$$r_\mathsf{A} f^* \xrightarrow{r_\mathsf{A} f^* \eta_B} r_\mathsf{A} f^* r_\mathsf{B} \quad : \mathsf{B} \to \mathsf{A}_0 \tag{1.1.88.2}$$

is an isomorphism. Now simply precompose with $f_!$ to obtain the desired result (the additional r_A is harmless since the functors already land in A_0). $\square$

* *

1.1.89 Proposition (Universal property of a reflective subcategory of presheaves)**.** *Let* $\mathsf{P}_0(\mathsf{C}) \subseteq \mathsf{P}(\mathsf{C})$ *be a reflective subcategory with reflector* r*. For any cocomplete category* E*, pullback along* $r \circ よ_{\mathsf{C}} : \mathsf{C} \to \mathsf{P}_0(\mathsf{C})$ *defines an equivalence between the following categories of functors:*

(1.1.89.1) *Functors* $\mathsf{P}_0(\mathsf{C}) \to \mathsf{E}$ *which are cocontinuous.*

(1.1.89.2) *Functors* $\mathsf{P}(\mathsf{C}) \to \mathsf{E}$ *which are cocontinuous and send reflections to isomorphisms.*

(1.1.89.3) *Functors* $\mathsf{C} \to \mathsf{E}$ *whose unique cocontinuous extension to* $\mathsf{P}(\mathsf{C})$ *send reflections to isomorphisms.*

Proof. By the universal property of a reflective subcategory (**??**), functors $\mathsf{P}_0(\mathsf{C}) \to \mathsf{E}$ are equivalent via pullback along r to functors $\mathsf{P}(\mathsf{C}) \to \mathsf{E}$ sending reflections to isomorphisms. This equivalence respects cocontinuity by (1.1.80). Now use the universal property of presheaves to identify cocontinuous functors $\mathsf{P}(\mathsf{C}) \to \mathsf{E}$ with functors $\mathsf{C} \to \mathsf{E}$ via pullback along $よ_{\mathsf{C}}$ (**??**). □

* *

⋆ **1.1.90 Definition** (Representable morphism)**.** A morphism $X \to Y$ in $\mathsf{P}(\mathsf{C})$ is called *representable* when the fiber product $X \times_Y c$ is representable for every map $c \to Y$ from an object $c \in \mathsf{C} \subseteq \mathsf{P}(\mathsf{C})$.

1.1.91 Exercise. Show that representability is preserved under pullback and closed under composition.

⋆ **1.1.92 Definition** (Induced property)**.** Let $\mathcal{P}$ be a property of morphisms in C which is preserved under pullback. A representable morphism $X \to Y$ in $\mathsf{P}(\mathsf{C})$ is said to have the 'induced' property $\overline{\mathcal{P}}$ (usually just said $\mathcal{P}$) when for every map $c \to Y$ from an object $c \in \mathsf{C}$, the pullback $X \times_Y c \to c$ has $\mathcal{P}$.

1.1.93 Warning. When discussing induced properties, we often shorten 'representable and $\overline{\mathcal{P}}$' to just '$\mathcal{P}$'. This is potentially dangerous: sometimes there is a reasonable generalization of $\mathcal{P}$ to (not necessarily representable) morphisms in $\mathsf{P}(\mathsf{C})$ which agrees with the induction (1.1.92) for representable morphisms (in which case 'representable and $\mathcal{P}$' is strictly stronger than just '$\mathcal{P}$').

1.1.94 Exercise. Let $\mathcal{P}$ be a property of morphisms in C which is preserved under pullback. Show that the induced property for morphisms in $\mathsf{P}(\mathsf{C})$ is preserved under pullback. Show that if $\mathcal{P}$ moreover closed under composition, then so is the induced property for morphisms in $\mathsf{P}(\mathsf{C})$.

1.1.95 Lemma. *Let* $\mathcal{P}$ *be a property of morphisms in* C *preserved under pullback, and let* $X \to Y \to B$ *be morphisms in* $\mathsf{P}(\mathsf{C})$*. The morphism* $X \to Y$ *has* $\mathcal{P}$ *iff every pullback* $X \times_B c \to Y \times_B c$ *has* $\mathcal{P}$*.*

Proof. Let $c \to Y$ be a morphism from $c \in \mathsf{C}$, and consider the following diagram.

$$\begin{tikzcd}
X \times_Y c \arrow[r] \arrow[d] & X \times_B c \arrow[d] \\
c = Y \times_Y c \arrow[r] \arrow[d] & Y \times_B c \arrow[d] \\
Y = Y \times_Y Y \arrow[r] & Y \times_B Y
\end{tikzcd} \tag{1.1.95.1}$$

The bottom square and the composite square are both pullbacks (**??**), so the top square is a pullback by cancellation (1.1.52). $\square$

* *

1.1.96 Definition (Monoidal category)**.** A *monoidal structure* $\otimes$ on a category C consists of the following data:

(1.1.96.1) A functor $\otimes : \mathsf{C} \times \mathsf{C} \to \mathsf{C}$.

(1.1.96.2) A natural isomorphism of functors $\otimes \circ (\mathbf{1}_{\mathsf{C}} \times \otimes) = \otimes \circ (\otimes \times \mathbf{1}_{\mathsf{C}})$, namely a chosen isomorphism $X \otimes (Y \otimes Z) = (X \otimes Y) \otimes Z$ functorial in $X, Y, Z \in \mathsf{C}$.

(1.1.96.3) The cyclic composition

$$\begin{tikzcd}
& (X \otimes Y) \otimes (Z \otimes W) \arrow[dl, leftrightarrow] \arrow[dr, leftrightarrow] & \\
((X \otimes Y) \otimes Z) \otimes W \arrow[d, leftrightarrow] & & X \otimes (Y \otimes (Z \otimes W)) \arrow[d, leftrightarrow] \\
(X \otimes (Y \otimes Z)) \otimes W \arrow[rr, leftrightarrow] & & X \otimes ((Y \otimes Z) \otimes W)
\end{tikzcd}$$

must be the identity map for all $X, Y, Z, W \in \mathsf{C}$.

(1.1.96.4) An object $\mathbf{1} \in \mathsf{C}$.

(1.1.96.5) Natural isomorphisms of functors $(\mathbf{1} \otimes -) = \mathbf{1}_{\mathsf{C}} = (- \otimes \mathbf{1})$, namely chosen isomorphisms $\mathbf{1} \otimes X = X = X \otimes \mathbf{1}$ functorial in X.

(1.1.96.6) The cyclic composition

$$\begin{tikzcd}
(X \otimes \mathbf{1}) \otimes Y \arrow[rr, leftrightarrow] \arrow[dr, leftrightarrow] & & X \otimes (\mathbf{1} \otimes Y) \arrow[dl, leftrightarrow] \\
& X \otimes Y &
\end{tikzcd}$$

must be the identity map for all $X, Y \in \mathsf{C}$.

A *monoidal category* $(\mathsf{C}, \otimes)$ is a category C equipped with a monoidal structure $\otimes$.

1.1.97 Definition (Enriched category)**.** Let $(\mathsf{C}, \otimes)$ be a monoidal category. The notion of a $(\mathsf{C}, \otimes)$-*enriched category* is a generalization of the notion of a category (1.1.1). A $(\mathsf{C}, \otimes)$-enriched category D has a set of objects, but morphisms in D consist of objects $\operatorname{Hom}(X, Y) \in \mathsf{C}$ for pairs $X, Y \in \mathsf{D}$. Composition in D consists of maps $\operatorname{Hom}(X, Y) \otimes \operatorname{Hom}(Y, Z) \to \operatorname{Hom}(X, Z)$, and associativity of composition involves the associator isomorphisms (1.1.96.2) of $(\mathsf{C}, \otimes)$ and implies that iterated composition

$$\operatorname{Hom}(X_0, X_1) \otimes \cdots \otimes \operatorname{Hom}(X_{n-1}, X_n) \to \operatorname{Hom}(X_0, X_n) \tag{1.1.97.1}$$

is well defined. Identity morphisms in D are maps $\mathbf{1}_X : \mathbf{1}_{\mathsf{C}} \to \operatorname{Hom}(X, X)$, and composition with $\mathbf{1}_X$ or $\mathbf{1}_Y$ on $\operatorname{Hom}(X, Y)$ must yield $\mathbf{1}_{\operatorname{Hom}(X,Y)}$ when combined with the unitor isomorphisms (1.1.96.5) of C.

A $(\mathsf{Set}, \times)$-enriched category is simply a category in the usual sense. A lax monoidal functor $(\mathsf{A}, \otimes) \to (\mathsf{B}, \otimes)$ turns $(\mathsf{A}, \otimes)$-enriched categories into $(\mathsf{B}, \otimes)$-enriched categories. In particular, we may regard a $(\mathsf{C}, \otimes)$-enriched category as having an 'underlying category' in the presence of a chosen lax monoidal functor $(\mathsf{C}, \otimes) \to (\mathsf{Set}, \times)$.

1.1.98 Exercise. Let C be a pointed category (1.1.49). Show that C is naturally enriched over $(\mathsf{Set}_*, \times)$ (pointed sets with the product symmetric monoidal structure).

1.1.99 Definition (Complex)**.** Let C be a pointed category (1.1.49)(1.1.98) together with an auto-equivalence $\Sigma : \mathsf{C} \to \mathsf{C}$ called 'suspension'. A *complex* in (C, Σ) is a pair (X, d) consisting of an object $X \in \mathsf{C}$ and a morphism $d : X \to \Sigma X$ whose square

$$X \xrightarrow{d} \Sigma X \xrightarrow{\Sigma d} \Sigma^2 X \tag{1.1.99.1}$$

vanishes (is the basepoint in $\operatorname{Hom}(X, \Sigma^2 X)$. Complexes in (C, Σ) form a category $\mathsf{Kom}(\mathsf{C}, \Sigma)$ in which a morphism $(X, d) \to (Y, d)$ is a map $X \to Y$ which commutes with d.

A functor $F : (\mathsf{C}, \Sigma) \to (\mathsf{D}, \Sigma)$ (meaning equipped with an isomorphism $F \circ \Sigma = \Sigma \circ F$) induces a functor $\mathsf{Kom}(\mathsf{C}, \Sigma) \to \mathsf{Kom}(\mathsf{D}, \Sigma)$. In particular, the shift functor Σ on C induces an autoequivalence of $\mathsf{Kom}(\mathsf{C}, \Sigma)$, also denoted Σ.

1.1.100 Definition (Sign functor)**.** The sign function $\operatorname{sgn} : \mathbb{R}^\times \to \mathbb{Z}^\times$ is given by $\operatorname{sgn}(\lambda) = \lambda/|\lambda|$. The sign functor sgn is a functor from the category of one-dimensional real vector spaces and isomorphisms to the category of free abelian groups of rank one and isomorphisms. It is defined by declaring that $\operatorname{sgn}(\mathbb{R}) = \mathbb{Z}$ and $\operatorname{sgn}(\mathbb{R} \xrightarrow{\lambda} \mathbb{R}) = (\mathbb{Z} \xrightarrow{\operatorname{sgn}(\lambda)} \mathbb{Z})$.

⋆ **1.1.101 Definition** (Orientation line)**.** Let V be a finite-dimensional real vector space. Its top wedge power $\wedge^{\dim V} V$ is a one-dimensional real vector space. The orientation line of V is

$$\mathfrak{o}(V) = \operatorname{sgn}(\wedge^{\dim V} V)[\dim V] \tag{1.1.101.1}$$

where sgn is the sign functor (1.1.100) and $[\dim V]$ indicates placement in homological degree $\dim V$. There are canonical associative isomorphisms $\mathfrak{o}(V \oplus W) = \mathfrak{o}(V) \otimes \mathfrak{o}(W)$ which are symmetric with respect to the super tensor product on graded $\mathbb{Z}$-modules. Thus the orientation line is a symmetric monoidal functor

$$((\mathsf{Vect}_{\mathbb{R}})_{\simeq}, \oplus) \to ((\mathsf{Ab}^{\mathbb{Z}})_{\simeq}, \otimes) \tag{1.1.101.2}$$

$$V \mapsto \mathfrak{o}(V) \tag{1.1.101.3}$$

There is a canonical isomorphism $\mathfrak{o}(V) = \mathfrak{o}(V^*)$.

Complex vector spaces are canonically oriented by taking, for any ordered $\mathbb{C}$-basis $v_1, \ldots, v_n \in V$, the generator $v_1 \wedge iv_1 \wedge \cdots \wedge v_n \wedge iv_n$ of $\wedge_{\mathbb{R}}^{2\dim_{\mathbb{C}} V} V$, which is independent up to positive scaling of the choice of basis. This establishes an isomorphism of symmetric monoidal functors between the pre-composition of the orientation line functor with the forgetful functor $\mathsf{Vect}_{\mathbb{C}} \to \mathsf{Vect}_{\mathbb{R}}$ and the functor $V \mapsto \mathbb{Z}[2\dim_{\mathbb{C}} V]$. This isomorphism is not unique: we have followed the usual convention by orienting $\mathbb{C}$ using $1 \wedge i$, but we could just as well have taken its opposite. This freedom is precisely the automorphism group of the symmetric monoidal functor $V \mapsto \mathbb{Z}[2\dim_{\mathbb{C}} V]$, namely $\mathbb{Z}/2$ generated by $(-1)^{\dim_{\mathbb{C}} V}$.

The definition of the orientation line of a vector space (1.1.101) carries over without change to the setting of vector bundles.

1.1.102 Definition (Mittag-Leffler inverse system)**.** An inverse system of sets $\cdots \to S_2 \to S_1 \to S_0$ is said to satisfy the *Mittag-Leffler condition* when the infinite decreasing intersection $S_i' = \bigcap_{j \geq i} \operatorname{im}(S_j \to S_i)$ is achieved at some finite stage: $S_i' = \operatorname{im}(S_j \to S_i)$ for some $j = j(i)$.

1.1.103 Lemma (Mittag-Leffler)**.** *Let $\{A_i\}_i \to \{B_i\}_i \to \{C_i\}_i$ be a sequence of maps inverse systems of abelian groups which is exact in the middle. If $\{A_i\}_i$ is Mittag-Leffler, then the sequence of inverse limits $\varprojlim_i A_i \to \varprojlim_i B_i \to \varprojlim_i C_i$ is also exact in the middle.*

1.2 2-categories

A 2-category is like a category, except that $\mathrm{Hom}(X, Y)$ is a groupoid (1.1.9) instead of a set. Because of this, the associativity axiom needs modification: a natural isomorphism between the two ways of composing a triple of morphisms $\mathrm{Hom}(X, Y) \times \mathrm{Hom}(Y, Z) \times \mathrm{Hom}(Z, W) \to \mathrm{Hom}(X, W)$ is specified, and these 'associators' are required to satisfy a certain 'pentagon identity' for quadruples of morphisms (1.2.1.4) which ensures that composition of any tuple of morphisms is well defined up to well defined isomorphism.

The theory of 2-categories contains the theory of categories as a special case, namely when all morphism groupoids are discrete (1.1.30). Most concepts and results in category theory carry over directly to 2-category theory, albeit with the caveat that there are often many more diagrams to chase. A detailed treatment of the theory of 2-categories can be found in Johnson–Yau [35] (though the reader should beware of various terminological differences with our presentation here).

1.2.1 Definition (2-category)**.** A *2-category* C consists of the following data:

(1.2.1.1) A set C, whose elements are called the *objects* of $\mathcal{C}$.

(1.2.1.2) For every pair of objects $X, Y \in \mathsf{C}$, a groupoid $\mathrm{Hom}(X, Y)$, whose objects are called the *morphisms* $X \to Y$ in C and whose morphisms are called the *2-morphisms* in $\mathcal{C}$.

(1.2.1.3) For every triple of objects $X, Y, Z \in \mathsf{C}$, a functor

$$\mathrm{Hom}(X, Y) \times \mathrm{Hom}(Y, Z) \to \mathrm{Hom}(X, Z)$$

called *composition*.

(1.2.1.4) For every quadruple of objects $X, Y, Z, W \in \mathsf{C}$, a natural isomorphism between the two ways of composing twice to obtain a functor

$$\mathrm{Hom}(X, Y) \times \mathrm{Hom}(Y, Z) \times \mathrm{Hom}(Z, W) \to \mathrm{Hom}(X, W)$$

such that for every quadruple of morphisms a, b, c, d, the cyclic composition

$$\begin{tikzcd}
& (ab)(cd) \arrow[dl, leftrightarrow] \arrow[dr, leftrightarrow] & \\
((ab)c)d \arrow[d, leftrightarrow] & & a(b(cd)) \arrow[d, leftrightarrow] \\
(a(bc))d \arrow[rr, leftrightarrow] & & a((bc)d)
\end{tikzcd}$$

is the identity map.

(1.2.1.5) For every object $X \in \mathsf{C}$, an object $\mathbf{1}_X \in \mathrm{Hom}(X, X)$ together with natural isomorphisms between composition with $\mathbf{1}_X$ and the identity functors on $\mathrm{Hom}(X, Y)$ and $\mathrm{Hom}(Z, X)$, such that for every pair of morphisms a, b, the cyclic composition

$$\begin{tikzcd}
& ab \arrow[dl, leftrightarrow] \arrow[dr, leftrightarrow] & \\
(a\mathbf{1})b \arrow[rr, leftrightarrow] & & a(\mathbf{1}b)
\end{tikzcd}$$

is the identity map.

1.2.2 Example (Categories are 2-categories)**.** Every category may be regarded as a 2-category by regarding each set $\mathrm{Hom}(X, Y)$ as a groupoid as in (1.1.30).

1.2.3 Example (Homotopy category of a 2-category)**.** A 2-category C gives rise to a category $\pi_0\mathsf{C}$ by replacing each morphism groupoid $\mathrm{Hom}(X, Y)$ with its set of isomorphism classes.

1.2.4 Example (2-category of categories)**.** Categories form a 2-category Cat in which $\mathrm{Hom}(\mathsf{C}, \mathsf{D}) = \mathsf{Fun}(\mathsf{C}, \mathsf{D})_{\simeq}$. That is, a morphism $\mathsf{C} \to \mathsf{D}$ is a functor, and a 2-morphism is a natural isomorphism of functors. The homotopy category of the 2-category Cat is the category denoted hCat discussed in (1.1.35).

1.3 Simplicial objects

Simplicial sets were introduced by Eilenberg–Zilber [21].

⋆ **1.3.1 Definition** (Simplex category $\mathbf{\Delta}$)**.** Let $\mathbf{\Delta}$ denote the category whose objects are non-empty finite ordered sets and whose morphisms are weakly order preserving maps. In other words, every object of $\mathbf{\Delta}$ is isomorphic to $[n] = \{0, \ldots, n\}$ for some integer $n \geq 0$, and a morphism $f : [n] \to [m]$ is a map of sets satisfying $f(i) \leq f(j)$ for $i \leq j$.

1.3.2 Example (Simplices as categories)**.** We may regard $[n]$ as the category with set of objects $\{0, \ldots, n\}$ and a single morphism $i \to j$ for $i \leq j$. Now a map $[n] \to [m]$ in $\mathbf{\Delta}$ is the same as a functor $[n] \to [m]$. Thus $\mathbf{\Delta} \subseteq \mathsf{Cat}$ is a full subcategory; compare (1.1.3)(1.1.31).

1.3.3 Example (Complete simplex)**.** Given a finite set S, the *complete simplex* on S is the subspace of $\mathbb{R}^S$ defined by the conditions $\sum_s x_s = 1$ and $x_s \geq 0$. A map of finite sets $f : S \to T$ induces a map $\mathbb{R}^S \to \mathbb{R}^T$ by pushforward $y_t = \sum_{f(s)=t} x_s$, hence a map $\Delta^S \to \Delta^T$ by restriction. This defines a functor $\mathsf{Set}^{\mathsf{fin}} \to \mathsf{Top}$; by pre-composing with the forgetful functor $\mathbf{\Delta} \to \mathsf{Set}^{\mathsf{fin}}$, we obtain a functor $\mathbf{\Delta} \to \mathsf{Top}$.

⋆ **1.3.4 Definition** (Simplicial object)**.** For any category C, a *simplicial object* of C is a functor $\mathbf{\Delta}^{\mathsf{op}} \to \mathsf{C}$ (dually, a functor $\mathbf{\Delta} \to \mathsf{C}$ is termed a *cosimplicial object*). A simplicial object $X_\bullet : \mathbf{\Delta}^{\mathsf{op}} \to \mathsf{C}$ thus consists of a sequence of objects $X_0, X_1, \ldots$ of C and maps $f^* : X_m \to X_n$ associated to maps $f : [n] \to [m]$, satisfying $(fg)^* = g^* f^*$. Simplicial objects of C form a category denoted $\mathsf{sC} = \mathsf{Fun}(\mathbf{\Delta}^{\mathsf{op}}, \mathsf{C})$ (and $\mathsf{csC} = \mathsf{Fun}(\mathbf{\Delta}, \mathsf{C})$ for cosimplicial objects). Note that a simplicial object of C is, despite the terminology, evidently not an object of C.

⋆ **1.3.5 Definition** (Simplicial set)**.** The category of simplicial sets is $\mathsf{sSet} = \mathsf{Fun}(\mathbf{\Delta}^{\mathsf{op}}, \mathsf{Set})$.

The Yoneda functor of $\mathbf{\Delta}$ is an embedding $\mathbf{\Delta} \to \mathsf{sSet}$, and the image of $[n]$ under this embedding is also denoted Δ^n. For any simplicial set $X_\bullet$, the Yoneda Lemma (**??**) identifies elements of $X_n = X([n])$ with maps $\Delta^n \to X$; these are called the 'n-simplices of X'. One should view a simplicial set as a combinatorial/categorical specification of a way to 'glue' together these simplices along simplicial maps (more formally, the category sSet is the free cocompletion of $\mathbf{\Delta}$ (**??**)).

1.3.6 Exercise. Describe the k-simplices of Δ^1 (there are $k + 2$ of them).

⋆ **1.3.7 Definition** (Levelwise property)**.** Let $\mathcal{P}$ be a property of morphisms in a category C. A morphism of simplicial objects $X_\bullet \to Y_\bullet$ in C is called *(levelwise)* $\mathcal{P}$ when each of its constituent maps $X_k \to Y_k$ has $\mathcal{P}$.

1.3.8 Exercise (Simplicial mapping space)**.** Show that for every pair of simplicial sets $X, Y \in \mathsf{sSet}$, there is a simplicial set $\underline{\mathrm{Hom}}(X, Y)$ defined by the universal property that a map $Z \to \underline{\mathrm{Hom}}(X, Y)$ is the same as a map $Z \times X \to Y$. Show that there is a tautological composition map $\underline{\mathrm{Hom}}(X, Y) \times \underline{\mathrm{Hom}}(Y, Z) \to \underline{\mathrm{Hom}}(X, Z)$, which is associative for quadruples (X, Y, Z, W).

⋆ **1.3.9 Definition** (Non-degenerate simplex)**.** Let X be a simplicial set. A simplex $[n] \to X$ is called *non-degenerate* when it has no factorization as $[n] \to [m] \to X$ with $m < n$; otherwise, it is called *degenerate*.

⋆ **1.3.10 Definition** ([21, (8.3)])**.** Let X be a simplicial set. Every simplex $[n] \to X$ admits a *unique* factorization $[n] \twoheadrightarrow [r] \to X$ with $[r] \to X$ is non-degenerate.

Proof. The existence of a factorization of the desired form is trivial, so the content is to prove uniqueness.

A surjection out of $[n]$ is determined uniquely by the set of arrows in $0 \to \cdots \to n$ which are collapsed. Fix a pair of surjections $f : [n] \twoheadrightarrow [r]$ and $g : [n] \twoheadrightarrow [s]$, and let $[n] \twoheadrightarrow [a]$ be the surjection which collapses the union of the arrows collapsed by f and g. This determines a diagram of the following shape.

$$\begin{tikzcd} {[n]} \arrow[r, two heads, "f"] \arrow[d, two heads, "g"'] & {[r]} \arrow[d, two heads] \\ {[s]} \arrow[r, two heads] & {[a]} \end{tikzcd} \tag{1.3.10.1}$$

We will show that this diagram is pushout in the category of simplicial sets, from which the desired uniqueness assertion follows immediately. A simple inspection shows that (1.3.10.1) is a pushout in the simplex category $\mathbf{\Delta}$, but this does not imply that it is a pushout in the category of simplicial sets sSet (Yoneda typically does not preserve colimits).

To show that (1.3.10.1) is a pushout in sSet, it is equivalent (since colimits in diagram categories are computed pointwise (**??**)) to show that the induced diagram

$$\begin{tikzcd} \operatorname{Hom}([k],[n]) \arrow[r] \arrow[d] & \operatorname{Hom}([k],[r]) \arrow[d] \\ \operatorname{Hom}([k],[s]) \arrow[r] & \operatorname{Hom}([k],[a]) \end{tikzcd} \tag{1.3.10.2}$$

is a pushout for every $[k] \in \mathbf{\Delta}$. This can be checked by the following explicit argument.

Every surjection in $\mathbf{\Delta}$ has a section, and having a section is preserved by the functor $\operatorname{Hom}([k], -)$, so the maps in (1.3.10.2) are surjective. In particular, the induced map from the colimit C of the $(\bullet \leftarrow \bullet \rightarrow \bullet)$ part of the square to its lower right corner is surjective. To show injectivity of this map, we need to show that if two maps $[k] \to [n]$ agree upon post-composition with the surjection $[n] \twoheadrightarrow [a]$, then they coincide in the colimit C. Denote by A the endomorphism of $\operatorname{Hom}([k],[n])$ obtained by post-composing with $f : [n] \twoheadrightarrow [r]$ and then with the section $[r] \to [n]$ of f sending an element $i \in [r]$ to the smallest element of $f^{-1}(i) \subseteq [n]$. Similarly, define an endomorphism B of $\operatorname{Hom}([k],[n])$ using g in place of f. Now it is simple to check that if two elements of $\operatorname{Hom}([k],[n])$ coincide upon post-composition with $[n] \twoheadrightarrow [a]$, then they can be made to coincide in $\operatorname{Hom}([k],[n])$ by applying A and B sufficiently many times. This gives the desired injectivity assertion. □

1.3.11 Definition (Cardinality of a simplicial set)**.** The *cardinality* of a simplicial set is the cardinality of the set of its *non-degenerate* simplices. If a simplicial set has cardinality κ, then the set of (all of) its simplices has cardinality $\aleph_0 \cdot \kappa$, which equals $\max(\aleph_0, \kappa)$ when $\kappa > 0$.

1.3.12 Exercise. Show that there are exactly $\frac{(n+m)!}{n!m!}$ non-degenerate $(n+m)$-simplices in $\Delta^n \times \Delta^m$. Identify these simplices with paths from $(0,0)$ to (n,m) in the $n \times m$ unit grid.

* *

1.3.13 Definition (Truncated simplicial object)**.** Denote by $\boldsymbol{\Delta}_{\leq k} \subseteq \boldsymbol{\Delta}$ the full subcategory spanned by the objects $[a]$ for $a \leq k$. A k-truncated simplicial object of a category C is a functor $\boldsymbol{\Delta}_{\leq k}^{\mathsf{op}} \to \mathsf{C}$. There is a tautological restriction ('truncation') functor

$$\mathsf{Fun}(\boldsymbol{\Delta}^{\mathsf{op}}, \mathsf{C}) \to \mathsf{Fun}(\boldsymbol{\Delta}_{\leq k}^{\mathsf{op}}, \mathsf{C}) \tag{1.3.13.1}$$

from simplicial objects to k-truncated simplicial objects. When C is cocomplete, the truncation functor has a left adjoint given by left Kan extension.

$$\mathsf{Fun}(\boldsymbol{\Delta}_{\leq k}^{\mathsf{op}}, \mathsf{C}) \to \mathsf{Fun}(\boldsymbol{\Delta}^{\mathsf{op}}, \mathsf{C}) \tag{1.3.13.2}$$

$$X_\bullet \mapsto \Big([a] \mapsto \operatorname*{colim}_{([a]\downarrow\boldsymbol{\Delta}_{\leq k})^{\mathsf{op}}} X_\bullet\Big) \tag{1.3.13.3}$$

Since $\boldsymbol{\Delta}_{\leq k} \subseteq \boldsymbol{\Delta}$ is a full subcategory, this left adjoint is fully faithful (**??**). We implicitly identify k-truncated simplicial objects with the full subcategory of simplicial objects given by the essential image of this functor. Being k-truncated is thus a *property* of a simplicial object, and a simplicial object will be called *truncated* when it is k-truncated for some $k < \infty$. The truncation functor from simplicial objects to k-truncated simplicial objects is also called the k*-skeleton* functor.

⋆ **1.3.14 Definition** (Latching object)**.** Let $X_\bullet : \boldsymbol{\Delta}^{\mathsf{op}} \to \mathsf{C}$ be a simplicial object. The nth *latching object* of $X_\bullet$ is the colimit

$$L_n X_\bullet = \operatorname*{colim}_{([n]\downarrow\boldsymbol{\Delta}_{<n})^{\mathsf{op}}} X_\bullet. \tag{1.3.14.1}$$

There is a tautological map $L_n X_\bullet \to X_n$ called the nth *latching map* of $X_\bullet$. More generally, the nth latching map of a map of simplicial objects $X_\bullet \to Y_\bullet$ is the tautological map

$$X_n \bigsqcup_{L_n X_\bullet} L_n Y_\bullet \to Y_n \tag{1.3.14.2}$$

(when $X_\bullet = \varnothing$ is the initial object, this evidently reduces to the latching map of $Y_\bullet$).

The dual notion (i.e. for cosimplicial objects) is called *matching* and is denoted M^n.

1.3.15 Remark. We note that the full subcategory $([n] \twoheaddownarrow \boldsymbol{\Delta}_{<n}) \subseteq ([n] \downarrow \boldsymbol{\Delta}_{<n})$ spanned by surjections $[n] \twoheadrightarrow [a]$ is initial, since its inclusion has a right adjoint (sending a map $[n] \to [a]$ to the surjection $[n] \twoheadrightarrow \operatorname{im}([n] \to [a])$) (1.1.68). Thus the latching object is equivalently given by the colimit

$$L_n X_\bullet = \operatorname*{colim}_{([n]\twoheaddownarrow\boldsymbol{\Delta}_{<n})^{\mathsf{op}}} X_\bullet. \tag{1.3.15.1}$$

This category $([n] \downarrow \mathbf{\Delta}_{<n})$ has a quite simple structure. A surjection $f : [n] \twoheadrightarrow [a]$ is determined uniquely by the sequence of n bits $\varepsilon_i(f) = f(i) - f(i-1) \in \{0, 1\}$ for $i = 1, \ldots, n$. The category $([n] \downarrow \mathbf{\Delta}_{\leq n})$ is thus the poset category $\{0 \leftarrow 1\}^n$, and its full subcategory $([n] \downarrow \mathbf{\Delta}_{<n})$ is the complement of the initial vertex $(1, \ldots, 1)$ (corresponding to the identity surjection $[n] \twoheadrightarrow [n]$).

1.3.16 Lemma. *A simplicial object is k-truncated iff its latching maps in all degrees $> k$ are isomorphisms.*

Proof. The counit map $\mathrm{sk}_{r-1} X_\bullet \to X_\bullet$ in degree r is precisely the rth latching map $L_r X_\bullet \to X_r$. Thus if X is $(r-1)$-truncated, then the rth latching map is an isomorphism. Since being k-truncated implies being i-truncated for all $i \geq k$, we conclude that being k-truncated implies the latching maps in all degrees $> k$ are isomorphisms.

For the converse, we apply the criterion (1.1.81) for identifying a reflective subcategory. It thus suffices to show, for any pair of simplicial objects $X_\bullet$ and $Y_\bullet$ whose latching maps are isomorphisms in degrees $> k$, that a morphism $X_\bullet \to Y_\bullet$ is an isomorphism iff it is an isomorphism in degrees $\leq k$. This may be proven by induction. □

⋆ **1.3.17 Definition** (Reedy property [76])**.** Let $\mathcal{P}$ be a property of morphisms in a category C. A simplicial object $X_\bullet : \mathbf{\Delta}^{\mathsf{op}} \to \mathsf{C}$ is said to be *Reedy* $\mathcal{P}$ when its latching maps $L_i X_\bullet \to X_i$ have property $\mathcal{P}$. More generally, a morphism of simplicial objects $X_\bullet \to Y_\bullet$ is called Reedy $\mathcal{P}$ when its relative latching maps (1.3.14.2) have $\mathcal{P}$ (when $X_\bullet = \varnothing$ is the initial object, this is evidently the same as $Y_\bullet$ being Reedy $\mathcal{P}$).

1.3.18 Lemma. *Let $X^\bullet \to Y^\bullet$ be a map of cosimplicial objects. The map on nth matching objects $M^n X^\bullet \to M^n Y^\bullet$ is (functorially in $X^\bullet \to Y^\bullet$) a finite composition of pullbacks of ith matching maps $X^i \to M^i X^\bullet \times_{M^i Y^\bullet} Y^i$ for $i < n$.*

Proof. Write matching objects as limits over the categories $([n] \downarrow \mathbf{\Delta}_{<n})$ as in (1.3.15). Consider the category $([n] \downarrow \mathbf{\Delta}_{<n}) \times (x \to y)$ and the evident diagram from it associated to $X^\bullet \to Y^\bullet$. The limit of this diagram is $M^n X^\bullet$ (since $([n] \downarrow \mathbf{\Delta}_{<n}) \times x$ is initial) while the limit of its restriction to $([n] \downarrow \mathbf{\Delta}_{<n}) \times y$ is $M^n Y^\bullet$. Now let us build $([n] \downarrow \mathbf{\Delta}_{<n}) \times (x \to y)$ from its full subcategory $([n] \downarrow \mathbf{\Delta}_{<n}) \times y$ by iteratively adding maximal objects not already present. The effect on the limit of adding such a maximal object $([n] \twoheadrightarrow [i]) \times x$ is to form a pullback of the ith matching map of $X^\bullet \to Y^\bullet$ (use Mayer–Vietoris (**??**) twice). □

1.3.19 Exercise. Let $X_\bullet : \mathbf{\Delta}^{\mathsf{op}} \to \mathsf{C}$ be a simplicial object which is Reedy $\mathcal{P}$. Conclude from (1.3.18) that if $F : \mathsf{C} \to \mathsf{D}$ preserves pushouts of $\mathcal{P}$-morphisms, then it preserves the latching objects $L_i X_\bullet$ (in the sense that the natural map $L_i F(X_\bullet) \to F(L_i X_\bullet)$ is an isomorphism (1.1.69)).

* *

We now study simplicial abelian groups (and, more generally, simplicial objects in additive categories (**??**)).

The following classical result identifies simplicial abelian groups with complexes of abelian groups supported in non-negative homological degree. It is an 'abelian' analogue of the fundamental result on non-degenerate simplices for simplicial sets (1.3.10) and the resulting fact that every simplicial set is the ascending union of its skeleta, each of which is obtained from the previous by attaching some set of non-degenerate simplices.

⋆ **1.3.20 Dold–Kan Correspondence** ([14][46]). *The functor*

$$\mathrm{DK} : \mathsf{Kom}_{\geq 0}(\mathsf{Ab}) \to \mathsf{sAb} \tag{1.3.20.1}$$
$$K_\bullet \mapsto \Big([n] \mapsto \mathrm{Hom}(C^{\mathrm{cell}}_\bullet(\Delta^n), K_\bullet)\Big) \tag{1.3.20.2}$$

is an equivalence of categories.

Proof. The key to the Dold–Kan correspondence is to express the group of homomorphisms $\mathrm{Hom}(C^{\mathrm{cell}}_\bullet(\Delta^n), K_\bullet)$ using a 'shelling' (filtration by pushouts of horns) of Δ^n. Fix such a filtration (there are many) $\varnothing = F_0 \subseteq F_1 \subseteq \cdots \subseteq F_{2^n} = \Delta^n$, which necessarily contains exactly $\binom{n}{k}$ pushouts of k-dimensional horns for all k. Applying $\mathrm{Hom}(C^{\mathrm{cell}}_\bullet(-), K_\bullet)$ yields a sequence of maps.

$$\begin{multline}\mathrm{Hom}(C^{\mathrm{cell}}_\bullet(\Delta^n), K_\bullet) = \mathrm{Hom}(C^{\mathrm{cell}}_\bullet(F_{2^n}), K_\bullet) \to \cdots \\ \cdots \to \mathrm{Hom}(C^{\mathrm{cell}}_\bullet(F_1), K_\bullet) \to \mathrm{Hom}(C^{\mathrm{cell}}_\bullet(F_0), K_\bullet) = 0 \quad (1.3.20.3)\end{multline}$$

Every restriction map

$$\mathrm{Hom}(C^{\mathrm{cell}}_\bullet(F_i), K_\bullet) \to \mathrm{Hom}(C^{\mathrm{cell}}_\bullet(F_{i-1}), K_\bullet) \tag{1.3.20.4}$$

has a canonical section: when (F_i, F_{i-1}) is a pushout of a k-dimensional horn, a chain map $C^{\mathrm{cell}}_\bullet(F_{i-1}) \to K_\bullet$ may be extended to $C^{\mathrm{cell}}_\bullet(F_i)$ by declaring it should vanish on the new k-simplex, and this uniquely determines its value on the new $(k-1)$-simplex. The kernel of the restriction map is $\mathrm{Hom}(C^{\mathrm{cell}}_\bullet(F_i, F_{i-1}), K_\bullet) = \mathrm{Hom}(C^{\mathrm{cell}}_\bullet(\Delta^k, \Lambda^k_j), K_\bullet) = K_k$. A choice of shelling of Δ^n thus defines an isomorphism

$$\mathrm{Hom}(C^{\mathrm{cell}}_\bullet(\Delta^n), K_\bullet) \cong \bigoplus_k K_k^{\oplus\binom{n}{k}}. \tag{1.3.20.5}$$

It follows immediately that the Dold–Kan functor is faithful. □

1.3.21 Example. Let X be a topological space, and let $\mathsf{Vect}(X)$ denote the additive category of finite-dimensional real vector bundles on X. The category $\mathsf{Vect}(X)$ is idempotent complete (it suffices to treat the 'universal' case which consists of showing that $\ker \pi$ is a vector bundle over $\{\pi : \mathbb{R}^n \to \mathbb{R}^n \mid \pi^2 = \pi\}$). The Dold–Kan Correspondence (1.3.20) thus provides an equivalence between complexes of vector bundles supported in non-negative homological degrees $\mathsf{Kom}_{\geq 0}(\mathsf{Vect}(X))$ and simplicial vector bundles $\mathsf{sVect}(X)$.

1.3.22 Example. Consider an idempotent complete additive category A. Idempotent completeness is invariant under passing to opposites, so A^{op} is also idempotent complete. The Dold–Kan Correspondence (1.3.20) for A is an equivalence between $\mathsf{Kom}^{\geq 0}(\mathsf{A})$ and csA.

1.3.23 Exercise. Show that under the Dold–Kan Correspondence (1.3.20):
(1.3.23.1) $[\mathbb{Z}[k+1] \to \mathbb{Z}[k]] \in \mathsf{Kom}_{\geq 0}(\mathsf{Ab})$ corresponds to $C^k_{\text{cell}}(\Delta^\bullet) \in \mathsf{sAb}$.
(1.3.23.2) $\mathbb{Z}[k] \in \mathsf{Kom}_{\geq 0}(\mathsf{Ab})$ corresponds to $Z^k_{\text{cell}}(\Delta^\bullet) \in \mathsf{sAb}$.

1.3.24 Corollary. *Let C be an idempotent complete additive category, and let $\mathcal{P}$ be any property of morphisms in C which is closed under direct sums and retracts. For any map $A_\bullet \to B_\bullet$ in sC and any $n \geq 0$, the following are equivalent:*
(1.3.24.1) *The map $A_n \to B_n$ has $\mathcal{P}$.*
(1.3.24.2) *The map $N_iA_\bullet \to N_iB_\bullet$ has $\mathcal{P}$ for all $i \leq n$.*
In particular $A_k \to B_k$ has $\mathcal{P}$ for all $k \geq 0$ iff $N_kA_\bullet \to N_kB_\bullet$ has $\mathcal{P}$ for all $k \geq 0$.

Proof. A shelling of Δ^n fixes a functorial isomorphism $A_n = \bigoplus_{i=0}^n (N_iA_\bullet)^{\binom{n}{i}}$ (1.3.20.5). □

The next result is a linear analogue of the theory of non-degene simplices in simplicial sets (1.3.10) (compare (**??**)). It appears that it does not follow formally from (1.3.10), since the forgetful functor $\mathsf{Vect} \to \mathsf{Set}$ does not preserve colimits.

1.3.25 Corollary. *For any simplicial object $A_\bullet$ in an idempotent complete additive category, there is a functorial short exact sequence*

$$0 \to L_kA_\bullet \to A_k \to N_kA_\bullet \to 0 \tag{1.3.25.1}$$

for every $k \geq 0$. This short exact sequence has a functorial splitting associated to any choice of codimension one face of Δ^k.

Proof. □

1.3.26 Corollary. *Let $A_\bullet \to B_\bullet$ be a map of simplicial objects in an idempotent complete additive category. The cone of the kth latching map $A_k \sqcup_{L_kA_\bullet} L_kB_\bullet \to B_k$ is (functorially) homotopy equivalent to the cone of the map $N_kA_\bullet \to N_kB_\bullet$ on normalized chains in degree k.*

Proof. The map on short exact sequences (1.3.25)

$$\begin{tikzcd} L_kA_\bullet \arrow[r] \arrow[d] & A_\bullet \arrow[r] \arrow[d] & N_kA_\bullet \arrow[d] \\ L_kB_\bullet \arrow[r] & B_\bullet \arrow[r] & N_kB_\bullet \end{tikzcd} \tag{1.3.26.1}$$

induces a map from the total complex of the square on the left to the cone of $N_kA_\bullet \to N_kB_\bullet$. It suffices to show that this map is a homotopy equivalence and so is the natural map from the total complex of the square on the left to the cone of $A_k \sqcup_{L_k A_\bullet} L_k B_\bullet \to B_k$. The above map on short exact sequences is functorially split (1.3.25), hence may be written as

$$\begin{array}{ccccc} L_k A_\bullet & \longrightarrow & L_k A_\bullet \oplus N_k A_\bullet & \longrightarrow & N_k A_\bullet \\ \downarrow & & \downarrow & & \downarrow \\ L_k B_\bullet & \longrightarrow & L_k B_\bullet \oplus N_k B_\bullet & \longrightarrow & N_k B_\bullet \end{array} \tag{1.3.26.2}$$

with the evident inclusion and projection maps, from which the two desired homotopy equivalence assertions are immediate. □

1.3.27 Corollary. *A simplicial object in an additive category is n-truncated* (1.3.13) *iff the corresponding chain complex is supported in degrees $\leq n$.* □

Proof. Combine (1.3.16) and (1.3.26). □

1.4 Simplicial homotopy theory

1.4.1 Definition (Boundary and horns)**.** The boundary $\partial\Delta^n \subseteq \Delta^n$ consists of those simplices of Δ^n which omit at least one vertex of Δ^n. The ith horn $\Lambda^n_i \subseteq \Delta^n$ ($0 \le i \le n$ and $n \ge 1$) consists of those simplices of Δ^n which omit at least one vertex other than vertex i.

1.4.2 Exercise. Draw $\Lambda^n_i \subseteq \Delta^n$ and $\partial\Delta^n \subseteq \Delta^n$ for all $n \le 3$.

1.4.3 Exercise. Show that the map $\Lambda^n_i \to \Delta^n$ does not have a retraction (except for $n = 1$), but does after applying geometric realization (it will be helpful to use (**??**)).

1.4.4 Definition (Transfinite composition of morphisms)**.** Let α be an ordinal, and consider a sequence of objects $X_0, X_1, \ldots$ of a category C indexed by ordinals $\nu < \alpha$ along with morphisms

$$\operatorname*{colim}_{\mu<\nu} X_\mu \to X_\nu \tag{1.4.4.1}$$

for all $\nu < \alpha$. The induced morphism $X_0 \to \operatorname{colim}_{\nu<\alpha} X_\nu$ is called the *transfinite composition* of the morphisms (1.4.4.1).

1.4.5 Exercise. Let $\mathcal{M}$ be a set of morphisms, and let $\overline{\mathcal{M}}$ be the set of morphisms expressible as transfinite compositions of morphisms in $\mathcal{M}$. Show that $\overline{\mathcal{M}}$ is closed under transfinite composition. Show that if $\mathcal{M}$ is closed under pushouts then so is $\overline{\mathcal{M}}$. Show that if every morphism in $\mathcal{M}$ satisfies the left lifting property with respect to a morphism $X \to Y$, then so does every morphism in $\overline{\mathcal{M}}$.

1.4.6 Definition (Pair)**.** A *pair* of simplicial sets (X, A) is an injective map $A \to X$. A morphism of pairs $(X, A) \to (X', A')$ is commutative square. We often (but with some necessary exceptions) identify a simplicial set X with the pair $(X, \varnothing)$.

$*$ $*$

A fundamental concept in categorical homotopy theory is lifting properties.

1.4.7 Definition (Lifting property)**.** A *lift* for a commuting diagram of solid arrows

$$\begin{tikzcd} A \arrow[r] \arrow[d] & X \arrow[d] \\ B \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd} \tag{1.4.7.1}$$

is a dotted arrow making the diagram commute. A morphism $X \to Y$ is said to satisfy the *right lifting property* with respect to a morphism $A \to B$ (and $A \to B$ satisfies the *left lifting property* with respect to $X \to Y$) when every such diagram with these given vertical arrows has a lift. The right lifting property in the special case $X \to *$ will be called the *extension property*: X satisfies the extension property for $A \to B$ when every map $A \to X$ admits a factorization $A \to B \to X$.

1.4.8 Exercise. Show that the right lifting property with respect to any fixed morphism $A \to B$ is preserved under pullback and closed under composition.

1.4.9 Exercise. Show that the left lifting property with respect to any fixed morphism $X \to Y$ is preserved under pushout and closed under transfinite composition.

1.4.10 Definition (Kan fibration [44, 45])**.** A map of simplicial sets $X \to Y$ is called a *Kan fibration* when it has the right lifting property for every horn (Δ^n, Λ^n_i). That is, $X \to Y$ is a Kan fibration when for every commuting diagram of solid arrows

$$\begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d] & X \arrow[d] \\ \Delta^n \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd} \tag{1.4.10.1}$$

there exists a dotted arrow making the diagram commute.

A simplicial set X is called a *Kan complex* iff the map $X \to *$ is a Kan fibration. In other words, X is a Kan complex when it satisfies the extension property for (Δ^n, Λ^n_i), meaning every map $\Lambda^n_i \to X$ extends to Δ^n.

1.4.11 Exercise. Use the retraction property (1.4.3) to show that the singular simplicial set (**??**) of any topological space is a Kan complex.

1.4.12 Exercise. The set of components $\pi_0 X$ of a simplicial set X is the set vertices X_0 modulo the equivalence relation closure of the relation given by the edges ($x \sim x'$ iff there exists an edge $x \to x'$); this gives a functor $\pi_0 : \mathsf{sSet} \to \mathsf{Set}$. Show that if X is a Kan complex, then the edge relation is an equivalence relation.

1.4.13 Exercise. Show that every simplicial abelian group is a Kan complex by appealing to the Dold–Kan correspondence (1.3.20) and noting that $C^{\mathrm{cell}}_\bullet(\Lambda^n_i) \hookrightarrow C^{\mathrm{cell}}_\bullet(\Delta^n)$ has a retract. In fact, every simplicial group is a Kan complex (Moore [66, Théorème 3]).

1.4.14 Definition (Smash product of pairs)**.** For simplicial set pairs (X, A) and (Y, B), we term

$$(X, A) \wedge (Y, B) = (X \times Y, (X \times B) \cup_{A \times B} (A \times Y)). \tag{1.4.14.1}$$

their *smash product* (beware: like tensor product, the smash product is not the categorical product).

1.4.15 Exercise. Show that if (X', A') is a pushout of a pair (X, A), then $(X', A') \wedge (Y, B)$ is a pushout of $(X, A) \wedge (Y, B)$. Show that if (X, A) is filtered by pushouts of pairs in some colection $\mathcal{M}$, then $(X, A) \wedge (Y, B)$ is filtered by pushouts of pairs in $\mathcal{M} \wedge (Y, B)$.

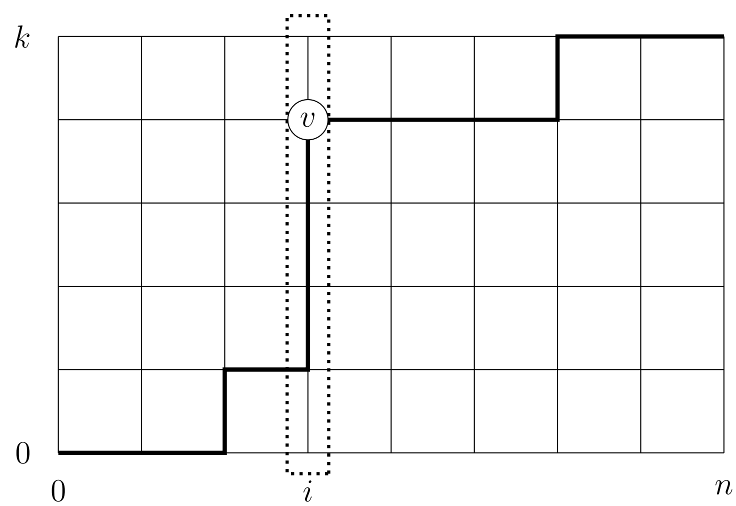
⋆ **1.4.16 Lemma.** *Every smash product $(\Delta^n, \Lambda^n_i) \wedge (\Delta^k, \partial\Delta^k)$ is filtered by pushouts of horns.*

Proof. The product $\Delta^n \times \Delta^k$ is the nerve of the category $[n] \times [k]$. Its non-degenerate $(n+k)$-simplices are thus in bijection with lattice paths from $(0,0)$ to (n,k). This set of $(n+k)$-simplices has a natural partial order in which $\sigma \succeq \tau$ iff the path corresponding to σ lies above that corresponding to τ (when the $[n]$ coordinate is drawn horizontally and the $[k]$ coordinate vertically). We filter the pair $(\Delta^n, \Lambda^n_i) \wedge (\Delta^k, \partial\Delta^k)$ by adding the non-degenerate $(n+k)$-simplices one at a time, according to any total order refining the aforementioned partial order. It suffices to show that each simplex addition in this filtration can be realized by filling some number of horns.

Let $Q \subseteq \Delta^n \times \Delta^k$ denote the union of $(\Delta^n \times \partial\Delta^k) \cup (\Lambda^n_i \times \Delta^k)$ and any set S of non-degenerate $(n+k)$-simplices with the property that $\sigma \succeq \tau \in S$ implies $\sigma \in S$. Our aim is to show that the pair $(Q \cup \sigma, Q)$ is filtered by pushouts of horns for any $\sigma \notin S$ which is maximal among simplices not in S. Equivalently, this means filtering the pair $(\sigma, \sigma \cap Q)$ by pushouts of horns.

Let us say that the pair $(\sigma, \sigma \cap Q)$ is *coned* at a vertex v of σ iff for every simplex $\tau \subseteq \sigma \cap Q$, the cone of τ with v is also $\subseteq \sigma \cap Q$. Denoting by $\sigma_{\hat{v}} \subseteq \sigma$ the span of all vertices other than v, any filtration of $(\sigma_{\hat{v}}, \sigma_{\hat{v}} \cap Q)$ by pushouts of $(\Delta^a, \partial\Delta^a)$ determines, by coning at v, a filtration of $(\sigma, \sigma \cap Q)$ by pushouts of horns with cone point v. It thus suffices show that $(\sigma, \sigma \cap Q)$ is coned at some vertex $v \in \sigma$.

Regarding σ as a lattice path from $(0,0)$ to (n,k), choose $v = (i,j) \in \sigma$ where i indexes the horn Λ^n_i and j is as large as possible given i.



(1.4.16.1)

Our goal is now to show that $(\sigma, \sigma \cap Q)$ is coned at v. We first describe the intersection $\sigma \cap Q$. A simplex $\tau \subseteq \sigma$ is contained in Q iff it satisfies at least one of the following conditions:

(1.4.16.2) The vertices of τ do not surject onto $[n] - \{i\}$.

(1.4.16.3) The vertices of τ do not surject onto $[k]$.

(1.4.16.4) The subset of the lattice path σ corresponding to τ misses at least one cliffbottom corner (a vertex $w \in \sigma$ for which both $w + (0,1)$ and $w - (1,0)$ are in σ).

Now suppose $\tau \subseteq \sigma$ lies in Q, and let us show that the simplex spanned by τ union v also lies in Q. The property of not surjecting onto $[n] - \{i\}$ is certainly preserved by adding v.

Missing a cliffbottom corner is also preserved by adding v since v is never a cliffbottom corner. Now suppose τ does not surject onto $[k]$ but $\tau \cup v$ does. This means τ does not contain any vertex with the same second coordinate as v. If the second coordinate of v is $< k$, then τ misses a cliffbottom corner, hence so does $\tau \cup v$. If the second coordinate of v is k and $i < n$, then $\tau \cup v$ cannot surject onto $[n] - \{i\}$, since it misses everything $> i$. This completes the proof in the case $i < n$. The case $i = n$ now follows by symmetry. □

1.4.17 Exercise. Let X and Y be simplicial sets. Use (1.4.16) (along with (1.4.15) and (**??**)) to show that if Y is a Kan complex then so is the simplicial mapping space $\underline{\mathrm{Hom}}(X, Y)$ (1.3.8).

1.4.18 Exercise (Homotopy category of Kan complexes hSpc)**.** For a Kan complex X and a simplicial set K, call maps $f, g : K \to X$ *homotopic* iff there exists a map $K \times \Delta^1 \to X$ whose restrictions to $K \times 0$ and $K \times 1$ coincide with f and g, respectively. Use (1.4.16) to show that homotopy is an equivalence relation on the set of maps $K \to X$. Conclude that Kan complexes and homotopy classes of maps form a category, denoted hSpc. A map of Kan complexes is called a *homotopy equivalence* iff it is an isomorphism in hSpc. A Kan complex is called *contractible* when it is homotopy equivalent to a point $*$.

1.4.19 Exercise. Let $X \to Y$ be a Kan fibration. Associate to any edge $y \to y'$ in Y a map $X_y \to X_{y'}$ by lifting the pair $X_y \times (\Delta^1, 0)$, and show that this map is well defined up to homotopy. Show that for any 2-simplex in Y with vertices y, y', y'', the resulting triangle commutes up to homotopy. Show that the map $X_y \to X_{y'}$ associated to an edge $y \to y'$ is a homotopy equivalence (a homotopy inverse may be constructed by lifting $X_{y'} \times (\Delta^1, 1)$). Conclude that this defines a diagram $Y \to \mathsf{hSpc}_{\simeq}$.

1.4.20 Definition (Trivial Kan fibration)**.** A map of simplicial sets is called a *trivial Kan fibration* iff it satisfies the right lifting property for every pair $(\Delta^n, \partial\Delta^n)$. A simplicial set is called a *trivial Kan complex* iff the map $X \to *$ is a trivial Kan fibration.

1.4.21 Exercise. Show that a trivial Kan fibration is a Kan fibration. In fact, show that a trivial Kan fibration satisfies the right lifting property for every pair (X, A).

1.4.22 Exercise. Show that a Kan complex is trivial iff it is contractible.

1.4.23 Exercise (Functor $\mathsf{sSet} \to \mathsf{hSpc}$)**.** Show that if $X \hookrightarrow Y$ is filtered by pushouts of horns and Z is a Kan complex, then the map $\underline{\mathrm{Hom}}(Y, Z) \to \underline{\mathrm{Hom}}(X, Z)$ is a trivial Kan fibration. Use the small object argument (**??**) to show that for every simplicial set X, there exists an inclusion $X \hookrightarrow \overline{X}$ which is filtered by pushouts of horns with $\overline{X}$ a Kan complex. Show that for any pair of such inclusions $X \hookrightarrow \overline{X}$ and $Y \hookrightarrow \overline{Y}$ and any map $X \to Y$, there exists a dotted arrow making the following diagram commute

$$\begin{tikzcd} X \arrow[r, hook] \arrow[d] & \overline{X} \arrow[d, dashed] \\ Y \arrow[r, hook] & \overline{Y} \end{tikzcd} \tag{1.4.23.1}$$

and that moreover this dotted arrow is unique up to homotopy rel X. Show that sending X to (any choice of) $\overline{X}$ and sending a map $X \to Y$ to (any choice of) extension $\overline{X} \to \overline{Y}$ gives a well defined functor $\mathsf{sSet} \to \mathsf{hSpc}$. A map of simplicial sets which is sent to an isomorphism by this functor is called a *homotopy equivalence*. Show that any inclusion of simplicial sets which is filtered by pushouts of horns is a homotopy equivalence.

1.4.24 Exercise. Show that a trivial Kan fibration is a homotopy equivalence.

* *

1.4.25 Definition (Differential graded category)**.** Let k be a field. A k-linear *differential graded category* (or *dg-category*) C is a category enriched over complexes of vector spaces over k. In other words, it consists of a set C of objects, a *morphism complex* $\operatorname{Hom}(X,Y) \in \mathsf{Kom}(\mathsf{Vect}_k)$ for each $X, Y \in \mathsf{C}$, and composition maps $\operatorname{Hom}(X,Y) \otimes \operatorname{Hom}(Y,Z) \to \operatorname{Hom}(X,Z)$ which are associative and unital.

When discussing dg-categories, we implicitly fix a choice of ground field k.

1.5 ∞-categories

An ∞-category is a generalization of a category. In an ∞-category, the morphisms from one object to another form a *space*, and composition is associative *up to coherent homotopy*. There are various different ways of turning this slogan into a precise mathematical definition; such a definition is termed a 'model' for the theory of ∞-categories. We use here the model known as *quasi-categories*. Quasi-categories were introduced by Bordman–Vogt [8], and their development into a working model of ∞-categories is due to Joyal [36, 37, 38] and Lurie [58].

In this section, we develop the basics of the theory of ∞-categories in elementary, intentionally unsophistocated, terms. References include Lurie [58], Riehl–Verity [78], and Land [50].

1.5.1 Definition (Inner/outer/left/right horn)**.** A horn $\Lambda^n_i \subseteq \Delta^n$ (1.4.1) is called *inner* when $0 < i < n$ and *outer* when $i \in \{0, n\}$. It is called *left* (resp. *right*) when $0 \leq i < n$ (resp. $0 < i \leq n$).

⋆ **1.5.2 Definition** (∞-category)**.** An ∞*-category* is a simplicial set which has the extension property (1.4.7) for all inner horns (1.5.1).

1.5.3 Remark (∞-categories vs quasi-categories)**.** A simplicial set satisfying the extension property for all inner horns is also called a *weak Kan complex* [8] or *quasi-category* [36], while the term '∞-category' may also refer to other sorts of structures (for example fibrant simplicial categories (1.5.8)) which make precise the one-sentence slogan 'definition' of an ∞-category at the beginning of this section (1.5). This terminological distinction allows one to formulate the thesis that quasi-categories are a 'model' of ∞-categories (like the Church–Turing thesis, this is not something which can be formally proven, rather only supported with evidence such as equivalences between various different reasonable models).

Let us see how categories are a special case of ∞-categories.

⋆ **1.5.4 Exercise** (Nerve of a category)**.** Let C be a category. The *nerve* of C is the simplicial set whose set of n-simplices is the set of functors the poset category $[n] = (0 \to \cdots \to n)$ to C. Show that a simplicial set is the nerve of a category iff every inner horn has a *unique* filling. In particular, conclude that the nerve of any category is an ∞-category.

We will henceforth identify a category with its nerve without further comment. Once we define equivalences of ∞-categories, it will become evident that this identification respects the principle of equivalence (note that the set of n-simplices of the nerve of a category is evidently *not* invariant under equivalence).

⋆ **1.5.5 Definition** (Objects, morphisms, and composition in an ∞-category)**.** An *object* x of an ∞-category C is a vertex of C, and a *morphism* $f : x \to y$ is an edge in C. The *identity morphism* $\mathbf{1}_x$ of an object $x \in \mathsf{C}$ is the degenerate edge over x. A 2-simplex in C with boundary

$$\begin{tikzcd} & y \arrow[dr, "g"] & \\ x \arrow[ur, "f"] \arrow[rr, "h"] & & z \end{tikzcd} \tag{1.5.5.1}$$

should be thought of as a *homotopy* between the ‘composition of f and g’ (which is not itself a morphism in C since it is not an edge) and h. A given horn Λ^2_1 typically has many different fillings to Δ^2, so we cannot call h ‘the’ composition of f and g (merely ‘a’ composition). The higher horn filling conditions do imply, however, that extending a given map $\Lambda^2_1 \to \mathsf{C}$ to Δ^2 is a contractible choice (precisely, $\operatorname{Hom}(\Delta^2, \mathsf{C}) \to \operatorname{Hom}(\Lambda^2_1, \mathsf{C})$ is a trivial Kan fibration (1.5.21)). They also encode the data to guarantee that composition is, in a certain sense, associative up to coherent homotopy.

1.5.6 Definition (Opposite ∞-category)**.** Given an ∞-category C, its opposite C^{op} is the opposite simplicial set (i.e. its pre-composition with $\mathsf{op} : \boldsymbol{\Delta} \to \boldsymbol{\Delta}$).

1.5.7 Definition (Full subcategory)**.** A *full subcategory* of an ∞-category C is a subcomplex $\mathsf{A} \subseteq \mathsf{C}$ with the property that a simplex $\Delta^n \to \mathsf{C}$ belongs to A iff all of its vertices belong to A. Full subcategories of C are evidently in bijection with subsets of C_0.

Here are some common constructions of ∞-categories.

1.5.8 Definition (Fibrant simplicial category)**.** A *fibrant simplicial category* is a category C enriched (1.1.97) in Kan complexes.

1.5.9 Definition (Nerve of a fibrant simplicial category; Cordier [9][58, 1.1.5])**.** The *(simplicial) nerve* of a fibrant simplicial category C is the simplicial set in which an n-simplex is a tuple of objects $X_0, \ldots, X_n \in \mathsf{C}$ along with maps $f_{ij} : (\Delta^1)^{\{i+1,\ldots,j-1\}} \to \mathsf{C}(X_i, X_j)$ satisfying $f_{ik}|_{\{t_j=1\}} = f_{ij} \times f_{jk}$, which we may express as commutativity of the following diagram:

$$\begin{tikzcd}
(\Delta^1)^{\{i+1,\ldots,j-1\}} \times (\Delta^1)^{\{j+1,\ldots,k-1\}} \arrow[r, "f_{ij}\times f_{jk}"] \arrow[d, "\times\{1\}^{\{j\}}"] & \mathsf{C}(X_i, X_j) \times \mathsf{C}(X_j, X_k) \arrow[d] \\
(\Delta^1)^{\{i+1,\ldots,k-1\}} \arrow[r, "f_{ik}"] & \mathsf{C}(X_i, X_k)
\end{tikzcd} \tag{1.5.9.1}$$

The pullback of such data along a map $s : \Delta^m \to \Delta^n$ is given by $Y_i = X_{s(i)}$ and $g_{ij} = f_{s(i)s(j)}$ pre-composed with the map $(\Delta^1)^{\{i+1,\ldots,j-1\}} \to (\Delta^1)^{\{s(i)+1,\ldots,s(j)-1\}}$ given on vertices by the formula $t_k = \max_{s(a)=k} t_a$ (interpreted to be 0 when $s^{-1}(k)$ is empty).

1.5.10 Exercise. Describe explicitly the 0-simplices (objects), 1-simplices (morphisms), and 2-simplices of the nerve of a fibrant simplicial category C. Consider the subcomplex of the nerve consisting of those simplices in which every f_{ij} is constant; how is this related to C?

1.5.11 Lemma. *The simplicial nerve of a fibrant simplicial category is an ∞-category.*

Proof. The extension problem for maps from an inner horn (Δ^n, Λ^n_i) to the simplicial nerve of C amounts to the extension problem for

$$f_{0n} : (\Delta^1, \partial\Delta^1)^{\{1,\ldots,i-1\}} \wedge (\Delta^1, \{1\})^i \wedge (\Delta^1, \partial\Delta^1)^{\{i+1,\ldots,n-1\}} \to \mathsf{C}(X_0, X_n). \tag{1.5.11.1}$$

This extension problem is solvable since $\mathsf{C}(X_0, X_n)$ is Kan and the domain pair is filtered by pushouts of horns (1.4.16). $\square$

⋆ **1.5.12 Example** (∞-category of spaces Spc)**.** The category of Kan complexes is naturally enriched over the category of Kan complexes (1.3.8)(1.4.17). Its simplicial nerve is called the *∞-category of spaces*, denoted Spc.

1.5.13 Definition (Differential graded category)**.** A $\mathbb{Z}$-linear *differential graded category* (or *dg-category*) C is a category enriched over complexes of $\mathbb{Z}$-modules. In other words, it consists of a set C of objects, a *morphism complex* $\operatorname{Hom}(X,Y) \in \mathsf{Kom}(\mathsf{Mod}_{\mathbb{Z}})$ for each $X, Y \in \mathsf{C}$, and composition maps $\operatorname{Hom}(X,Y) \otimes \operatorname{Hom}(Y,Z) \to \operatorname{Hom}(X,Z)$ which are associative and unital.

1.5.14 Definition (Nerve of a differential graded category [31, A.2.1][59, 1.3.1])**.** The *(differential graded) nerve* of a ($\mathbb{Z}$-linear) dg-category C is the simplicial set in which an n-simplex is a tuple of objects $X_0, \ldots, X_n \in \mathsf{C}$ along with maps $f_{ij} : C_\bullet^{\mathrm{cell}}((\Delta^1)^{\{i+1,\ldots,j-1\}}) \to \mathsf{C}(X_i, X_j)$ satisfying $f_{ik}|_{\{t_j=1\}} = f_{ij} \times f_{jk}$, as in (1.5.9).

1.5.15 Exercise. Show that the nerve of a differential graded category is an ∞-category (compare (1.5.11)).

* *

⋆ **1.5.16 Definition** (Functor)**.** A functor of ∞-categories $\mathsf{C} \to \mathsf{D}$ is a map of simplicial sets. Functors from C to D are the objects of an ∞-category $\mathsf{Fun}(\mathsf{C},\mathsf{D}) = \underline{\operatorname{Hom}}(\mathsf{C},\mathsf{D})$ (the simplicial mapping space (1.3.8)).

⋆ **1.5.17 Definition** (Diagram)**.** Let K be a simplicial set. A K-shaped diagram in an ∞-category C is a map of simplicial sets $K \to \mathsf{C}$. Such diagrams form an ∞-category $\mathsf{Fun}(K,\mathsf{C}) = \underline{\operatorname{Hom}}(K,\mathsf{C})$ (1.5.19).

1.5.18 Exercise. For any simplicial set K and any category C, show that $\mathsf{Fun}(K,\mathsf{C})$ is (the nerve of) the category of K-shaped diagrams in C from (1.1.51). In particular, conclude that for categories C and D, the category of functors $\mathsf{Fun}(\mathsf{C},\mathsf{D})$ defined here (1.5.16) coincides with that defined earlier (1.1.22).

1.5.19 Proposition. *$\mathsf{Fun}(K,\mathsf{C})$ is an ∞-category for any ∞-category C.*

Proof. This is very similar to (1.4.17). We are to show that C satisfies the extension property for pairs $(\Delta^n, \Lambda^n_i) \wedge K$ with $0 < i < n$. By filtering K by pushouts of pairs $(\Delta^k, \partial\Delta^k)$ (**??**), it suffices to show that C satisfies the extension property for pairs $(\Delta^n, \Lambda^n_i) \wedge (\Delta^k, \partial\Delta^k)$ with $0 < i < n$ and $k \geq 0$. It thus suffices to show that for $0 < i < n$, the smash product $(\Delta^n, \Lambda^n_i) \wedge (\Delta^k, \partial\Delta^k)$ is filtered by pushouts of inner horns. We verify this property next (1.5.20) (stated separately for later use). □

⋆ **1.5.20 Lemma.** *The smash product $(\Delta^n, \Lambda^n_i) \wedge (\Delta^k, \partial\Delta^k)$ with $0 < i < n$ is filtered by pushouts of inner horns.*

Proof. We saw earlier that $(\Delta^n, \Lambda^n_i) \wedge (\Delta^k, \partial\Delta^k)$ is filtered by pushouts of horns (Δ^m, Λ^m_j) (1.4.16). Let us argue that all the horns (Δ^m, Λ^m_j) appearing in this filtration are inner. The cone point $j \in \Delta^m$ of every such horn is the vertex v in (1.4.16.1); in particular, it projects to the cone point $i \in \Delta^n$. The image of the map $\Delta^m \to \Delta^n$ thus both contains i and cannot be contained in Λ^n_i, which together imply that $\Delta^m \twoheadrightarrow \Delta^n$ is in fact surjective. Thus $0 < i < n$ implies $0 < j < m$. □

1.5.21 Exercise. Show that for any ∞-category C, the map $\mathsf{Fun}(\Delta^n, \mathsf{C}) \to \mathsf{Fun}(\Lambda^n_i, \mathsf{C})$ is a trivial Kan fibration for any inner horn (Δ^n, Λ^n_i).

⋆ **1.5.22 Definition** (Inner fibration)**.** A map of simplicial sets is called an *inner fibration* when it satisfies the right lifting property with respect to inner horns.

1.5.23 Exercise. Show that for any inner fibration $Q \to X$, the simplicial set of sections $\mathsf{Sec}(X, Q)$ (a map $Z \to \mathsf{Sec}(X, Q)$ being a map $Z \times X \to Q$ over X) is an ∞-category.

* *

1.5.24 Definition (Homotopy category of an ∞-category)**.** Let C be an ∞-category. For objects $x, y \in \mathsf{C}$, the relation of *right homotopy* on the set of morphisms $x \to y$ is defined by $e \sim e'$ iff there exists a 2-simplex

$$\begin{tikzcd} & y \arrow[dr, "\mathbf{1}_y"] & \\ x \arrow[ur, "e"] \arrow[rr, "e'"] & & y \end{tikzcd} \tag{1.5.24.1}$$

Right homotopy is an equivalence relation: reflexivity holds by taking a degenerate 2-simplex over e, and symmetry and transitivity follow from the following two inner horn fillings (the boxed vertex is the cone point of the horn):

$$\begin{tikzcd} & \boxed{y} \arrow[r, "\mathbf{1}_y"] \arrow[drr, "\mathbf{1}_y"'] & y \arrow[dr, "\mathbf{1}_y"] & \\ x \arrow[ur, "e"] \arrow[urr, "e'"'] \arrow[rrr, "e"'] & & & y \end{tikzcd} \qquad \begin{tikzcd} & y \arrow[r, "\mathbf{1}_y"] \arrow[drr, "\mathbf{1}_y"'] & \boxed{y} \arrow[dr, "\mathbf{1}_y"] & \\ x \arrow[ur, "e"] \arrow[urr, "e'"'] \arrow[rrr, "e''"'] & & & y \end{tikzcd} \tag{1.5.24.2}$$

There is a corresponding equivalence relation *left homotopy*. Right homotopy implies left homotopy by filling the following inner horn (so by symmetry the converse is true as well)

$$\begin{tikzcd} & x \arrow[r, "e"] \arrow[drr, "e"'] & \boxed{y} \arrow[dr, "\mathbf{1}_y"] & \\ x \arrow[ur, "\mathbf{1}_x"] \arrow[urr, "e"'] \arrow[rrr, "e'"'] & & & y \end{tikzcd} \tag{1.5.24.3}$$

Since right homotopy and left homotopy are the same, we may simply call this relation *homotopy* of morphisms $x \to y$. Filling the following inner horn

$$\begin{tikzcd}
& \boxed{y} \arrow[r, "b"] \arrow[dr, "b'"] & z \arrow[d, "\mathbf{1}_z"] \\
x \arrow[ur, "a"] \arrow[urr, "c"] \arrow[rr, "c'"'] & & z
\end{tikzcd} \tag{1.5.24.4}$$

shows that if b and b' are homotopic, then any two fillings of $x \xrightarrow{a} y \xrightarrow{b} z$ and $x \xrightarrow{a} y \xrightarrow{b'} z$ give homotopic morphisms $x \to z$. By symmetry, we conclude that composition is well-defined on homotopy classes. The *homotopy category* hC has the same objects as C (i.e. the vertices of C) and has morphisms the homotopy classes of morphisms in C. There is a tautological functor $\mathsf{C} \to \mathsf{hC}$.

1.5.25 Exercise. Show that the homotopy category of a category is itself.

1.5.26 Exercise. Show that a functor $\mathsf{C} \to \mathsf{D}$ induces a functor $\mathsf{hC} \to \mathsf{hD}$. Show that the natural map $\mathsf{h}(\mathsf{C} \times \mathsf{D}) \to \mathsf{hC} \times \mathsf{hD}$ is an isomorphism. Conclude that a natural transformation $F \to G$ of functors $\mathsf{C} \to \mathsf{D}$ induces a natural transformation $\mathsf{h}F \to \mathsf{h}G$ of functors $\mathsf{hC} \to \mathsf{hD}$.

1.5.27 Exercise. Show that the homotopy category of ∞-category Spc (1.5.12) is the category denoted hSpc in (1.4.18). More generally, describe the homotopy category of (the nerve (1.5.9) of) a fibrant simplicial category.

1.5.28 Lemma. *The functor* $\mathsf{C} \to \mathsf{hC}$ *satisfies the right lifting property with respect to the pair* $(\Delta^2, \partial\Delta^2)$.

Proof. Fill the inner horn

$$\begin{tikzcd}
& y \arrow[r, "b"] \arrow[dr, "b"] & \boxed{z} \arrow[d, "\mathbf{1}_z"] \\
x \arrow[ur, "a"] \arrow[urr, "c'"] \arrow[rr, "c"'] & & z
\end{tikzcd} \tag{1.5.28.1}$$

to see that a 2-simplex with edges a, b, c exists in C iff the boundary commutes in hC. □

⋆ **1.5.29 Definition** (Isomorphism in an ∞-category)**.** A morphism in an ∞-category C is called an isomorphism (resp. split monomorphism, split epimorphism) iff its image in the homotopy category hC is.

1.5.30 Exercise. Show that a functor of ∞-categories sends isomorphisms to isomorphisms.

1.5.31 Exercise. As a continuation of (1.5.26), show that a natural isomorphism $F \to G$ of functors $\mathsf{C} \to \mathsf{D}$ induces a natural isomorphism $\mathsf{h}F \to \mathsf{h}G$ of functors $\mathsf{hC} \to \mathsf{hD}$.

1.5.32 Definition (Property of morphisms in an ∞-category)**.** A property of morphisms in an ∞-category C is a property of morphisms in its homotopy category hC.

* *

1.5.33 Definition (Join)**.** For simplicial sets X and Y, their join $X \star Y$ is defined by the universal property that map $Z \to X \star Y$ is a map $p : Z \to \Delta^1$ and a pair of maps $p^{-1}(0) \to X$ and $p^{-1}(1) \to Y$.

1.5.34 Exercise. Show that $(X \star Y)^{\mathsf{op}} = Y^{\mathsf{op}} \star X^{\mathsf{op}}$.

1.5.35 Exercise. Show that $\Delta^n \star \Delta^m = \Delta^{n+m+1}$ for $n, m \geq -1$ (where $\Delta^{-1} = \varnothing$).

1.5.36 Exercise. Show that the set of non-degenerate or empty simplices of $X \star Y$ is the product of the sets of non-degenerate or empty simplices of X and Y.

⋆ **1.5.37 Definition** (Right and left cone)**.** The right and left cones of a simplicial set K are the joins $K^{\triangleright} = K \star \Delta^0$ and $K^{\triangleleft} = \Delta^0 \star K$, respectively.

1.5.38 Exercise. Prove that the geometric realization of $K^{\triangleright}$ is contractible for every simplicial set K.

⋆ **1.5.39 Definition** (Slice category)**.** Given a diagram $K \to \mathsf{C}$, the over-category $\mathsf{C}_{/K}$ is defined by the universal property that a map $Z \to \mathsf{C}_{/K}$ is a map $Z \star K \to \mathsf{C}$ extending the given map $K \to \mathsf{C}$. Dually, the under-category $\mathsf{C}_{K/}$ represents extensions to $K \star Z \to \mathsf{C}$.

1.5.40 Definition (Join of pairs)**.** The join of simplicial set pairs is

$$X \star (Y, B) = (X \star Y, X \star B), \tag{1.5.40.1}$$
$$(X, A) \star (Y, B) = (X \star Y, (X \star B) \cup_{A \star B} (A \star Y)). \tag{1.5.40.2}$$

Beware that join of pairs is not compatible with identifying X and $(X, \varnothing)$.

⋆ **1.5.41 Definition** (Left and right fibrations)**.** A map of simplicial sets is called a *left* (resp. *right*) *fibration* when it satisfies the right lifting property with respect to left (resp. right) horns (Δ^n, Λ^n_i), namely $0 \leq i < n$ (resp. $0 < i \leq n$) (1.5.1).

A left fibration over a simplicial set X is 'equivalent' in a certain sense to a diagram $X \to \mathsf{Spc}$ (1.5.53)(**??**). The proof of this will come quite a bit later, so for the moment we will regard it just as intuition.

1.5.42 Exercise. Show that a left fibration of ∞-categories reflects split monomorphisms, hence reflects isomorphisms (note the use of (1.5.28)).

1.5.43 Exercise. Show that for any diagram $L \to \mathsf{C}$ and any monomorphism $K \to L$, the right lifting property for $\mathsf{C}_{/L} \to \mathsf{C}_{/K}$ with respect to a pair (X, A) is equivalent to the extension property for maps $(X, A) \star (L, K) \to \mathsf{C}$. Conclude from (**??**) and (**??**) that the restriction map $\mathsf{C}_{/L} \to \mathsf{C}_{/K}$ is a right fibration. Conclude moreover that if $K \to L$ is filtered by pushouts of left horns, then $\mathsf{C}_{/L} \to \mathsf{C}_{/K}$ is a trivial Kan fibration.

* *

⋆ **1.5.44 Lemma** (Isomorphism as an extension property; Joyal [36])**.** *A morphism e in an ∞-category is an isomorphism iff every left outer horn $\Lambda^n_0 \subseteq \Delta^n$ with* 01 *edge e can be filled.*

Proof. If $f : x \to y$ satisfies the hypothesized horn filling condition for $n = 2, 3$, then filling the following two horns produces an inverse g to f in hC.

$$\begin{tikzcd} & y \arrow[dr, dashed, "\exists g"] & \\ \boxed{x} \arrow[ur, "f"] \arrow[rr, "\mathbf{1}_x"'] & & x \end{tikzcd} \qquad \begin{tikzcd} & y \arrow[r, "g"] \arrow[drr, "\mathbf{1}_y"] & x \arrow[dr, "f"] & \\ \boxed{x} \arrow[ur, "f"] \arrow[urr, "\mathbf{1}_x"'] \arrow[rrr, "f"'] & & & y \end{tikzcd} \tag{1.5.44.1}$$

Conversely, let us show that every map $\Lambda^n_0 \to \mathsf{C}$ with 01 edge an isomorphism extends to Δ^n. The extension problem for $(\Delta^n, \Lambda^n_0) \to \mathsf{C}$ is equivalent to the lifting property

$$\begin{tikzcd} \Lambda^1_0 \arrow[r] \arrow[d] & \mathsf{C}_{/\Delta^{n-2}} \arrow[d] \\ \Delta^1 \arrow[r] \arrow[ur, dashed] & \mathsf{C}_{/\partial\Delta^{n-2}} \end{tikzcd} \tag{1.5.44.2}$$

in view of the identity $(\Delta^n, \Lambda^n_0) = (\Delta^1, \Lambda^1_0) \star (\Delta^{n-2}, \partial\Delta^{n-2})$ (**??**). Since $\mathsf{C}_{/\partial\Delta^{n-2}} \to \mathsf{C}$ is a right fibration (1.5.43), it reflects isomorphisms (1.5.42), so the bottom edge in (1.5.44.2) is an isomorphism. Now the map $\mathsf{C}_{/\Delta^{n-2}} \to \mathsf{C}_{/\partial\Delta^{n-2}}$ is a right fibration (1.5.43), so it suffices to show that for any right fibration of ∞-categories $\mathsf{A} \to \mathsf{B}$, the lifting problem

$$\begin{tikzcd} * \arrow[r] \arrow[d, "0"'] & \mathsf{A} \arrow[d] \\ \Delta^1 \arrow[r] \arrow[ur, dashed] & \mathsf{B} \end{tikzcd} \tag{1.5.44.3}$$

has a solution provided the bottom arrow is an isomorphism in B (in fact, it need only be a split monomorphism). Since the edge $e : \Delta^1 \to \mathsf{B}$ is a split monomorphism in B, there exists by (1.5.28) a map $\Delta^2 \to \mathsf{B}$ in which the 02 edge is degenerate and the 01 edge is e. The degenerate edge certainly lifts to A, so it suffices to solve the lifting problem for the pair $(\Delta^2, 02)$, which is filtered by pushouts of right horns. □

1.5.45 Definition (∞-groupoid)**.** An ∞-*groupoid* is an ∞-category in which every morphism is an isomorphism (by (1.5.44), this is equivalent to being a Kan complex).

1.5.46 Definition (Core of an ∞-category)**.** For an ∞-category C, its core is the subcomplex $\mathsf{C}_{\simeq} \subseteq \mathsf{C}$ defined as those simplices all of whose edges are isomorphisms. A functor $\mathsf{C} \to \mathsf{D}$ evidently restricts to a functor $\mathsf{C}_{\simeq} \to \mathsf{D}_{\simeq}$.

The characterization of isomorphisms in an ∞-category by an extension property (1.5.44) leads naturally to the notion of a 'marked simplicial set'.

1.5.47 Definition (Marked simplicial set)**.** A *marked simplicial set* is a pair (X, S) consisting of a simplicial set X and a set $S \subseteq X_1$ of its edges (called the 'marked edges') containing all degenerate edges. A morphism of marked simplicial sets $(X, S) \to (X', S')$ is a morphism of simplicial sets $X \to X'$ which sends every marked edge of X to a marked edge of X'. The category of marked simplicial sets is denoted sSet^+.

By default, a simplicial set X will be regarded as being equipped with the *trivial marking*, consisting of only the degenerate edges, unless specified otherwise (this defines a fully faithful functor $\mathsf{sSet} \hookrightarrow \mathsf{sSet}^+$); for emphasis, the trivial marking is also denoted $X^\flat$. We denote by $X^\sharp$ the simplicial set X with all its edges marked. Note that the product of marked simplicial sets $(X, S) \times (X', S')$ is the product of underlying simplicial sets $X \times X'$ with a marking of those edges whose images in X and X' are both marked.

1.5.48 Definition (Marked horn)**.** The *marked horn* $(\Delta^n, \Lambda^n_i)^\sim$ is the usual horn (Δ^n, Λ^n_i) with a marking of the edge 01 if $i = 0$ and of the edge $(n-1, n)$ if $i = n$.

1.5.49 Example (Marking isomorphisms in an ∞-category)**.** Let C be an ∞-category. We denote by $\mathsf{C}^\natural$ the result of marking all the isomorphisms in C. Thus (1.5.44) says that $\mathsf{C}^\natural$ satisfies the extension property with respect to all marked horns. Conversely, if a marked simplicial set (X, S) satisfies the extension property with respect to all marked horns, then X is an ∞-category and every marked edge is an isomorphism (though S need not contain all isomorphisms).

1.5.50 Proposition (Isomorphisms in diagram categories)**.** *The functor* $\mathsf{Fun}(K, \mathsf{C}) \to \mathsf{Fun}(K_0, \mathsf{C}) = \prod_{k \in K} \mathsf{C}$ *reflects isomorphisms.*

Proof. We seek to show the extension property for maps $(\Delta^n, \Lambda^n_0) \to \mathsf{Fun}(K, \mathsf{C})$ in which the image of the edge 01 in $\mathsf{Fun}(K_0, \mathsf{C}) = \prod_{k \in K} \mathsf{C}$ is an isomorphism. Equivalently, this is the extension property for maps of marked simplicial sets $(\Delta^n, \Lambda^n_0)^\sim \wedge K \to \mathsf{C}^\natural$. It thus suffices to show that the smash product $(\Delta^n, \Lambda^n_0)^\sim \wedge (\Delta^k, \partial\Delta^k)$ is filtered by pushouts of marked horns. We verify this property next (1.5.51) (stated separately for later use). □

⋆ **1.5.51 Lemma.** *The smash product* $(\Delta^n, \Lambda^n_0)^\sim \wedge (\Delta^k, \partial\Delta^k)$ *is filtered by pushouts of marked left horns.*

Proof. This argument is similar to (1.5.20).

We saw earlier that $(\Delta^n, \Lambda^n_0) \wedge (\Delta^k, \partial\Delta^k)$ is filtered by pushouts of horns (Δ^m, Λ^m_j) (1.4.16). Let us argue that all the horns (Δ^m, Λ^m_j) appearing in this filtration are marked left horns. The cone point $j \in \Delta^m$ of every such horn is the vertex v in (1.4.16.1); in particular, it projects to the cone point $0 \in \Delta^n$. The image of the map $\Delta^m \to \Delta^n$ thus both contains 0 and cannot be contained in Λ^n_0, which together imply that $\Delta^m \twoheadrightarrow \Delta^n$ is in fact surjective. This implies $0 \leq j < m$.

Let us now further show that in the case $j = 0$, the edge $01 \subseteq \Delta^m$ is marked in the product $(\Delta^n, \Lambda^n_0)^\sim \wedge (\Delta^k, \partial\Delta^k)$. Property (1.4.16.3) says that $\Delta^m \to \Delta^k$ must be surjective, so if the image of j in Δ^k (i.e. the vertical coordinate of v) is >0, then the horn (Δ^m, Λ^m_j) is inner. Thus $j=0$ occurs precisely when $v=(0,0)$. By definition of v, this means the lattice path in question (corresponding to σ) contains $(1,0)\in\Delta^n\times\Delta^k$. Now properties (1.4.16.2) and (1.4.16.4) together imply that $\Delta^m\subseteq\Delta^n\times\Delta^k$ must contain this point $(1,0)$. We conclude that the edge $01\subseteq\Delta^m$ is the product of the edge $01\subseteq\Delta^n$ and the degenerate edge over $0\in\Delta^k$, and hence is marked in the product $(\Delta^n,\Lambda^n_0)^\sim\wedge(\Delta^k,\partial\Delta^k)$. □

1.5.52 Exercise. Show that for any inner fibration $Q\to X$, the functor $\mathsf{Sec}(Q/X)\to\prod_{x\in X}Q_x$ reflects isomorphisms.

1.5.53 Exercise. Let $X\to Y$ be a left fibration. Associate to any edge $y\to y'$ in Y a map $X_y\to X_{y'}$ by lifting the pair $X_y\times(\Delta^1,0)$, and show that this map is well defined up to homotopy. Show that for any 2-simplex in Y with vertices y,y',y'', the resulting triangle commutes up to homotopy. Conclude that this defines a diagram $Y\to\mathsf{hSpc}$.

1.5.54 Exercise (Alternative model for slice categories)**.** Let C be an ∞-category, and let $c\in\mathsf{C}$ be an object. Recall that the slice category $\mathsf{C}_{/c}$ is defined by the property that map $Z\to\mathsf{C}_{/c}$ from a simplicial set Z is the same as a map $Z^\triangleright\to\mathsf{C}$ sending the cone point to c. Define an 'alternative model' slice category $\mathsf{C}^{/c}$ by the property that a map $Z\to\mathsf{C}^{/c}$ is a map $Z\times\Delta^1\to\mathsf{C}$ sending $Z\times 1$ to c.

Let us show that $\mathsf{C}_{/c}$ and $\mathsf{C}^{/c}$ are equivalent over C. We construct a simplicial set Q with trivial Kan fibrations $\mathsf{C}_{/c}\leftarrow Q\to\mathsf{C}^{/c}$ over C. Define Q by the property that a map $Z\to Q$ is a map $(Z\times\Delta^1)^\triangleright\to\mathsf{C}$ sending $(Z\times 1)^\triangleright$ to c. Now Q maps to $\mathsf{C}_{/c}$ and $\mathsf{C}^{/c}$ by restricting to $(Z\times 0)^\triangleright$ and $Z\times\Delta^1$, respectively.

The map $Q\to\mathsf{C}^{/c}$ being a trivial Kan fibration amounts to the extension property for maps $((\Delta^k,\partial\Delta^k)\wedge(\Delta^1,1))\star(*,\varnothing)\to\mathsf{C}$. This extension property holds since this pair is filtered by pushouts of inner horns (1.5.51)(**??**).

The map $Q\to\mathsf{C}_{/c}$ being a trivial Kan fibration amounts to the extension property for maps $((\Delta^k,\partial\Delta^k)\wedge(\Delta^1,\partial\Delta^1))^\triangleright\to\mathsf{C}$ sending $(\Delta^k\times 1)^\triangleright$ to c. This extension property holds since this pair is filtered by pushouts of right horns whose marked edge maps to c (**??**)(**??**).

1.5.55 Example (Inverting an isomorphism)**.** Given an isomorphism e in an ∞-category C, in what sense is its inverse e^{-1} defined and unique, and in what sense is $(e^{-1})^{-1}=e$? Here is one possible answer to this question.

Let Iso denote the category with two objects a and b and a single morphism between any pair of objects (thus a and b are isomorphic). Given an ∞-category C, a functor

$$\mathsf{Iso}\to\mathsf{C} \tag{1.5.55.1}$$

a describes a pair of (homotopy coherently) inverse morphisms in C. Note that this picture is symmetric via the obvious involution of the category Iso exchanging the objects a and b. Now to express mathematically the claim that an isomorphism in C has a homotopically unique inverse, let us argue that the restriction map

$$\mathsf{Fun}(\mathsf{Iso},\mathsf{C})\to\mathsf{Fun}(\Delta^1,\mathsf{C}) \tag{1.5.55.2}$$

is a trivial Kan fibration over the full subcategory of $\mathsf{Fun}(\Delta^1, \mathsf{C})$ spanned by the isomorphisms in C. Note that a map from $Z \in \mathsf{sSet}$ to this full subcategory is a map of marked simplicial sets $Z \times (\Delta^1)^\sharp \to \mathsf{C}^\natural$. The desired trivial Kan fibration property thus amounts to the extension property for maps $(\mathsf{Iso}^\natural, (\Delta^1)^\sharp) \wedge (\Delta^k, \partial\Delta^k) \to \mathsf{C}^\natural$. It thus suffices by (1.5.51) to filter $(\mathsf{Iso}^\natural, (\Delta^1)^\sharp)$ by pushouts of marked horns. The nerve of Iso has precisely two non-degenerate simplices of every dimension. Let $\mathsf{Iso}_k \subseteq \mathsf{Iso}$ denote the $(k-1)$-skeleton union either one of the non-degenerate k-simplices (doesn't matter which). Now the pullback of $\mathsf{Iso}_k \subseteq \mathsf{Iso}$ under the inclusion of a non-degenerate $(k+1)$-simplex into Iso is an outer horn (inspection). The pair $(\mathsf{Iso}_{k+1}, \mathsf{Iso}_k)$ is thus a pushout of an outer horn, so $(\mathsf{Iso}, \Delta^1) = (\mathsf{Iso}, \mathsf{Iso}_1)$ is filtered by pushouts of outer horns (which are moreover marked since all morphisms in Iso are isomorphisms).

* *

We now recall how the homotopy category of an ∞-category is naturally enriched (1.1.97) over the homotopy category of spaces hSpc (1.4.18).

1.5.56 Definition (Mapping space Hom_C)**.** Given objects $x, y \in \mathsf{C}$, the *mapping space* $\mathrm{Hom}_\mathsf{C}(x, y) \in \mathsf{hSpc}$ has a few different presentations as explicit Kan complexes.

(1.5.56.1) $\mathrm{Hom}^{\mathrm{cyl}}_\mathsf{C}(x, y)$ is defined by the property that a map $Z \to \mathrm{Hom}^{\mathrm{cyl}}_\mathsf{C}(x, y)$ is a map $Z \times \Delta^1 \to \mathsf{C}$ sending $Z \times 0$ to x and sending $Z \times 1$ to y.

(1.5.56.2) $\mathrm{Hom}^{\mathrm{R}}_\mathsf{C}(x, y)$ is defined by the property that a map $Z \to \mathrm{Hom}^{\mathrm{R}}_\mathsf{C}(x, y)$ is a map $Z^\triangleright \to \mathsf{C}$ sending Z to x and sending the cone point to y (and dually $\mathrm{Hom}^{\mathrm{L}}_\mathsf{C}(x, y)$ is defined via maps $Z^\triangleleft \to \mathsf{C}$).

Note that $\mathrm{Hom}^{\mathrm{R}}_\mathsf{C}(x, y)$ is the fiber of $\mathsf{C}_{/y} \to \mathsf{C}$ over x, while $\mathrm{Hom}^{\mathrm{cyl}}_\mathsf{C}(x, y)$ is the fiber of $\mathsf{C}^{/y} \to \mathsf{C}$ (1.5.54) over x. Thus (1.5.54) provides a canonical homotopy equivalence between $\mathrm{Hom}^{\mathrm{R}}_\mathsf{C}(x, y)$ and $\mathrm{Hom}^{\mathrm{cyl}}_\mathsf{C}(x, y)$ (and, by symmetry, $\mathrm{Hom}^{\mathrm{L}}_\mathsf{C}(x, y)$).

1.5.57 Exercise (Enrichment of hC over hSpc)**.** For objects $x, y, z \in \mathsf{C}$, let the simplicial set $\mathrm{Hom}^{\mathrm{cyl}}_\mathsf{C}(x, y, z)$ represent the functor sending Z to the set of maps $Z \times \Delta^2 \to \mathsf{C}$ sending $Z \times i$ to x, y, z for $i = 0, 1, 2$, respectively. Show that the forgetful map $\mathrm{Hom}^{\mathrm{cyl}}_\mathsf{C}(x, y, z) \to \mathrm{Hom}^{\mathrm{cyl}}_\mathsf{C}(x, y) \times \mathrm{Hom}^{\mathrm{cyl}}_\mathsf{C}(y, z)$ is a trivial Kan fibration. Conclude that the forgetful map $\mathrm{Hom}^{\mathrm{cyl}}_\mathsf{C}(x, y, z) \to \mathrm{Hom}_\mathsf{C}(x, z)$ defines a 'composition' morphism

$$\mathrm{Hom}_\mathsf{C}(x, y) \times \mathrm{Hom}_\mathsf{C}(y, z) \to \mathrm{Hom}_\mathsf{C}(x, z) \tag{1.5.57.1}$$

in hSpc. Show that composition is unital (composition with $\mathbf{1}_x$ or $\mathbf{1}_y$ gives the identity map $\mathrm{Hom}_\mathsf{C}(x, y) \to \mathrm{Hom}_\mathsf{C}(x, y)$). Define a simplicial set $\mathrm{Hom}^{\mathrm{cyl}}_\mathsf{C}(x, y, z, w)$ and use it to show composition is associative. Conclude that this defines an enrichment of hC over hSpc, equipped with the monoidal structure $\times$ and the functor $\pi_0 : \mathsf{hSpc} \to \mathsf{Set}$.

* *

⋆ **1.5.58 Definition** (Equivalence of ∞-categories)**.** A functor of ∞-categories $F : \mathsf{C} \to \mathsf{D}$ is called an *equivalence* iff there exists a functor $G : \mathsf{D} \to \mathsf{C}$ such that $G \circ F \simeq \mathbf{1}_{\mathsf{C}}$ and $F \circ G \simeq \mathbf{1}_{\mathsf{D}}$ (isomorphisms in the functor categories $\mathsf{Fun}(\mathsf{C}, \mathsf{C})$ and $\mathsf{Fun}(\mathsf{D}, \mathsf{D})$, respectively).

1.5.59 Remark. Consider the category hCat_∞ whose objects are ∞-categories and whose morphisms are isomorphism classes of functors. A functor of ∞-categories is an equivalence iff it is an isomorphism in hCat_∞. It follows that equivalences of ∞-categories satisfy the 2-out-of-3 property (1.1.45).

1.5.60 Exercise (Functor $\mathsf{sSet} \to \mathsf{hCat}_\infty$; compare (1.4.23))**.** Show that if $X \hookrightarrow Y$ is filtered by pushouts of inner horns and C is an ∞-category, then the map $\mathsf{Fun}(Y, \mathsf{C}) \to \mathsf{Fun}(X, \mathsf{C})$ is a trivial Kan fibration. Use the small object argument (**??**) to show that for every simplicial set X, there exists an inclusion $X \hookrightarrow \overline{X}$ which is filtered by pushouts of inner horns with $\overline{X}$ an ∞-category.

⋆ **1.5.61 Definition** (Isofibration)**.** A functor of ∞-categories $F : \mathsf{C} \to \mathsf{D}$ is called an *isofibration* when the map $F^\natural : \mathsf{C}^\natural \to \mathsf{D}^\natural$ (1.5.49) satisfies the right lifting property with respect to marked horns (1.5.48).

1.5.62 Exercise. Show that if $F : \mathsf{C} \to \mathsf{D}$ is an isofibration, then $d \in \mathsf{D}$ is in the image of F iff it is in the essential image of F.

1.5.63 Exercise. Show that for any isofibration $\mathsf{C} \to \mathsf{D}$, the induced map $\mathsf{Fun}(K, \mathsf{C}) \to \mathsf{Fun}(K, \mathsf{D})$ is an isofibration (use the characterization of isomorphism in functor categories (1.5.50) and the fact that $(\Delta^k, \partial\Delta^k) \wedge (\Delta^n, \Lambda^n_i)^\sim$ is filtered by pushouts of marked horns (1.5.51)). Similarly, show that for any monomorphism $K \to L$ and any ∞-category C, the restriction map $\mathsf{Fun}(L, \mathsf{C}) \to \mathsf{Fun}(K, \mathsf{C})$ is an isofibration.

1.5.64 Definition (Categorical fiber)**.** Let $F : \mathsf{C} \to \mathsf{D}$ be a functor of ∞-categories. The *categorical fiber* $F^{-1}(d)$ over an object $d \in \mathsf{D}$ is the full subcategory of $\mathsf{C}_{F(\cdot)/d}$ spanned by isomorphisms $F(c) \to d$.

⋆ **1.5.65 Definition** (Final object)**.** An object $c \in \mathsf{C}$ is called a *final object* iff the extension property holds for maps $(\Delta^n, \partial\Delta^n) \to \mathsf{C}$ which send the final vertex $n \in \Delta^n$ to c (for $n \geq 1$). Dually, an initial object in C is a final object in C^{op}.

1.5.66 Exercise. Show that the full subcategory of C spanned by final objects is either a trivial Kan complex or empty.

1.5.67 Exercise. Show that if $c \in \mathsf{C}$ is a final object, then $\mathsf{C}_{/c} \to \mathsf{C}$ is a trivial Kan fibration. Conclude that every diagram $K \to \mathsf{C}$ extends to a diagram $K^\triangleright \to \mathsf{C}$ sending the final vertex to c.

1.5.68 Exercise. Show that if $c \in \mathsf{C}$ is final, then so is its image in hC.

1.5.69 Exercise. Show that if $x \to y$ is an isomorphism in C, then x is final iff y is final.

1.5.70 Exercise. Let $F : \mathsf{C} \to \mathsf{D}$ be a map of ∞-categories. Consider the lifting property for diagrams

$$\begin{tikzcd} \partial\Delta^r \arrow[r] \arrow[d] & \mathsf{C} \arrow[d, "F"] \\ \Delta^r \arrow[r] & \mathsf{D} \end{tikzcd} \tag{1.5.70.1}$$

where the map $\Delta^r \to \mathsf{D}$ sends the final vertex $r \in \Delta^r$ to a final object of D. Show that if this lifting property holds for all $r \geq 1$, then F reflects final objects. Show that if this lifting property holds for all $r \geq 0$, then F reflects and lifts final objects.

1.5.71 Exercise. Show that an object of $\prod_i \mathsf{C}_i$ is final iff its image in every C_i is final.

1.5.72 Proposition (Final objects in diagram categories)**.** *The functor* $\mathsf{Fun}(K, \mathsf{C}) \to \mathsf{Fun}(K_0, \mathsf{C})$ *reflects and lifts final objects.*

Proof. It suffices to show that $\mathsf{Fun}(K, \mathsf{C}) \to \mathsf{Fun}(K_0, \mathsf{C})$ satisfies the right lifting property with respect to maps from pairs $(\Delta^r, \partial\Delta^r)$ which send the final vertex $r \in \Delta^r$ to a final object of $\mathsf{Fun}(K_0, \mathsf{C})$ (1.5.70). By filtering the pair (K, K_0) by pushouts of pairs $(\Delta^k, \partial\Delta^k)$ with $k \geq 1$, we reduce to the extension property for maps

$$(\Delta^r, \partial\Delta^r) \wedge (\Delta^k, \partial\Delta^k) \to \mathsf{C} \tag{1.5.72.1}$$

whose specialization to every vertex lying over $r \in \Delta^r$ is final. The smash product $(\Delta^r, \partial\Delta^r) \wedge (\Delta^k, \partial\Delta^k)$ is filtered by pushouts of pairs $(\Delta^a, \partial\Delta^a)$. Each map $\Delta^a \to \Delta^r \times \Delta^k$ appearing in this filtration must send the final vertex $a \in \Delta^a$ to the final vertex $r \in \Delta^r$ (otherwise $\Delta^a \subseteq \partial\Delta^r \times \Delta^k$). We are thus reduced to the extension problem $(\Delta^a, \partial\Delta^a) \to \mathsf{C}$ for maps sending the final vertex $a \in \Delta^a$ to a final object, which is solvable for $a \geq 1$ (which is guaranteed by $k \geq 1$). ☐

* *

⋆ **1.5.73 Definition** (Limit and colimit)**.** A *limit diagram* is a diagram $K^{\triangleleft} \to \mathsf{C}$ which is a final object in $\mathsf{C}_{/K}$. The *limit* of a diagram $p : K \to \mathcal{C}$ is the image $\lim_K p \in \mathsf{C}$ of a final object in $\mathsf{C}_{/K}$ (if one exists; otherwise the limit is not defined).

1.5.74 Proposition (Recognizing products in a fibrant simplicial category)**.** *Let* C *be a fibrant simplicial category, and let* $X \to X_\alpha$ *be a collection of maps in* C *indexed by* α*. If the map* $\mathsf{C}(Z, X) \to \prod_\alpha \mathsf{C}(Z, X_\alpha)$ *is a homotopy equivalence for every* $Z \in \mathsf{C}$*, then* X *is the product* $\prod_\alpha X_\alpha$ *in (the simplicial nerve of)* C*.*

Proof. The family of maps $X \to X_\alpha$ determines a lift of $X \in \mathsf{C}$ to the slice category $\mathsf{C}_{/A}$, where A denotes the set of indices α, regarded as a disjoint union of 0-simplices. Our task is to show that this lift is a final object of $\mathsf{C}_{/A}$. In other words, we are to show the extension property for maps $(\Delta^n, \partial\Delta^n) \star A \to \mathsf{C}$ ($n \geq 1$) whose restriction to $n \star A = A^{\triangleleft}$ is the given family of maps $X \to X_\alpha$. This amounts to the extension property for diagrams

$$\begin{tikzcd} (\Delta^1, \partial\Delta^1)^{\{1,\ldots,n-1\}} \arrow[r] \arrow[d, "\times\{1\}^{\{n\}}"] & \mathsf{C}(Z,X) \arrow[d] \\ (\Delta^1, \partial\Delta^1)^{\{1,\ldots,n-1\}} \wedge (\Delta^1, 0)^{\{n\}} \arrow[r] & \prod_\alpha \mathsf{C}(Z, X_\alpha) \end{tikzcd} \tag{1.5.74.1}$$

where Z is the object assigned to the initial vertex $0 \in \Delta^n$. This extension property holds since the right vertical map is a homotopy equivalence (choose to extend the top horizontal map in the correct homotopy class rel boundary so that the subsequent extension problem $(\Delta^1, \partial\Delta^1)^{\{1,\ldots,n\}} \to \prod_\alpha \mathsf{C}(Z, X_\alpha)$ has a solution). □

1.5.75 Example. Products (resp. coproducts) of Kan complexes are products (resp. coproducts) in the ∞-category Spc (1.5.12) by (1.5.74).

* *

We now discuss final maps of simplicial sets following Lurie [58, 4.1.1].

⋆ **1.5.76 Definition** (Final; Joyal)**.** A map of simplicial sets $K' \to K$ is called ∞-*final* when the pullback map $\underline{\mathrm{Sec}}(K, \mathsf{E}) \to \underline{\mathrm{Sec}}(K', \mathsf{E})$ is a homotopy equivalence for every right fibration $\mathsf{E} \to K$.

1.5.77 Exercise. Show that if $K' \subseteq K$ is filtered by pushouts of right horns, then $\underline{\mathrm{Sec}}(K, \mathsf{E}) \to \underline{\mathrm{Sec}}(K', \mathsf{E})$ is a trivial Kan fibration, hence $K' \to K$ is ∞-final.

1.5.78 Exercise. Show that ∞-final maps are closed under composition. Show that a retract of an ∞-final map is ∞-final.

1.5.79 Exercise. For maps $K \xrightarrow{f} L \xrightarrow{g} M$, show that if f and $g \circ f$ are ∞-final, then so is g. Show by example that if g and $g \circ f$ are ∞-final, it need not be the case that f is ∞-final.

1.5.80 Lemma. *An ∞-final map of simplicial sets is a homotopy equivalence.*

Proof. Let $K \to K'$ be ∞-final. As a special case of (1.5.76), the pullback map $\underline{\mathrm{Hom}}(K', X) \to \underline{\mathrm{Hom}}(K, X)$ is a homotopy equivalence for every Kan complex X. In particular, it is a bijection on connected components, which implies the map $\mathrm{Hom}_{\mathsf{hSpc}}(K', -) \to \mathrm{Hom}_{\mathsf{hSpc}}(K, -)$ is an isomorphism of functors on hSpc. □

1.5.81 Lemma. *A product of ∞-final maps is ∞-final.*

Proof. It suffices to show that if $K' \to K$ is ∞-final, then so is $K' \times L \to K \times L$ for any simplicial set L. Given a right fibration $\mathsf{E} \to K \times L$, we can form its 'pushforward' to K, which is the right fibration $\mathsf{G} \to K$ defined by the universal property that a map from a simplicial set Z to G is a map $Z \to K$ together with a lift of $Z \times L \to K \times L$ to E. The right lifting property for $\mathsf{G} \to K$ with respect to (Δ^n, Λ^n_i) follows from the right lifting property for $\mathsf{E} \to K \times L$ with respect to pairs $(\Delta^n, \Lambda^n_i) \times L$, so $\mathsf{G} \to K$ is indeed a right fibration (1.5.51). Since $K' \to K$ is ∞-final, the pullback map

$$\underline{\mathrm{Sec}}(K \times L, \mathsf{E}) = \underline{\mathrm{Sec}}(K, \mathsf{G}) \to \underline{\mathrm{Sec}}(K', \mathsf{G}) = \underline{\mathrm{Sec}}(K' \times L, \mathsf{E}) \tag{1.5.81.1}$$

is a homotopy equivalence. □

1.5.82 Definition (Cartesian functor)**.** A functor $F : \mathsf{C} \to \mathsf{D}$ is called *cartesian* iff for every object $c \in \mathsf{C}$ and every morphism $d \to F(c)$ in D, the right fibration $\mathsf{C}_{/c} \times_{\mathsf{D}_{/F(c)}} \mathsf{D}_{/(d \to F(c))} \to \mathsf{C}$ (1.5.83) is representable and the map from the image in D of its representing object to d is an isomorphism.

1.5.83 Exercise. Use (1.5.43) to show that $\mathsf{C}_{/c} \times_{\mathsf{D}_{/F(c)}} \mathsf{D}_{/(d \to F(c))} \to \mathsf{C}$ is a right fibration.

1.5.84 Lemma. *A morphism $c \to c'$ in C is cartesian with respect to $F : \mathsf{C} \to \mathsf{D}$ iff the diagram*

$$\begin{tikzcd} c' \arrow[r] \arrow[d] & c \arrow[d] \\ F^*F(c') \arrow[r] & F^*F(c) \end{tikzcd} \tag{1.5.84.1}$$

is a pullback square in $\mathsf{P}(\mathsf{C})$, where $F^* : \mathsf{P}(\mathsf{D}) \to \mathsf{P}(\mathsf{C})$ denotes pullback of presheaves and we implicitly apply Yoneda functors.

Proof. □

1.5.85 Lemma. *Let $F : \mathsf{C} \to \mathsf{D}$ be a functor, and fix a diagram*

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & Y' \arrow[d] \\ X \arrow[r] & Y \end{tikzcd} \tag{1.5.85.1}$$

in C whose image under F is a pullback and whose bottom arrow $X \to Y$ is cartesian. In this case, the diagram (1.5.85.1) is a pullback iff $X' \to Y'$ is cartesian.

Proof. Consider the diagrams (1.5.84.1) associated to the morphisms $X \to Y$ and $X' \to Y'$, which fit together into a cube.

$$\begin{tikzcd} X' \arrow[rr] \arrow[dr] \arrow[dd] & & Y' \arrow[dr] \arrow[dd] & \\ & X \arrow[rr] \arrow[dd] & & Y \arrow[dd] \\ F^*F(X') \arrow[rr] \arrow[dr] & & F^*F(Y') \arrow[dr] & \\ & F^*F(X) \arrow[rr] & & F^*F(Y) \end{tikzcd} \tag{1.5.85.2}$$

The assumptions that $X \to Y$ is cartesian and that $F(1.5.85.1)$ is a pullback imply that two faces of this cube are pullbacks. By cancellation for fiber products (1.1.52), $X' \to Y'$ being cartesian and (1.5.85.1) being a pullback are both equivalent to the composite square

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & Y' \arrow[d] \\ F^*F(X) \arrow[r] & F^*F(Y) \end{tikzcd} \tag{1.5.85.3}$$

being a pullback. □

1.5.86 Lemma. *Let $F : \mathsf{C} \to \mathsf{D}$ be cartesian and suppose D has pullbacks. Then cartesian morphisms are preserved under pullback, and F sends pullbacks of cartesian morphisms to pullbacks in D.*

Proof. Fix a cartesian morphism $X \to Y$ and an arbitrary morphism $Y' \to Y$. Define a cartesian morphism $X' \to Y'$ as the cartesian lift of $F(X) \times_{F(Y)} F(Y') \to F(Y)$. There is a morphism $X' \to X$ completing the diagram (1.5.85.1) since $X \to Y$ is cartesian. Now by construction $X' \to Y'$ is cartesian and $F(1.5.85.1)$ is a pullback, so (1.5.85.1) is a pullback (1.5.85). By construction, the morphism $X' \to Y'$ is cartesian and the image $F(1.5.85.1)$ is a pullback. □

1.5.87 Definition (Relative limit)**.** Let $X \to Y$ be an inner fibration. Given a simplicial set K and a diagram of solid arrows

$$\begin{tikzcd} K \arrow[r] \arrow[d] & X \arrow[d] \\ K^{\triangleleft} \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd} \tag{1.5.87.1}$$

we can consider the simplicial set of dotted lifts. This is an ∞-category since $(K^{\triangleleft}, K) \wedge (\Delta^n, \Lambda^n_i)$ is filtered by pushouts of inner horns when $0 < i < n$ (1.5.20). A final object in this ∞-category is called the *relative limit* of the diagram.

1.5.88 Definition (Relative functor category)**.** Let $X \to Y$ be a map of simplicial sets, and let C be an ∞-category. The *relative functor category* $\mathsf{Fun}_Y(X, \mathsf{C})$ is the simplicial set defined by the universal property that a map $Z \to \mathsf{Fun}_Y(X, \mathsf{C})$ is a pair of maps $Z \to Y$ and $X \times_Y Z \to \mathsf{C}$.

Formation of the relative functor category is compatible with pullback: if $X' \to Y'$ is a pullback of $X \to Y$, then the natural map $\mathsf{Fun}_{Y'}(X', \mathsf{C}) \to \mathsf{Fun}_Y(X, \mathsf{C}) \times_Y Y'$ is an isomorphism. In the case $Y = *$, the relative functor category reduces to the usual functor category $\mathsf{Fun}(X, \mathsf{C})$. The fiber of the map $\mathsf{Fun}_Y(X, \mathsf{C}) \to Y$ over a point $y \in Y$ is thus the functor category $\mathsf{Fun}(X_y, \mathsf{C})$.

1.5.89 Definition (Weak Kan extension)**.** Let $f : \mathsf{A} \to \mathsf{B}$ be a functor. Given a functor $G : \mathsf{A} \to \mathsf{E}$, a *weak left Kan extension* of G along f is a functor $f_!G : \mathsf{B} \to \mathsf{E}$ and a natural transformation $\eta : G \to f_! G \circ f$ such that the pair $(f_! G, \eta)$ is an initial object in the category $\mathsf{Fun}(\mathsf{B}, \mathsf{E})_{G/f^*(\cdot)}$.

$$\begin{tikzcd} \mathsf{A} \arrow[r, "f"] \arrow[d, "G"'] \arrow[dr, phantom, "\Rightarrow"] & \mathsf{B} \arrow[dl, dashed, "f_! G"] \\ \mathsf{E} & \end{tikzcd} \tag{1.5.89.1}$$

When every G has a weak left Kan extension, the resulting left adjoint to f^* is denoted $f_! : \mathsf{Fun}(\mathsf{A}, \mathsf{E}) \to \mathsf{Fun}(\mathsf{B}, \mathsf{E})$. The dual notion is called weak right Kan extension and is denoted f_*, which is right adjoint to f^*.

The category of functors $H : \mathsf{B} \to \mathsf{E}$ equipped with a natural transformation $G \to H \circ f$ is the category of maps

$$(\mathsf{A} \times \Delta^1) \cup^f_{\mathsf{A}\times 1} \mathsf{B} \to \mathsf{E} \tag{1.5.89.2}$$

whose restriction to A equals G. We would like to replace the 'mapping cylinder' $(\mathsf{A}\times\Delta^1)\cup^f_{\mathsf{A}\times 1}\mathsf{B}$ in this situation with an ∞-category.

1.5.90 Exercise (Semi-orthogonal gluing)**.** Given a functor of ∞-categories $f : \mathsf{A} \to \mathsf{B}$, let $\langle \mathsf{A}, \mathsf{B}\rangle_f$ denote the simplicial set defined by the property that a map $Z \to \langle \mathsf{A}, \mathsf{B}\rangle_f$ is a map $p : Z \to \Delta^1$ and a diagram

$$\begin{tikzcd} p^{-1}(0) \arrow[r] \arrow[d] & \mathsf{A} \arrow[d, "f"] \\ Z \arrow[r] & \mathsf{B} \end{tikzcd} \tag{1.5.90.1}$$

Show that $\langle \mathsf{A}, \mathsf{B}\rangle_f$ is an ∞-category.

1.5.91 Lemma. *For any functor of ∞-categories $f : \mathsf{A} \to \mathsf{B}$, the tautological map*

$$(\mathsf{A} \times \Delta^1) \cup^f_{\mathsf{A}\times 1} \mathsf{B} \to \langle \mathsf{A}, \mathsf{B}\rangle_f \tag{1.5.91.1}$$

is a categorical equivalence.

Proof. For f a monomorphism, the map (1.5.91.1) is a monomorphism as well, and we will show that is filtered by pushouts of inner horns. We begin by classifying the non-degenerate simplices of $\langle \mathsf{A}, \mathsf{B}\rangle_f$. A simplex $\sigma : \Delta^k \to \langle \mathsf{A}, \mathsf{B}\rangle_f$ is (given injectivity of $\mathsf{A} \to \mathsf{B}$) a simplex $\pi_{\mathsf{B}}\sigma : \Delta^k \to \mathsf{B}$ along with a map $p_\sigma : \Delta^k \to \Delta^1$ such that $\pi_{\mathsf{B}}\sigma(p_\sigma^{-1}(0)) \subseteq \mathsf{A}$. The simplex σ is non-degenerate in precisely the following two situations:

(1.5.91.2) The simplex $\pi_{\mathsf{B}}\sigma$ is non-degenerate.

(1.5.91.3) The simplex $\pi_{\mathsf{B}}\sigma$ is the composition of a surjection $\Delta^k \twoheadrightarrow \Delta^{k-1}$ (say identifying vertices i and $i{+}1$) with a non-degenerate simplex $\tau : \Delta^{k-1} \to \mathsf{B}$ and $p_\sigma([0 \cdots i]) = 0$ and $p_\sigma([i{+}1 \cdots k]) = 1$.

We group the non-degenerate simplices of $\langle \mathsf{A}, \mathsf{B}\rangle_f$ by their associated (as above) non-degenerate simplex of B.

We now define a filtration of the pair $(\langle \mathsf{A},\mathsf{B}\rangle_f, (\mathsf{A}\times\Delta^1)\cup^f_{\mathsf{A}\times 1}\mathsf{B})$ as follows. We filter B by simplices $(\Delta^r,\partial\Delta^r)$, and in this order we add all the associated non-degenerate simplices of $\langle \mathsf{A},\mathsf{B}\rangle_f$. Adding the non-degenerate simplices of $\langle \mathsf{A},\mathsf{B}\rangle_f$ associated to a given non-degenerate $\tau:\Delta^r\to\mathsf{B}$ amounts to attaching the pair $((\Delta^a,\partial\Delta^a)\wedge(\Delta^1,1))\star(\Delta^b,\partial\Delta^b)$ where $\Delta^r=\Delta^a\star\Delta^b$ and $\Delta^a\subseteq\Delta^r$ is the maximal prefix satisfying $\tau(\Delta^a)\subseteq\mathsf{A}$. The smash product $(\Delta^a,\partial\Delta^a)\wedge(\Delta^1,1)$ is filtered by right horns by (1.5.51), and these become inner upon join with $(\Delta^b,\partial\Delta^b)$ by (**??**). □

⋆ **1.5.92 Definition** (Kan extension)**.** Fix functors $f:\mathsf{A}\to\mathsf{B}$ and $G:\mathsf{A}\to\mathsf{E}$. An extension of G to $\langle \mathsf{A},\mathsf{B}\rangle_f$ is called a *(pointwise) left Kan extension* when its pre-composition with the tautological map

$$(\mathsf{A}_{f(\cdot)/b})^{\triangleright}\to\langle \mathsf{A},\mathsf{B}\rangle_f \tag{1.5.92.1}$$

is a colimit diagram in E for every $b\in\mathsf{B}$.

1.5.93 Lemma. *If $f:\mathsf{A}\to\mathsf{B}$ is fully faithful and $g:\mathsf{A}\to\mathsf{E}$ has a left Kan extension $f_!g$, then the natural transformation $g\to f^*f_!g$ is an isomorphism.*

Proof. Consider the functor $\langle \mathsf{A},\mathsf{B}\rangle_f\to\mathsf{E}$ underlying the left Kan extension $f_!g$. For $a\in\mathsf{A}$, consider the following tautological diagram.

$$\begin{tikzcd} \Delta^1 \arrow[r, "\mathbf{1}_{f(a)}:f(a)\to f(a)"] \arrow[d, "a\times"] & (\mathsf{A}_{f(\cdot)/f(a)})^{\triangleright} \arrow[d] \arrow[dr] & \\ (\mathsf{A}\times\Delta^1)\cup^f_{\mathsf{A}\times 1}\mathsf{B} \arrow[r] & \langle \mathsf{A},\mathsf{B}\rangle_f \arrow[r] & \mathsf{E} \end{tikzcd} \tag{1.5.93.1}$$

The diagonal arrow is a colimit diagram by definition of Kan extension, and the point $\mathbf{1}_{f(a)}\in\mathsf{A}_{f(\cdot)/f(a)}$ is a final object since f is fully faithful, so we conclude that the composition $\Delta^1\to\mathsf{E}$ is an isomorphism. Considering now the composition $\Delta^1\to\mathsf{E}$ through the rest of the diagram, this means precisely that the unit transformation $g(a)\to(f_!g)(f(a))$ is an isomorphism. □

1.5.94 Corollary. *Let $f:\mathsf{A}\to\mathsf{B}$ be fully faithful. A functor $\mathsf{B}\to\mathsf{E}$ is a left Kan extension along f iff the composition $(\mathsf{A}_{f(\cdot)/b})^{\triangleright}\to\mathsf{B}\to\mathsf{E}$ is a colimit diagram for every $b\in\mathsf{B}$.*

Proof. Since the unit transformation $g\to f^*f_!g$ is an isomorphism for every g (1.5.93), the functor $(\mathsf{A}\times\Delta^1)\cup^f_{\mathsf{A}\times 1}\mathsf{B}\to\mathsf{E}$ underlying a left Kan extension factors through the projection $(\mathsf{A}\times\Delta^1)\cup^f_{\mathsf{A}\times 1}\mathsf{B}\to\langle \mathsf{A},\mathsf{B}\rangle_f\to\mathsf{B}$. □

1.5.95 Corollary. *If $f:\mathsf{A}\to\mathsf{B}$ is fully faithful, then the left Kan extension functor $f_!:\mathsf{Fun}(\mathsf{A},\mathsf{E})\to\mathsf{Fun}(\mathsf{B},\mathsf{E})$ is fully faithful on its domain of definition.*

Proof. Combine (1.5.93) with (**??**). □

* *

We now discuss universal properties of presheaf categories, which assert (in various precise senses) that passing to a presheaf category freely adjoins colimits.

1.5.96 Exercise. Let $p : K \to \mathsf{C}$ be any diagram. Apply the small object argument (**??**) to express p as the composition of a map $K \hookrightarrow \widehat{K}$ which is filtered by pushouts of right horns and a map $\hat{p} : \widehat{K} \to \mathsf{C}$ which is a right fibration. Note that $K \hookrightarrow \widehat{K}$ is final since it is filtered by pushouts of right horns (1.5.77) and hence that $\operatorname{colim} p \to \operatorname{colim} \hat{p}$ is an isomorphism (**??**). Combine this with the fact that a right fibration is its own colimit (**??**) to conclude that $\operatorname{colim}_{\mathsf{P}(\mathsf{C})} p$ is the right fibration $\hat{p} : \widehat{K} \to \mathsf{C}$.

⋆ **1.5.97 Definition** (Finite presheaf)**.** A presheaf $F \in \mathsf{P}(\mathsf{C})$ is called *finite* when it is a finite colimit of representable presheaves. The full subcategory spanned by finite presheaves is denoted $\mathsf{P}_{\mathsf{fin}}(\mathsf{C}) \subseteq \mathsf{P}(\mathsf{C})$.

1.5.98 Lemma (Classification of morphisms in $\mathsf{P}_{\mathsf{fin}}(\mathsf{C})$)**.** *Let $p : K \to \mathsf{C}$ be a finite diagram. Every morphism out of p in $\mathsf{P}_{\mathsf{fin}}(\mathsf{C})$ is isomorphic in $(p \downarrow \mathsf{P}_{\mathsf{fin}}(\mathsf{C}))$ to the tautological map $p \to q$ associated to a diagram $q : L \to \mathsf{C}$, an injection $K \hookrightarrow L$, and an isomorphism $q|_K = p$.*

Proof. Fix a finite diagram $q : L \to \mathsf{C}$ and an arbitrary morphism $p \to q$ in $\mathsf{P}_{\mathsf{fin}}(\mathsf{C})$. The object $q \in \mathsf{P}_{\mathsf{fin}}(\mathsf{C})$ is represented by the right fibration $\hat{q} : \widehat{L} \to \mathsf{C}$ obtained from $q : L \to \mathsf{C}$ by applying the small object argument (1.5.96). Our morphism $p \to q$ is thus induced by a map $K \to \widehat{L}$ over C (**??**). Since K is finite and right horns are finite, this morphism necessarily factors through the result $\overline{L} \subseteq \widehat{L}$ of attaching just finitely many right horns to L. The morphism $\operatorname{colim} q \to \operatorname{colim} \bar{q}$ is an isomorphism for the same reason $\operatorname{colim} q \to \operatorname{colim} \hat{q}$ is (1.5.96). Thus our morphism $p \to q$ is represented by the morphism $K \to \overline{L}$ of finite simplicial sets over C. This map may not be injective, so we may replace it with the mapping cylinder $K = K \times 0 \subseteq (K \times \Delta^1) \cup_{K \times 1} \overline{L}$. □

⋆ **1.5.99 Proposition.** *The full subcategory of finite presheaves $\mathsf{P}_{\mathsf{fin}}(\mathsf{C}) \subseteq \mathsf{P}(\mathsf{C})$ is closed under finite colimits in $\mathsf{P}(\mathsf{C})$.*

Proof. It suffices to show that a pushout of finite presheaves is finite (**??**). So, consider morphisms $X \leftarrow Y \to Z$ in $\mathsf{P}_{\mathsf{fin}}(\mathsf{C})$. Represent Y by a finite diagram $p : K \to \mathsf{C}$. By the classification of morphisms in $\mathsf{P}_{\mathsf{fin}}(\mathsf{C})$ (1.5.98), the morphisms $Y \to X$ and $Y \to Z$ are of the form $p \to q$ and $p \to r$ for finite diagrams $q : L \to \mathsf{C}$ and $r : M \to \mathsf{C}$ with $K \subseteq L$ and $K \subseteq M$ with $q|_K = p = r|_K$. We may thus consider the pushout diagram $q \sqcup_p r : L \sqcup_K M \to \mathsf{C}$, which represents an object of $\mathsf{P}_{\mathsf{fin}}(\mathsf{C})$. There is now a tautological square diagram containing p, q, r, and $q \sqcup_p r$, and this diagram is a pushout in $\mathsf{P}(\mathsf{C})$ by Mayer–Vietoris (**??**). □

* *

⋆ **1.5.100 Definition** (Local presheaf)**.** Let C be an ∞-category, and let Λ be a *set* of morphisms in $\mathsf{P}(\mathsf{C})$. We denote by $\mathsf{P}_{\Lambda}(\mathsf{C}) \subseteq \mathsf{P}(\mathsf{C})$ the full subcategory spanned by right Λ-local objects (1.1.82).

⋆ **1.5.101 Proposition.** *The full subcategory* $\mathsf{P}_\Lambda(\mathsf{C}) \subseteq \mathsf{P}(\mathsf{C})$ *is reflective.*

Proof. Represent Λ as a set of diagrams $\{A_\alpha \hookrightarrow X_\alpha \to \mathsf{C}\}_\alpha$ for simplicial set pairs (X_α, A_α). A right fibration $Q \to \mathsf{C}$ is Λ-local iff it satisfies the right lifting property with respect to all pairs $(X_\alpha, A_\alpha) \wedge (\Delta^k, \partial\Delta^k)$ (mapping to C via the given maps $A_\alpha \hookrightarrow X_\alpha \to \mathsf{C}$). For any right fibration $Q \to \mathsf{C}$ satisfying this lifting property, if a map $K' \to \mathsf{C}$ is obtained from $K \to \mathsf{C}$ (not necessarily a right fibration) by forming the pushout of such a right lifting problem against a right horn or a pair $(X_\alpha, A_\alpha) \wedge (\Delta^k, \partial\Delta^k)$, then the restriction map $\mathsf{Fun}_{/\mathsf{C}}(K', Q) \to \mathsf{Fun}_{/\mathsf{C}}(K, Q)$ is a trivial Kan fibration.

We can now argue that every object of $\mathsf{P}(\mathsf{C})$ has a reflection in $\mathsf{P}_\Lambda(\mathsf{C})$. Represent an arbitrary object of $\mathsf{P}(\mathsf{C})$ by a diagram $K \to \mathsf{C}$. The small object argument (**??**) produces a factorization $K \to \overline{K} \to \mathsf{C}$ where $K \to \overline{K}$ is filtered by pushouts of right horns and pairs $(X_\alpha, A_\alpha) \wedge (\Delta^k, \partial\Delta^k)$ over C and $\overline{K} \to \mathsf{C}$ has the right lifting property with respect to right horns and pairs $(X_\alpha, A_\alpha) \wedge (\Delta^k, \partial\Delta^k)$ over C. Thus $\overline{K}$ lies in $\mathsf{P}_\Lambda(\mathsf{C})$ and the restriction map $\mathrm{Hom}_{\mathsf{P}(\mathsf{C})}(\overline{K}, Q) = \mathsf{Fun}_{/\mathsf{C}}(\overline{K}, Q) \to \mathsf{Fun}_{/\mathsf{C}}(K, Q) = \mathrm{Hom}_{\mathsf{P}(\mathsf{C})}(K, Q)$ is a trivial Kan fibration for all left fibrations $Q \to \mathsf{C}$ in $\mathsf{P}_\Lambda(\mathsf{C})$. □

1.5.102 Warning. It is tempting to claim the converse to (1.5.101) (every reflective subcategory of $\mathsf{P}(\mathsf{C})$ is of the form $\mathsf{P}_\Lambda(\mathsf{C})$ for some set of morphisms Λ, namely the collection of all reflections) by applying (1.1.83), however this argument is flawed since the collection of all reflections need not be a set.

1.5.103 Lemma. *The reflector* $r_\Lambda : \mathsf{P}(\mathsf{C}) \to \mathsf{P}_\Lambda(\mathsf{C})$ *sends morphisms in* Λ *to isomorphisms.*

Proof. Let $\ell \in \Lambda$. The morphism $r_\Lambda\ell$ is an isomorphism iff $\mathrm{Hom}(r_\Lambda\ell, X)$ is an isomorphism for every $X \in \mathsf{P}_\Lambda(\mathsf{C})$. We have $\mathrm{Hom}(r_\Lambda\ell, X) = \mathrm{Hom}(\ell, X)$ for $X \in \mathsf{P}_\Lambda(\mathsf{C})$, and $\mathrm{Hom}(\ell, X)$ is an isomorphism for all $X \in \mathsf{P}_\Lambda(\mathsf{C})$ by definition of $\mathsf{P}_\Lambda(\mathsf{C})$. □

1.5.104 Lemma. *A functor* $F : \mathsf{P}(\mathsf{C}) \to \mathsf{E}$ *sending reflections* $X \to r_\Lambda X$ *to isomorphisms sends all morphisms in* Λ *to isomorphsms.*

Proof. Suppose F sends reflections to isomorphisms. Given a morphism $\ell : X \to Y$ in Λ, consider the diagram

$$\begin{tikzcd} F(X) \arrow[r, "F(\ell)"] \arrow[d, "F(X\to rX)"] & F(Y) \arrow[d, "F(Y\to rY)"] \\ F(rX) \arrow[r, "F(r\ell)"] & F(rY) \end{tikzcd} \tag{1.5.104.1}$$

obtained by applying F to square $\ell \to r\ell$. The two vertical arrows are isomorphisms by hypothesis on F. The bottom horizontal arrow $F(r\ell)$ is an isomorphism since $r\ell$ is an isomorphism (1.5.103). Thus the top horizontal map $F(\ell)$ is also an isomorphism, as desired. □

1.5.105 Lemma. *A cocontinuous functor* $F : \mathsf{P}(\mathsf{C}) \to \mathsf{E}$ *sends reflections* $X \to r_\Lambda X$ *to isomorphisms iff it sends all morphisms in* Λ *to isomorphsms.*

Proof. One direction is given by (1.5.104), so we just need to prove the other.

Suppose F is cocontinuous and sends morphisms in Λ to isomorphisms. The construction of the reflector $r : \mathsf{P}(\mathsf{C}) \to \mathsf{P}_\Lambda(\mathsf{C})$ by the small object argument (1.5.101) exhibits the reflection $X \to rX$ as the colimit $X \to \varinjlim_i X_i$ of a diagram over a well ordered set whose transition maps $\varinjlim_{i<i_0} X_i \to X_{i_0}$ are pushouts of pairs $(Y, A) \wedge (\Delta^k, \partial\Delta^k)$ mapping to C via maps $A \hookrightarrow Y \to \mathsf{C}$ in Λ. Now F is cocontinuous, so to show that F sends such a reflection to an isomorphism, it suffices to show that it sends (the presheaf on C represented by) any such pair $(Y, A) \wedge (\Delta^k, \partial\Delta^k)$ to an isomorphism. Now this pair is simply the kth iterated codiagonal of the morphism $A \to Y$ in $\mathsf{P}(\mathsf{C})$, and F preserves codiagonals since it is cocontinuous, so we are done since F sends each map $A \to Y$ to an isomorphism by hypothesis. □

1.5.106 Proposition (Universal property of local presheaves)**.** *For any cocomplete ∞-category E, pullback along $\mathsf{C} \xrightarrow{y} \mathsf{P}(\mathsf{C}) \xrightarrow{r_\Lambda} \mathsf{P}_\Lambda(\mathsf{C})$ defines equivalences between the following ∞-categories of functors:*

(1.5.106.1) *Functors $\mathsf{P}_\Lambda(\mathsf{C}) \to \mathsf{E}$ which are cocontinuous.*

(1.5.106.2) *Functors $\mathsf{P}(\mathsf{C}) \to \mathsf{E}$ which send reflections $X \to rX$ to isomorphisms.*

(1.5.106.3) *Functors $\mathsf{P}(\mathsf{C}) \to \mathsf{E}$ which send morphisms in Λ to isomorphisms.*

(1.5.106.4) *Functors $\mathsf{C} \to \mathsf{E}$ whose unique cocontinuous extension to $\mathsf{P}(\mathsf{C})$ satisfy the above two equivalent conditions.*

Proof. Combine the universal property of a reflective subcategory of presheaves (1.1.89) with the equivalence for cocontinuous functors $\mathsf{P}(\mathsf{C}) \to \mathsf{E}$ of sending reflections to isomorphisms and sending morphisms in Λ to isomorphisms (1.5.105). □

1.5.107 Definition (Finite local presheaves)**.** We denote by $\mathsf{P}_{\Lambda,\mathsf{fin}}(\mathsf{C}) \subseteq \mathsf{P}_\Lambda(\mathsf{C})$ the full subcategory spanned by finite colimits of objects of C. By reflectivity of $\mathsf{P}_\Lambda(\mathsf{C}) \subseteq \mathsf{P}(\mathsf{C})$, this is the same as the image of the finite presheaves $\mathsf{P}_{\mathsf{fin}}(\mathsf{C}) \subseteq \mathsf{P}(\mathsf{C})$ under the reflector $\mathsf{P}(\mathsf{C}) \to \mathsf{P}_\Lambda(\mathsf{C})$.

* *

We now study ∞-sifted colimits following [79] and [58, 5.5.8].

⋆ **1.5.108 Definition** (∞-sifted)**.** A simplicial set K is called ∞-*sifted* when the diagonal map $K \to K^n$ is ∞-final (1.5.76) for all $n \geq 0$.

1.5.109 Exercise. Use (1.5.80) to show that a sifted simplicial set is contractible.

1.5.110 Lemma. *A simplicial set K is ∞-sifted iff it is non-empty and the diagonal map $K \to K \times K$ is ∞-final.*

Proof. Suppose K is non-empty and $K \to K^2$ is ∞-final, and let us show that $K \to K^n$ is ∞-final for all $n \geq 0$ (the other direction is trivial). The case $n = 1$ is trivial, and the cases $n \geq 2$ follow by induction upon expressing the diagonal map $\Delta_n : K \to K^n$ as the composition of Δ_{n-1} and $\mathbf{1}_{K^{n-2}} \times \Delta_2$ and recalling that ∞-final maps are closed under composition and products (1.5.81).

For the case $n = 0$, it suffices (and, in fact, is necessary (1.5.80)) to show that K is contractible. Since $K \to K \times K$ is ∞-final, it is a homotopy equivalence, hence acts bijectively on homotopy groups/sets (**??**). On the other hand, the homotopy group/set functors preserve products, so the diagonal maps of the homotopy groups/sets of K are bijections. This implies they are trivial, so K is contractible by Whitehead's Theorem (**??**). □

1.5.111 Exercise. Let $K \to L$ be ∞-final. Show that if K is ∞-sifted then L is ∞-sifted. Show by example that if L is ∞-sifted then K need not be ∞-sifted.

⋆ **1.5.112 Definition** (Formal ∞-sifted colimits)**.** We define $\mathsf{Sif}(\mathsf{C}) \subseteq \mathsf{P}(\mathsf{C})$ to consist of those presheaves which are ∞-sifted colimits of representable presheaves.

1.5.113 Lemma. *A presheaf lies in $\mathsf{Sif}(\mathsf{C})$ iff the total space of its corresponding right fibration over C is ∞-sifted.*

Proof. For any right fibration $\pi : \mathsf{E} \to \mathsf{C}$, the corresponding object of $\mathsf{P}(\mathsf{C})$ is the colimit $\operatorname{colim}^{\mathsf{P}(\mathsf{C})}_{\mathsf{E}} \pi$. Thus if E is ∞-sifted then the corresponding object of $\mathsf{P}(\mathsf{C})$ lies in $\mathsf{Sif}(\mathsf{C}) \subseteq \mathsf{P}(\mathsf{C})$.

Conversely, suppose K is ∞-sifted and $p : K \to \mathsf{C}$ is a diagram. Apply the small object argument (**??**) to factor p as the composition $K \to \hat{K} \to \mathsf{C}$ of a right fibration $\hat{p} : \hat{K} \to \mathsf{C}$ and a map $K \to \hat{K}$ filtered by pushouts of right horns. The map $K \to \hat{K}$ is ∞-final (1.5.77), so $\operatorname{colim}^{\mathsf{P}(\mathsf{C})}_{K} p = \operatorname{colim}^{\mathsf{P}(\mathsf{C})}_{\hat{K}} \hat{p}$ is the object corresponding to the right fibration $\hat{p} : \hat{K} \to \mathsf{C}$. The total space $\hat{K}$ is ∞-sifted since K is ∞-sifted and $K \to \hat{K}$ is ∞-final (1.5.111). □

1.5.114 Lemma. *Let $f : \mathsf{C} \to \mathsf{D}$ be a functor, and let $F \to G$ be a morphism in $\mathsf{P}(\mathsf{C})$. The left Kan extension functor $f_! : \mathsf{P}(\mathsf{C}) \to \mathsf{P}(\mathsf{D})$ preserves all pullbacks of $F \to G$ iff it preserves the pullback diagrams*

$$\begin{tikzcd} F \times_G c' \arrow[r] \arrow[d] & F \times_G c \arrow[d] \\ c' \arrow[r] & c \end{tikzcd} \tag{1.5.114.1}$$

for all morphisms $c' \to c \to G$ from $c', c \in \mathsf{C}$.

Proof. The diagram (1.5.114.1) is the pullback of $F \to G$ along $c' \to c \to G$, which is more fully illustrated as follows.

$$\begin{tikzcd} F \times_G c' \arrow[r] \arrow[d] & F \times_G c \arrow[r] \arrow[d] & F \arrow[d] \\ c' \arrow[r] & c \arrow[r] & G \end{tikzcd} \tag{1.5.114.2}$$

Now if $f_!$ preserves every pullback of $F \to G$, then it preserves the right fiber square and composite fiber square above, hence preserves the left fiber square (1.5.114.1) by cancellation (1.1.52).

It remains to show that if $f_!$ preserves all fiber squares (1.5.114.1), then it preserves all pullbacks of $F \to G$. Consider the pullback of $F \to G$ under a morphism $Z \to G$ from arbitrary $Z \in \mathsf{P}(\mathsf{C})$. Writing Z as a colimit of representables and appealing to the fact that presheaf pullback is cocontinuous (**??**) and $f_!$ is cocontinuous, we may reduce to the case that Z is representable. That is, we are to show that $f_!$ preserves the pullback square

$$\begin{tikzcd} F \times_G c_0 \arrow[r] \arrow[d] & F \arrow[d] \\ c_0 \arrow[r] & G \end{tikzcd} \tag{1.5.114.3}$$

for any map $c_0 \to G$ from $c_0 \in \mathsf{C}$. Present G via the tautological colimit diagram $G = \operatorname{colim}_{(\mathsf{C}\downarrow G)} c$ and note $F = \operatorname{colim}_{(\mathsf{C}\downarrow G)} F \times_G c$ (**??**). Thus we would like to show that the following pullback square

$$\begin{tikzcd} F \times_G c_0 \arrow[r] \arrow[d] & \operatorname*{colim}_{(\mathsf{C}\downarrow G)} F \times_G c \arrow[d] \\ c_0 \arrow[r] & \operatorname*{colim}_{(\mathsf{C}\downarrow G)} c \end{tikzcd} \tag{1.5.114.4}$$

in $\mathsf{P}(\mathsf{C})$ is preserved by $f_!$. Note that we can see the square (1.5.114.4) to be a pullback using (1.5.116) since the diagram $(\mathsf{C} \downarrow G) \to \mathsf{Fun}(\Delta^1, \mathsf{P}(\mathsf{C}))$ given by $F \times_G c \to c$ sends edges in $(\mathsf{C} \downarrow G)$ to pullback squares in $\mathsf{P}(\mathsf{C})$. This property is preserved by $f_!$ by assumption (1.5.114.1), so since $f_!$ is cocontinuous, we conclude it sends (1.5.114.4) to a pullback square. □

⋆ **1.5.115 Corollary** (Pullbacks and properties preserved by presheaf left Kan extension)**.** *Let $f : \mathsf{C} \to \mathsf{D}$ be a functor. If f preserves pullbacks, then $f_! : \mathsf{P}(\mathsf{C}) \to \mathsf{P}(\mathsf{D})$ preserves pullbacks of representable morphisms (in particular, sends representable morphisms to representable morphisms). More generally, for properties of morphisms $\mathcal{P}$ (in C) and $\mathcal{Q}$ (in D) preserved under pullback, if f sends pullbacks of $\mathcal{P}$-morphisms to pullbacks of $\mathcal{Q}$-morphisms (in particular, sends $\mathcal{P}$-morphisms to $\mathcal{Q}$-morphisms), then so does $f_!$.*

Proof. We apply (1.5.114). If $F \to G$ is a $\mathcal{P}$-morphism in $\mathsf{P}(\mathsf{C})$, then the pullbacks (1.5.114.1) are pullbacks of $\mathcal{P}$-morphisms in C, hence are preserved by f by assumption. Thus (1.5.114) guarantees that $f_!$ preserves all pullbacks of $F \to G$. To see that $f_!(F \to G)$ is a $\mathcal{Q}$-morphism, note that every morphism $d \to f_!G$ from $d \in \mathsf{D}$ factors through $f_!(c \to G)$ for some $c \in \mathsf{C}$, and the pullback $f_!(F \times_G c \to c)$ of $f_!(F \to G)$ is a $\mathcal{Q}$-morphism in D by hypothesis on f. □

1.5.116 Lemma. *Let $p \to q$ be a morphism in $\mathsf{Fun}(K, \mathsf{Spc})$. If the map $p \to q$ sends edges in K to pullback squares in Spc, then the diagram*

$$\begin{tikzcd} p(v) \arrow[r] \arrow[d] & \operatorname*{colim}_K p \arrow[d] \\ q(v) \arrow[r] & \operatorname*{colim}_K q \end{tikzcd} \tag{1.5.116.1}$$

is a pullback in Spc.

Proof. Using the small object argument (**??**), represent $p \to q$ as a composition of left fibrations $P \to Q \to K$. Consider the diagram

$$\begin{tikzcd} P_k \arrow[r] \arrow[d] & P_{k'} \arrow[d] \\ Q_k \arrow[r] & Q_{k'} \end{tikzcd} \tag{1.5.116.2}$$

associated to an edge $k \to k'$ in K. Given an edge $q \to q'$ in Q lying over $k \to k'$, there is a map $P_q \to P_{q'}$ induced by the fact that $P \to Q$ is a left fibration. Equivalently, this is the map from the homotopy fiber of $P_k \to Q_k$ over q to the homotopy fiber of $P_{k'} \to Q_{k'}$ over q' induced by the edge $q \to q'$. Since the square (1.5.116.2) is a pullback, this map is a homotopy equivalence. Now any left fibration $P \to Q$ for which the maps $P_q \to P_{q'}$ associated to edges $q \to q'$ in Q are homotopy equivalences is a Kan fibration (**??**).

Now the desired diagram (1.5.116.1) may be realized concretely as the pullback of simplicial sets

$$\begin{tikzcd} P \times_K v \arrow[r] \arrow[d] & P \arrow[d] \\ Q \times_K v \arrow[r] & Q \end{tikzcd} \tag{1.5.116.3}$$

which remains a pullback in Spc since $P \to Q$ is a Kan fibration. □

Chapter 2

Topology

⋆ **2.0.1 Definition** (Topological space)**.** A *topology* on a set X is a collection of subsets $T \subseteq 2^X$ (called 'open subsets') satisfying the following axioms:

(2.0.1.1) $\varnothing$ and X are open.

(2.0.1.2) If U and V are open, then is $U \cap V$ is open.

(2.0.1.3) If U_α are open, then $\bigcup_\alpha U_\alpha$ is open.

A subset is called *closed* when its complement is open. A *topological space* is a set equipped with a topology. A map between topological spaces is called *continuous* when the inverse image of every open subset is open. The category of topological spaces is denoted Top.

2.0.2 Definition (Neighborhood)**.** Let X be a topological space, and let $x \in X$ be a point. A *neighborhood* of x is a subset $N \subseteq X$ containing an open subset $U \subseteq X$ which contains x. A *neighborhood base* at x is a collection $\mathcal{N}$ of neighborhoods of x with the property that every open subset $U \subseteq X$ containing x contains some $N \in \mathcal{N}$. To say that x has 'arbitrarily small' neighborhoods with some property means that the collection of all neighborhoods of x with this property is a neighborhood base of x.

2.0.3 Definition (Locally compact)**.** A topological space is called *locally compact* iff every point has arbitrarily small compact neighborhoods.

2.1 Properties of morphisms

We recall here some important properties of morphisms (1.1.40) of topological spaces.

$\star$ **2.1.1 Exercise.** Show that the following properties of morphisms of topological spaces $f : X \to Y$ are closed under composition (1.1.43).

(2.1.1.1) f is open (i.e. the image of any open set is open).

(2.1.1.2) f is closed (i.e. the image of any closed set is closed).

(2.1.1.3) f is an embedding (i.e. a homeomorphism onto its image).

(2.1.1.4) f has local sections (i.e. there is an open cover $Y = \bigcup_i U_i$ such that each inclusion $U_i \to Y$ factors through $X \to Y$).

$$\begin{tikzcd} & X \arrow[d] \\ U_i \arrow[ur, dashed] \arrow[r] & Y \end{tikzcd}$$

Show that the following properties of morphisms of topological spaces $f : X \to Y$ are preserved under pullback (1.1.54).

(2.1.1.5) f is open.

(2.1.1.6) f is an embedding.

(2.1.1.7) f is a closed embedding.

(2.1.1.8) f is has local sections.

Show that being closed is not preserved under pullback, but that it is preserved under pullback along open embeddings.

2.1.2 Exercise (Locally closed embedding)**.** A map of topological spaces $X \to Y$ is called a *locally closed embedding* iff it can be factored as a closed embedding $X \to U$ followed by an open embedding $U \to Y$. Show that locally closed embeddings are preserved under pullback and closed under composition.

2.1.3 Exercise (Locally trivial)**.** A map of topological spaces $X \to Y$ is called *locally trivial* iff there exists an open cover $Y = \bigcup_i U_i$ such that each restriction $X \times_Y U_i \to U_i$ is isomorphic to the projection $U_i \times F_i \to U_i$ for some topological space F_i. Show that being locally trivial is preserved under pullback.

$\star$ **2.1.4 Exercise** (Local isomorphism)**.** A map of topological spaces $X \to Y$ is called a *local isomorphism* iff there exists a collection of open embeddings $\{V_i \to X\}$ which is jointly surjective (an 'open covering') such that each composition $V_i \to X \to Y$ is an open embedding. Show that local isomorphisms are preserved under pullback and closed under composition.

$\star$ **2.1.5 Definition** (Target-local property)**.** Let $\mathcal{P}$ be a property of morphisms of topological spaces. We say $\mathcal{P}$ is *local on the target* when for every open cover $Y = \bigcup_i U_i$, a morphism $X \to Y$ has $\mathcal{P}$ iff every pullback $X \times_Y U_i \to U_i$ has $\mathcal{P}$. In particular, $\mathcal{P}$ is preserved under pullback by open embeddings.

2.1.6 Exercise. Show that the properties of morphisms of topological spaces (2.1.1.1)–(2.1.1.4) and (2.1.2)–(2.1.4) are local on the target.

2.1.7 Exercise. Let $\mathcal{P}$ be a property of morphisms of topological spaces which is preserved under pullback. Show that $\mathcal{P}$ is local on the target iff it satisfies the following two properties.

(2.1.7.1) For every map $X \to Y$ and every map $Z \to Y$ admitting local sections, if $X \times_Y Z \to Z$ has $\mathcal{P}$ then so does $X \to Y$.

(2.1.7.2) If a collection of maps $f_i : X_i \to Y_i$ all have $\mathcal{P}$, then so does their disjoint union $\bigsqcup_i f_i : \bigsqcup_i X_i \to \bigsqcup_i Y_i$.

2.1.8 Exercise. Let $\mathcal{P}$ be a property of morphisms of topological spaces which is local on the target (hence, in particular, preserved under pullback by open embeddings). Show that $\mathcal{P}$ is preserved under pullback by local isomorphisms.

⋆ **2.1.9 Definition** (Source-local property)**.** Let $\mathcal{P}$ be a property of morphisms of topological spaces. We say $\mathcal{P}$ is *local on the source* when for every open cover $X = \bigcup_i V_i$ and every collection of open sets $U_i \subseteq Y$ on the same index set, a map $f : X \to Y$ with $f(V_i) \subseteq U_i$ satisfies $\mathcal{P}$ iff all its restrictions $V_i \to U_i$ satisfy $\mathcal{P}$.

2.1.10 Exercise. Show that being open (2.1.1.1) is local on the source.

2.1.11 Exercise. Show that being a local isomorphism (2.1.4) is local on the source. Conversely, show that if $\mathcal{P}$ is local on the source and contains all isomorphisms, then it contains all local isomorphisms. This justifies the term 'local isomorphism'.

2.1.12 Exercise. Show that a property which is local on the source is also local on the target.

2.1.13 Exercise. Show that if $\mathcal{P}$ is local on the source, then $\varnothing \to Y$ has $\mathcal{P}$ for every topological space Y.

* *

Recall that closed maps of topological spaces are not preserved under pullback (2.1.1). 'Universally closed' is the weakest property which is preserved under pullback and implies closed (2.1.14). It turns out that this notion is a relative form of compactness. We will see that, like compactness, it has equivalent characterizations in terms of coverings and subnet convergence (2.1.23).

2.1.14 Definition (Universally closed)**.** A map of topological spaces $X \to Y$ is called *universally closed* when for every map $Z \to Y$, the pullback $X \times_Y Z \to Z$ is closed.

2.1.15 Exercise. Show that being universally closed is preserved under pullback, closed under composition, and local on the target.

2.1.16 Exercise. Show that an embedding of topological spaces is closed iff it is universally closed.

2.1.17 Exercise. Show that if the composition $X \to Y \to Z$ is universally closed and $X \to Y$ is surjective, then $Y \to Z$ is universally closed.

2.1.18 Definition (Limit pointed topological space)**.** A *limited pointed topological space* $(X, 0)$ is a topological space X together with a point $0 \in X$ whose complement is dense ($\overline{X \setminus 0} = X$); we set $X^* = X \setminus 0$. A map of limit pointed topological spaces $f : (X, 0_X) \to (Y, 0_Y)$ is a map satisfying $f^{-1}(0_Y) = 0_X$ (equivalently, $f(0_X) = 0_Y$ and $f(X^*) \subseteq Y^*$).

A limit pointed topological space X is called *discrete* when X^* has the discrete topology. Given any limit pointed topological space X, we can consider the topology on it obtained by adjoining all subsets of X^* as open sets; this is called the discretization X^δ. There is an evident map of limit pointed topological spaces $X^\delta \to X$, composition with which induces, for discrete limit pointed topological spaces Y, a bijection between maps of limit pointed topological spaces $Y \to X^\delta$ and $Y \to X$.

2.1.19 Exercise. Show that there is a unique topology on $\mathbb{Z}_{\geq 0} \cup \{\infty\}$ with the property that a map $f : \mathbb{Z}_{\geq 0} \cup \{\infty\} \to X$ is continuous iff the sequence $f(0), f(1), f(2), \ldots$ converges to $f(\infty)$. Show that $\mathbb{Z}_{\geq 0} \cup \{\infty\}$ with this topology is a discrete limit pointed topological space.

2.1.20 Definition (Swarm)**.** A *swarm* in a topological space X is a limited pointed topological space S and a map $S^* \to X$. A *completed swarm* is a map $S \to X$. A *limit* of a swarm $S^* \to X$ is a point $x \in X$ for which sending $0 \mapsto x$ defines a completed swarm, and a swarm is *convergent* iff it has a limit. A *subswarm* of a swarm $S^* \to X$ is its pre-composition with a map of limit pointed topological spaces $T \to S$.

A *relative swarm* on a map $X \to Y$ is a commuting diagram of solid arrows

$$\begin{tikzcd} S^* \arrow[r] \arrow[d] & X \arrow[d] \\ S \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd} \tag{2.1.20.1}$$

and a *completed relative swarm* is a relative swarm along with a dotted arrow making the diagram commute. The definition of limits, convergence, and subswarms carry over analogously to the relative context.

A (relative) swarm is called discrete when its underlying limit pointed topological space is discrete. Pre-composition with discretization is a subswarm.

2.1.21 Exercise. Show that the closure of a subset A of a topological space X is the set of limits of swarms $S^* \to X$ landing inside A.

2.1.22 Definition (Compact)**.** A topological space X is called *compact* when for every open cover $X = \bigcup_i U_i$ there exists a finite subcollection which cover X.

$\star$ **2.1.23 Proposition.** *For a map of topological spaces $f : X \to Y$, the following are equivalent:*

(2.1.23.1) (Universally closed) *For every map $Z \to Y$, the pullback $X \times_Y Z \to Z$ is closed.*

(2.1.23.2) (Finite subcover property) *For every $\{U_i \subseteq X\}_i$ covering $f^{-1}(y)$, there exists a finite subcollection which cover $f^{-1}(V)$ for some open neighborhood $y \in V \subseteq Y$.*

(2.1.23.3) (Subswarm lifting property) *Every relative swarm on $X \to Y$ has a convergent subswarm.*

(2.1.23.4) *The map $X \to Y$ is closed and has compact fibers.*

These conditions are a relative version of compactness: a topological space X is compact iff the map $X \to *$ is universally closed.

Proof. Let us show the subswarm lifting property (2.1.23.3) implies universal closedness (2.1.23.1). Since the subswarm lifting property is evidently preserved under pullback, it suffices to show that it implies that $X \to Y$ is closed. Let $A \subseteq X$ be closed, and let us show that $f(A)$ is closed. Suppose $S^* \to Y$ is a swarm contained in $f(A)$ converging to some $y \in Y$, and let us show that $y \in f(A)$. By passing to a subswarm, we can assume that S^* has the discrete topology, and hence we can lift $S^* \to Y$ to X so that it lands inside A. This is now a relative swarm, which (after passing to a further subswarm) has a limit by the subswarm lifting property, which lies in A since A is closed. We have thus shown $y \in f(A)$ as desired. □

Now for some properties of morphisms of topological spaces which are defined using the relative diagonal (1.1.57).

2.1.24 Exercise. Show that the diagonal of any map of topological spaces is an embedding. Show that the diagonal of any injective map of topological spaces is an isomorphism.

2.1.25 Exercise. Show that the diagonal of a local isomorphism of topological spaces is an open embedding.

⋆ **2.1.26 Exercise** (Separated)**.** Show that for a morphism of topological spaces $f : X \to Y$, the following are equivalent:

(2.1.26.1) Every pair of distinct points $x_1, x_2 \in X$ in the same fiber $f(x_1) = f(x_2)$ have disjoint open neighborhoods $U_1 \cap U_2 = \varnothing$, $x_i \in U_i \subseteq X$.

(2.1.26.2) The relative diagonal $X \to X \times_Y X$ is a closed embedding.

(2.1.26.3) Every relative swarm on $X \to Y$ has at most one limit.

A morphism satisfying these conditions is called *separated*; this is a relative version of the Hausdorff property (X is Hausdorff iff $X \to *$ is separated). Show that being separated is preserved under pullback, closed under composition, and local on the target.

⋆ **2.1.27 Exercise** (Proper)**.** A map of topological spaces is called *proper* iff all its iterated diagonals are universally closed. Show that a map has proper diagonal iff it is separated. Conclude that a map is proper iff it is separated and universally closed (in particular, $X \to *$ is proper iff X is compact Hausdorff).

2.1.28 Exercise. Show that a map of topological spaces is a proper local isomorphism iff it is locally trivial (2.1.3) with finite fibers.

Now that we have seen the notions of separatedness and properness, let us have a more abstract discussion of properties of morphisms of topological spaces defined in terms of their diagonal (1.1.59).

2.1.29 Exercise. Let $\mathcal{P}$ be a property of morphisms of topological spaces which is local on the target. Show that $\mathcal{P}_\Delta$ is also local on the target.

One reason to consider properties of the diagonal is to apply cancellation (1.1.62).

2.1.30 Exercise. Prove both directly and using cancellation that if $X \to Y \to Z$ are maps of topological spaces whose composition $X \to Z$ is separated, then the first map $X \to Y$ is separated.

2.1.31 Exercise. Prove both directly and using cancellation that if $X \to Y \to Z$ are maps of topological spaces whose composition $X \to Z$ is an embedding, then the first map $X \to Y$ is an embedding.

2.1.32 Exercise. Prove both directly and using cancellation that if $X \to Y \to Z$ are maps of topological spaces with $X \to Z$ an open embedding and $Y \to Z$ is a local isomorphism, then $X \to Y$ is an open embedding. Conclude that any section of a local isomorphism is an open embedding.

2.1.33 Exercise. Prove both directly and using cancellation that if $X \to Y \to Z$ are maps of topological spaces with $X \to Z$ universally closed and $Y \to Z$ separated, then $X \to Y$ is universally closed. Deduce that a compact subspace of a Hausdorff space is closed.

2.2 Sheaves

A presheaf F on a topological space X assigns to each open subset $U \subseteq X$ a set $F(U)$ and to each inclusion $V \subseteq U$ a 'restriction' map $F(U) \to F(V)$, compatible with composition for triples $W \subseteq V \subseteq U$. A presheaf F is called a sheaf when, roughly speaking, an element of $F(U)$ amounts to *local data* on U, where 'locality' is understood via the restriction maps. Sheaves originated in work of Leray [54, 64], though the modern definition of a sheaf was formulated a bit later, notably by Cartan. It makes sense to consider presheaves and sheaves valued in any category, not just the category of sets. Sheaves valued in 2-categories were first considered by Giraud [25], who introduced sheaves valued in the 2-category of groupoids. More generally, one can consider sheaves valued in any ∞-category.

Here we review the basic theory of sheaves, sheafification, pushforward, pullback, etc. We will also explain the meaning and the utility of ∞-categories in the context of sheaves.

2.2.1 Definition (Category of open subsets)**.** Let X be a topological space. We denote by $\mathsf{Open}(X)$ the poset of open subsets of X, regarded as a category as in (1.1.3); that is, an object of $\mathsf{Open}(X)$ is an open subset $U \subseteq X$, and there is a single morphism from U to V when $U \subseteq V$.

⋆ **2.2.2 Definition** (Presheaf on a topological space)**.** A *presheaf* on a topological space X valued in C is a functor $\mathsf{Open}(X)^{\mathsf{op}} \to \mathsf{C}$. The category of such presheaves is denoted $\mathsf{P}(X;\mathsf{C}) = \mathsf{Fun}(\mathsf{Open}(X)^{\mathsf{op}}, \mathsf{C})$. By default, a presheaf without further qualification is a presheaf of sets (that is, valued in Set); notationally $\mathsf{P}(X) = \mathsf{P}(X;\mathsf{Set})$. Dually, a *precosheaf* is a functor $\mathsf{Open}(X) \to \mathsf{C}$.

2.2.3 Example. Here are some examples of presheaves.

(2.2.3.1) For any topological space X, we can assign to $U \subseteq X$ the set $C(U)$ of continuous functions $U \to \mathbb{R}$, and to an inclusion $U \subseteq V$ the restriction map $C(V) \to C(U)$. Thus $U \mapsto C(U)$ is a presheaf on X.

(2.2.3.2) $U \mapsto C(U \times U)$ is a presheaf on any topological space.

(2.2.3.3) Associating to $U \subseteq X$ the set of embeddings of U into $\mathbb{R}^n$ (some fixed n) is a presheaf (the restriction of an embedding is an embedding).

(2.2.3.4) The *constant presheaf* assigns to every $U \subseteq X$ a fixed set S and to every inclusion the identity map $\mathbf{1}_S$.

(2.2.3.5) Associating to U the set of isomorphism classes of vector bundles on U is a presheaf.

(2.2.3.6) On a smooth manifold, the assignment $U \mapsto C^\infty(U)$ is a presheaf.

(2.2.3.7) On a smooth manifold, assigning to U the set of smooth embeddings of U into a fixed $\mathbb{R}^n$ is a presheaf.

(2.2.3.8) On a smooth manifold, assigning to U the set of smooth immersions of U into a fixed $\mathbb{R}^n$ is a presheaf.

We have omitted an explicit description of the restriction maps for most of these examples since they are quite obvious. The same holds for most presheaves we will encounter.

⋆ **2.2.4 Definition** (Čech descent)**.** Let F be a presheaf on a topological space X. Given any open covering $X = \bigcup_i U_i$, there is a natural map

$$F(X) \to \lim\Big(\prod_i F(U_i) \rightrightarrows \prod_{i,j} F(U_i \cap U_j)\Big), \tag{2.2.4.1}$$

and we say that F satisfies *descent* for the open cover $X = \bigcup_i U_i$ when this map is an isomorphism.

⋆ **2.2.5 Definition** (Sheaf on a topological space)**.** A presheaf F on a topological space X is called a *sheaf* when it satisfies descent (2.2.4) for all open covers of all open subsets of X. The category of sheaves $\mathsf{Shv}(X; \mathsf{C})$ is the full subcategory of $\mathsf{P}(X; \mathsf{C})$ spanned by sheaves. As with presheaves, the default is $\mathsf{C} = \mathsf{Set}$ unless specified otherwise. Dually, a *cosheaf* is a precosheaf satisfying descent (for its opposite presheaf).

2.2.6 Example. The presheaf of continuous real valued functions (2.2.3.1) is a sheaf on any topological space X. This amounts to two separate assertions: (i) to specify a function, it is equivalent to specify it on an open cover subject to the requirement of agreement on overlaps, and (ii) a function is continuous iff it is locally continuous. There is nothing special about the target being $\mathbb{R}$: the same holds for continuous functions $C(-, Z)$ valued in any fixed topological space Z.

2.2.7 Exercise. Of the remaining presheaves (2.2.3.2)–(2.2.3.8), which are sheaves? For those which are not, what exactly fails?

2.2.8 Exercise. Show that the identity functor $\mathsf{Top} \to \mathsf{Top}$ is a cosheaf.

It turns out that the sheaf property, namely Čech descent (2.2.4), admits an equivalent formulation in terms of so-called 'covering sieves'. This alternative formulation is often useful.

2.2.9 Definition (Sieve)**.** A *sieve* S on a topological space X is a set of open subsets $U \subseteq X$ with the property that $V \subseteq U \in S$ implies $V \in S$. A *covering sieve* S on X is a sieve for which $\bigcup_{U \in S} U = X$. The set of covering sieves on a topological space X is denoted $J(X)$.

2.2.10 Definition (Sieve descent)**.** Let F be a presheaf on a topological space X. Associated to any covering sieve $S \in J(X)$ is a map

$$F(X) \to \lim_{U \in S} F(U). \tag{2.2.10.1}$$

When this map is an isomorphism, we say that F satisfies *descent* for the covering sieve S.

2.2.11 Exercise (Sieve descent equals Čech descent)**.** Show that a presheaf satisfies descent for all covering sieves on X iff it satisfies descent for all open coverings of X.

* *

2.2.12 Exercise (Étale space)**.** Fix a topological space X. Let $\mathsf{Top}_{/X}^{\mathsf{lociso}} \subseteq \mathsf{Top}_{/X}$ denote the full subcategory spanned by local isomorphisms to X. Consider the functor

$$\mathsf{Top}_{/X}^{\mathsf{lociso}} \to \mathsf{Shv}(X) \tag{2.2.12.1}$$

sending a local isomorphism $A \to X$ to its sheaf of sections. Show that this functor is an equivalence of categories, and that it identifies pullback of sheaves with pullback of local isomorphisms. The local isomorphism over X corresponding to a sheaf F on X is called the *étale space* of F.

⋆ **2.2.13 Proposition** (Sheafification)**.** *The inclusion* $\mathsf{Shv}(X) \subseteq \mathsf{P}(X)$ *is a reflective subcategory whose left adjoint, termed* sheafification *and denoted* $F \mapsto F^{\#}$*, is given by applying* $\dagger$ *twice.*

⋆ **2.2.14 Definition** (Čech nerve)**.** Let $X = \bigcup_{i \in I} U_i$ be an open cover. Denote by 2^I_{fin} the category of finite subsets of I, and consider the functor $(2^I_{\mathsf{fin}})^{\mathsf{op}} \to \mathsf{Top}$ given by $A \mapsto \bigcap_{i \in A} U_i$ (in particular $\varnothing \mapsto X$). We may regard 2^I_{fin} as the cone $(2^I_{\mathsf{fin}} \setminus \varnothing)^{\triangleleft}$ and thus obtain a comparison map

$$N(X, \{U_i\}_{i \in I}) = \underset{\substack{\varnothing \neq A \subseteq I \\ |I| < \infty}}{\overset{\mathsf{P}(\mathsf{Open}(X))}{\operatorname{colim}}} \bigcap_{i \in A} U_i \to X. \tag{2.2.14.1}$$

The *Čech nerve* $N(X, \{U_i\}_{i \in I})$ of the open cover $X = \bigcup_{i \in I} U_i$ is the above formal colimit (i.e. colimit in $\mathsf{P}(\mathsf{Open}(X))$) over $(2^I_{\mathsf{fin}} \setminus \varnothing)^{\mathsf{op}}$.

2.2.15 Definition (Homotopy coherent pushforward and pullback)**.** Consider the category $\mathsf{Open} \rtimes \mathsf{Top}$ of pairs (X, U) where X is a topological space and $U \subseteq X$ is an open subset, in which a morphism $(X, U) \to (Y, V)$ is a map $f : X \to Y$ with $f(U) \subseteq V$ (equivalently $U \subseteq f^{-1}(V)$). The functor

$$\mathsf{Open} \rtimes \mathsf{Top} \to \mathsf{Top} \tag{2.2.15.1}$$
$$(X, U) \mapsto X \tag{2.2.15.2}$$

is (by inspection) cartesian, the cartesian edges being the morphisms $(X, f^{-1}(V)) \to (Y, V)$ for $f : X \to Y$. This cartesian functor encodes the categories $\mathsf{Open}(X)$ for topological spaces X and the functors $f^{-1} = \mathsf{Open}(f) : \mathsf{Open}(Y) \to \mathsf{Open}(X)$ for maps $f : X \to Y$.

2.3 Topological stacks

In (2.2), we studied sheaves on a fixed topological space. We now turn to sheaves on the category of all topological spaces, where the discussion takes a markedly different, more geometric, flavor. We will call a sheaf on the category of topological spaces a *topological stack* (we find this terminology the most descriptive, though it is not standard). The Yoneda functor gives a fully faithful embedding from the category of topological spaces into the category of topological stacks

$$\mathsf{Top} \subseteq \mathsf{Shv}(\mathsf{Top}), \tag{2.3.0.1}$$

and it is helpful to regard topological stacks as 'generalized topological spaces'. We will see how to generalize many natural notions and constructions from topological spaces to topological stacks. Arbitrary topological stacks are a bit like arbitrary topological spaces: they can be very pathological and are not of so much interest. There is a particularly nice class of topological stacks, namely those which admit a *representable atlas*; they are equivalent, in a certain sense, to 'topological groupoids' as introduced by Ehresmann [20] and developed by Haefliger [27, 28, 29] and others. Examples of such topological stacks include orbifolds [80, 86] and graphs/complexes of groups [29].

References for the theory we are about to discuss include Noohi [71] and Heinloth [30]. It is a topological analogue of the theory of algebraic stacks originating from Grothendieck, Deligne–Mumford [12], and Artin [4], for which a comprehensive reference is Laumon–Moret-Bailly [51]. This topological analogue is an easier, more elementary, version of the algebraic theory; it was documented only much later in Noohi [71]. An intuitive geometric introduction may be found in Behrend [7].

We will work in the generality of sheaves of ∞-groupoids $\mathsf{Shv}(-) = \mathsf{Shv}(-;\mathsf{Spc})$. We emphasize, however, that the reader may restrict to the technically and conceptually simpler setting of sheaves of groupoids $\mathsf{Shv}(-;\mathsf{Grpd}) \subseteq \mathsf{Shv}(-;\mathsf{Spc})$ and retain the essence of the discussion (in fact, this is the setting addressed by all of the aforementioned references).

* *

⋆ **2.3.1 Definition** (Topological stack)**.** A *topological stack* is a sheaf on Top valued in the ∞-category Spc (that is, a functor $\mathsf{Top}^{\mathsf{op}} \to \mathsf{Spc}$ satisfying descent for open covers (2.2.4)(2.2.10)(**??**)). Topological stacks form the ∞-category $\mathsf{Shv}(\mathsf{Top})$.

2.3.2 Proposition (Universal property of topological stacks)**.** *For any cocomplete ∞-category E, pullback along the functors $\mathsf{Top} \xrightarrow{y_{\mathsf{Top}}} \mathsf{P}(\mathsf{Top}) \xrightarrow{\#} \mathsf{Shv}(\mathsf{Top})$ defines equivalences between the following ∞-categories of functors:*

(2.3.2.1) *Cocontinuous functors $\mathsf{Shv}(\mathsf{Top}) \to \mathsf{E}$.*

(2.3.2.2) *Cocontinuous functors $\mathsf{P}(\mathsf{Top}) \to \mathsf{E}$ which send sheafifications to isomorphisms.*

(2.3.2.3) *Cocontinuous functors $\mathsf{P}(\mathsf{Top}) \to \mathsf{E}$ which send Čech nerves $N(X, \{U_i\}_i) \to X$ to isomorphisms.*

(2.3.2.4) *Cosheaves $\mathsf{Top} \to \mathsf{E}$.*

Proof. This is a special case of the universal property of local presheaves (1.5.106), given the fact that a presheaf is a sheaf iff it is right local (1.1.82) with respect to Čech nerves $N(X,\{U_i\}_i)\to X$ (**??**). $\square$

⋆ **2.3.3 Definition** (Point of a topological stack)**.** A *point* x of a topological stack X is a map $x:*\to X$, i.e. it is an object $x\in X(*)$ (also simply written $x\in X$).

* *

Recall the notion of a representable morphism of presheaves (1.1.90) and induced properties (1.1.92).

2.3.4 Example (Open cover)**.** A morphism of topological stacks $U\to X$ is called an open embedding (1.1.92) when for every morphism $Z\to X$ from a topological space Z, the pullback $U\times_X Z\to Z$ is an open embedding of topological spaces. A collection of open embeddings $\{U_i\to X\}_i$ is called an open cover when for every morphism $Z\to X$ from a topological space Z, the collection of pullbacks $\{U_i\times_X Z\to Z\}_i$ is an open cover.

* *

⋆ **2.3.5 Definition** (Admits local sections)**.** A map of topological stacks $X\to Y$ is said to *admit local sections* iff for every map $U\to Y$ from a topological space U, there exists an open cover $U=\bigcup_i U_i$ so that each restriction $U_i\to Y$ lifts to X.

$$\begin{tikzcd} & & X \arrow[d] \\ U_i \arrow[r] \arrow[urr, dashed] & U \arrow[r] & Y \end{tikzcd} \tag{2.3.5.1}$$

2.3.6 Exercise. Show that a representable map of topological stacks admits local sections in the sense of (2.3.5) iff it does so in the sense of (1.1.92).

2.3.7 Exercise. Show that admitting local sections is preserved under pullback and closed under composition.

⋆ **2.3.8 Lemma.** *For any map of topological stacks $U\to X$ admitting local sections, the natural map*

$$\operatorname*{colim}^{\mathsf{Shv(Top)}}\Big(\cdots \substack{\to\\\to\\\to\\\to} U\times_X U\times_X U \substack{\to\\\to\\\to} U\times_X U \rightrightarrows U\Big)\to X \tag{2.3.8.1}$$

is an isomorphism.

* *

2.3.9 Exercise. Let X be a topological stack. A subset $E \subseteq |X(*)|$ ($|\cdot|$ denotes isomorphism classes) determines an assignment to every map $f : Z \to X$ of a subset $Z_{E,f} \subseteq Z$ which is compatible with pullback in the sense that $Z'_{E,f\circ g} = g^{-1}(Z_{E,f})$ for every map $g : Z' \to Z$. Show that this defines a bijection between subsets of $|X|$ and pullback compatible assignments of subsets of Z to maps $Z \to X$.

2.3.10 Exercise (Classification of embedded substacks)**.** Let X be a topological stack, let $E \subseteq |X(*)|$ be any subset, and let X_E denote the topological stack for which a map $Z \to X_E$ is a map $Z \to X$ whose specialization to every point of Z lies in E. Show that for $f : Z \to X$, the natural diagram

$$\begin{tikzcd} Z_{E,f} \arrow[r] \arrow[d] & X_E \arrow[d] \\ Z \arrow[r, "f"] & X \end{tikzcd} \tag{2.3.10.1}$$

is a pullback square. Conclude that $X_E \to X$ is an embedding (2.1.1.3)(1.1.92) and that $X_E \to X$ satisfies a property $\mathcal{P}$ preserved under pullback iff every $Z_{E,f} \subseteq Z$ satisfies $\mathcal{P}$. Moreover, show that every embedding $X' \to X$ is uniquely isomorphic to a unique $X_E \to X$.

* *

⋆ **2.3.11 Definition** (Universally closed)**.** A map of topological stacks $X \to Y$ is said to be *universally closed* when it satisfies the subswarm lifting property (2.1.23.3), namely that for any commuting diagram of solid arrows

$$\begin{tikzcd} T^* \arrow[r, dashed] \arrow[d, dashed] & S^* \arrow[r] \arrow[d] & X \arrow[d] \\ T \arrow[r, dashed] \arrow[urr, dashed] & S \arrow[r] & Y \end{tikzcd} \tag{2.3.11.1}$$

in which (S, S^*) is a limit pointed topological space (2.1.18), there exists a map of limit pointed topological spaces $(T, T^*) \to (S, S^*)$ and a diagonal dotted arrow making the diagram commute.

2.3.12 Exercise. Show that universal closedness (2.3.11) is preserved under pullback, closed under composition, and local on the target (**??**).

2.3.13 Exercise. Show that a representable morphism of topological stacks is universally closed in the sense of (2.3.11) (satisfies the subswarm lifting property) iff it is universally closed in the sense of (1.1.92) (every pullback to a topological space is universally closed).

⋆ **2.3.14 Definition** (Proper)**.** A map of topological stacks is called *proper* when its iterated diagonals are all universally closed.

⋆ **2.3.15 Definition** (Separated)**.** A map of topological stacks is called *separated* when its diagonal is proper.

* *

⋆ **2.3.16 Definition** (Atlas)**.** Let X be a topological stack. A morphism $U \to X$ admitting local sections (where U is a topological space) is called an *atlas* for X.

2.3.17 Proposition (Proper atlas from proper diagonal)**.** *Let X have a representable atlas and proper diagonal, and let $U \to X$ be a map from a locally compact* (2.0.3) *Hausdorff topological space U. Suppose $p \in U$ is such that $\underline{\mathrm{Aut}}(p) = p \times_X p \subseteq p \times_X U$ is open (note that $p \to U$ is a closed embedding, hence so is its pullback $p \times_X p \to p \times_X U$). For every sufficiently small open neighborhood $V \subseteq U$ of p, we have $p \times_X p = p \times_X V$ and the map $V \to X$ is proper over an open substack of X containing the image of p.*

Proof. By hypothesis, $p \times_X (U \setminus p) \subseteq p \times_X U$ is closed, and $p \times_X U \to U$ is proper since it is a pullback of the diagonal of X. Thus the image of $p \times_X (U \setminus p) \to U$ is a closed set disjoint from p. By replacing U with the complement of this closed set, we may assume wlog that $p \times_X p = p \times_X U$ (p is unique in its orbit).

Since U is locally compact, there exists a compact neighborhood $K \subseteq U$ of p. Since $K \to *$ is proper and $X \to *$ has proper diagonal, it follows that the map $K \to X$ is proper (1.1.62). Suppose $V \subseteq U$ is open and contained in K. Hence $V \subseteq K$ is open, so $K \setminus V \to K$ is a closed embedding, hence proper, so $K \setminus V \to X$ is also proper. It is representable, hence its image (embedded substack of X) is closed (consider its pullback under any map from a topological space Z to X). Let $Y \subseteq X$ denote the complement of the image of $K \setminus V \to X$ (thus Y is an open substack of X); note that Y contains the image of p since p is unique in its orbit. Thus $V \times_X Y = K \times_X Y \to Y$ is a pullback of $K \to X$, hence is proper. □

* *

⋆ **2.3.18 Definition** (n-Artin morphism)**.** A morphism of topological stacks is called (-1)-*Artin* iff it is an isomorphism. A morphism of topological stacks $X \to Y$ is called n*-Artin* (for integer $n \geq 0$) when for every map $U \to Y$ from a topological space U, the pullback $X \times_Y U$ admits an $(n-1)$-Artin atlas. It is immediate that n-Artin morphisms are preserved under pullback.

2.3.19 Lemma. *n-Artin morphisms are closed under composition.*

Proof. Fix n-Artin maps $X \to Y \to Z$, and consider a map $U \to Z$ from representable U. Since $Y \to Z$ is n-Artin, there exists an $(n-1)$-Artin atlas $V \to Y \times_Z U$. Since $X \to Y$ is n-Artin, there exists an $(n-1)$-Artin atlas $W \to X \times_Y V$.

$$\begin{tikzcd} W \arrow[r] & X \times_Y V \arrow[r] \arrow[d] & X \times_Z U \arrow[r] \arrow[d] & X \arrow[d] \\ & V \arrow[r] & Y \times_Z U \arrow[r] \arrow[d] & Y \arrow[d] \\ & & U \arrow[r] & Z \end{tikzcd} \tag{2.3.19.1}$$

The maps $W \to X \times_Y V \to X \times_Z U$ are both $(n-1)$-Artin and admit local sections, hence so does their composition (by induction on n). This is the desired $(n-1)$-Artin atlas for $X \times_Z U$. □

2.3.20 Lemma. *The diagonal of an n-Artin morphism is* $\max(n-1, 0)$*-Artin.*

Proof. Let $X \to Y$ is n-Artin.

By (**??**), every pullback of $X \to X \times_Y X$ to a topological space U is a pullback of the diagonal of the pullback $X \times_Y U \to U$. We may thus assume wlog that Y is representable.

Since Y is representable, there exists an $(n-1)$-Artin atlas $U \to X$. We consider the following fiber square.

$$\begin{tikzcd} U \times_X U \arrow[r] \arrow[d] & X \arrow[d] \\ U \times_Y U \arrow[r] & X \times_Y X \end{tikzcd} \tag{2.3.20.1}$$

The pullback $U \times_Y U$ is representable since U and Y are representable. The pullback $U \times_X U$ has a $\max(-1, n-2)$-Artin atlas since U is representable and $U \to X$ is $(n-1)$-Artin. Thus the morphism $U \times_X U \to U \times_Y U$ is $\max(0, n-1)$-Artin (**??**). The map $U \times_Y U \to X \times_Y X$ admits local sections since $U \to X$ does (1.1.56), hence $X \to X \times_Y X$ is $\max(0, n-1)$-Artin (**??**). □

2.4 Mapping stacks

Here we study topological stacks parameterizing maps between topological spaces.

⋆ **2.4.1 Definition** (Mapping stack $\underline{\mathrm{Hom}}(X, Y)$)**.** For topological spaces X and Y, the topological stack $\underline{\mathrm{Hom}}(X, Y)$ is defined by declaring a map $Z \to \underline{\mathrm{Hom}}(X, Y)$ to be a continuous map $Z \times X \to Y$.

2.4.2 Example. The set of maps $* \to \underline{\mathrm{Hom}}(X, Y)$ is the set $\mathrm{Hom}(X, Y)$ of continuous maps $X \to Y$.

2.4.3 Exercise. Show that $\underline{\mathrm{Hom}}(X, Y)$ is a sheaf on Top.

2.4.4 Exercise. Show that the natural map $\underline{\mathrm{Hom}}(X, Y \times Y') \to \underline{\mathrm{Hom}}(X, Y) \times \underline{\mathrm{Hom}}(X, Y')$ is an isomorphism.

2.4.5 Exercise. Show that there is a tautological 'evaluation' map $X \times \underline{\mathrm{Hom}}(X, Y) \to Y$.

⋆ **2.4.6 Definition** (Condition on morphisms)**.** A *condition* $\mathcal{C}$ on morphisms $X \to Y$ is a subset $\mathrm{Hom}_{\mathcal{C}}(X, Y) \subseteq \mathrm{Hom}(X, Y)$. Equivalently, a condition is the assignment to every map $f : Z \to \underline{\mathrm{Hom}}(X, Y)$ of a subset $Z_{\mathcal{C},f} \subseteq Z$ which is compatible with pullback in the sense that $Z'_{\mathcal{C},f\circ g} = g^{-1}(Z_{\mathcal{C},f})$ for any map $g : Z' \to Z$ (2.3.9). Given a condition $\mathcal{C}$, we can consider the embedded substack $\underline{\mathrm{Hom}}_{\mathcal{C}}(X, Y) \subseteq \underline{\mathrm{Hom}}(X, Y)$ parameterizing those maps $Z \times X \to Y$ whose specialization to every $z \in Z$ lies in $\mathrm{Hom}_{\mathcal{C}}(X, Y)$; this defines a bijection between conditions on morphisms $X \to Y$ and embedded substacks of $\underline{\mathrm{Hom}}(X, Y)$ (2.3.10).

Given a property of morphisms of topological stacks $\mathcal{P}$, we say that a condition $\mathcal{C}$ satisfies $\mathcal{P}$ when the morphism $\underline{\mathrm{Hom}}_{\mathcal{C}}(X, Y) \to \underline{\mathrm{Hom}}(X, Y)$ has $\mathcal{P}$. Concretely, this just means that the inclusion $Z_{\mathcal{C},f} \to Z$ has $\mathcal{P}$ for every map $f : Z \times X \to Y$.

2.4.7 Exercise. Show that $f(A) \subseteq V$ is a closed condition on maps $f : X \to Y$ for any subset $A \subseteq X$ and any closed subset $V \subseteq Y$.

2.4.8 Exercise. Show that $f|_A = \mathbf{1}_A$ is a closed condition on maps $f : X \to X$ for any subset $A \subseteq X$ provided X is Hausdorff.

2.4.9 Lemma. *For $K \subseteq X$ compact and $U \subseteq Y$ open, the condition $f(K) \subseteq U$ is open.*

Proof. Equivalently, we show that the condition $f(K) \cap V \neq \varnothing$ is closed for $V \subseteq Y$ closed. This condition may be alternatively stated as $f^{-1}(V) \cap K \neq \varnothing$. Given a map $F : Z \times X \to Y$, the subset $Z_F \subseteq Z$ of maps satisfying this condition is the image of $F|^{-1}_{Z\times K}(V)$ under the projection $Z \times K \to Z$. The inverse image $F|^{-1}_{Z\times K}(V)$ is closed, so its projection to Z is closed since $K \to *$ is universally closed (2.1.23). □

2.4.10 Lemma. *The diagonal of* $\underline{\mathrm{Hom}}(X, Y)$ *is an embedding* (2.1.1.3)*.*

Proof. The diagonal of $\underline{\mathrm{Hom}}(X, Y)$ is the map $\underline{\mathrm{Hom}}(X, Y) \to \underline{\mathrm{Hom}}(X, Y \times Y)$ (2.4.4). Since the diagonal $Y \to Y \times Y$ is an embedding (2.1.24), it follows that $\underline{\mathrm{Hom}}(X, Y) = \underline{\mathrm{Hom}}_{\mathcal{C}}(X, Y \times Y)$ where $\mathcal{C}$ is the condition of having image contained in the diagonal. The inclusion of the subsheaf of maps satisfying any condition $\mathcal{C}$ is an embedding (2.4.6). □

2.4.11 Exercise. Show that if Y is Hausdorff, then the diagonal of $\underline{\mathrm{Hom}}(X, Y)$ is a closed embedding.

We now turn to representability of $\underline{\mathrm{Hom}}(X, Y)$. As remarked earlier, the set of maps $* \to \underline{\mathrm{Hom}}(X, Y)$ is simply the set of maps $X \to Y$. It follows that $\underline{\mathrm{Hom}}(X, Y)$ is representable iff there is a topology $\mathcal{T}$ on the set $\mathrm{Hom}(X, Y)$ such that a map $Z \times X \to Y$ is continuous iff the induced map $Z \to \mathrm{Hom}(X, Y)_{\mathcal{T}}$ is continuous.

2.4.12 Definition (Compact-open topology)**.** Let X and Y be topological spaces. The compact-open topology on the set $\mathrm{Hom}(X, Y)$ is the topology generated by declaring that, for all compact sets $K \subseteq X$ and open sets $U \subseteq Y$, the locus of maps $f : X \to Y$ satisfying $f(K) \subseteq U$ should be open. The resulting topological space is denoted $\mathrm{Hom}(X, Y)_{\mathrm{cptopen}}$.

For any map $Z \to \underline{\mathrm{Hom}}(X, Y)$, the induced map $Z \to \mathrm{Hom}(X, Y)_{\mathrm{cptopen}}$ is continuous by (2.4.9), which gives a tautological map $\underline{\mathrm{Hom}}(X, Y) \to \mathrm{Hom}(X, Y)_{\mathrm{cptopen}}$. Hence if $\underline{\mathrm{Hom}}(X, Y)$ is representable, necessarily by $\mathrm{Hom}(X, Y)_{\mathcal{T}}$ for some topology $\mathcal{T}$, then $\mathcal{T}$ is at least as fine as the compact-open topology.

⋆ **2.4.13 Proposition.** *If X is locally compact* (2.0.3)*, then the map $\underline{\mathrm{Hom}}(X, Y) \to \mathrm{Hom}(X, Y)_{\mathrm{cptopen}}$ is an isomorphism. In particular, $\underline{\mathrm{Hom}}(X, Y)$ is representable.*

Proof. We are to show that if $Z \to \mathrm{Hom}(X, Y)_{\mathrm{cptopen}}$ is continuous, then the resulting map $Z \times X \to Y$ is also continuous. What we must show is that if (z, x) is sent inside an open set $U \subseteq Y$, then a neighborhood of (z, x) is as well. Since X is locally compact, there is a compact neighborhood $K \subseteq X$ of x such that $z \times K$ is sent inside U. Since $Z \to \mathrm{Hom}(X, Y)_{\mathrm{cptopen}}$ is continuous in the compact-open topology, there is an open set $V \subseteq Z$ such that $V \times K$ is sent inside U. □

* *

The basic mapping stack $\underline{\mathrm{Hom}}(-, -)$ (2.4.1) admits several important generalizations, such as the stack of sections of a fixed map $E \to X$ or maps between fibers $X_b \to Y_b$ of maps $X, Y \to B$. Here is the most general notion we will consider.

⋆ **2.4.14 Definition** (Parameterized stack of sections $\underline{\mathrm{Sec}}$)**.** Let $E \to X \to B$ be morphisms in a category C which has all pullbacks of $X \to B$. We define a presheaf $\underline{\mathrm{Sec}}_B(X, E)$ on C by the property that a map $Z \to \underline{\mathrm{Sec}}_B(X, E)$ from $Z \in \mathsf{C}$ is a map $Z \to B$ along with a map $X \times_B Z \to E$ over X.

$$\begin{tikzcd} & E \arrow[d] \\ X \times_B Z \arrow[r] \arrow[ur] \arrow[d] & X \arrow[d] \\ Z \arrow[r] & B \end{tikzcd} \tag{2.4.14.1}$$

Our present interest will be in the case of topological spaces, in which case $\underline{\mathrm{Sec}}_B(X, E)$ is evidently a sheaf.

2.4.15 Example. A point $* \to \underline{\mathrm{Sec}}_B(X,E)$ is a point $b \in B$ together with a section of $E_b \to X_b$.

2.4.16 Exercise. Show that a diagram

$$\begin{array}{ccc} E & \longrightarrow & F \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ B & \longrightarrow & C \end{array} \tag{2.4.16.1}$$

induces a map $\underline{\mathrm{Sec}}_B(X,E) \to \underline{\mathrm{Sec}}_C(Y,F)$. Show that the tautological maps

$$\underline{\mathrm{Sec}}_{B'}(X \times_B B', E \times_B B') \xrightarrow{\sim} \underline{\mathrm{Sec}}_B(X,E) \times_B B' \tag{2.4.16.2}$$

$$\underline{\mathrm{Sec}}_B(X, E \times_X F) \xrightarrow{\sim} \underline{\mathrm{Sec}}_B(X,E) \times_B \underline{\mathrm{Sec}}(X,F) \tag{2.4.16.3}$$

are both isomorphisms.

2.4.17 Exercise. Show that for any embedding $E \to F$ (over X), the induced map $\underline{\mathrm{Sec}}_B(X,E) \to \underline{\mathrm{Sec}}_B(X,F)$ is also an embedding (compare (2.4.6)). Conclude that the diagonal of $\underline{\mathrm{Sec}}_B(X,E) \to B$ is an embedding (and so, in particular, representable).

2.4.18 Exercise. Let $s : B \to X$ be a section, and let $F \subseteq s^*E := E \times_X B$ be a closed substack. Show that the condition on $\underline{\mathrm{Sec}}_B(X,E)$ of sending s to F is a closed condition.

2.4.19 Lemma. *If $E \to F$ is a closed embedding (over X) and $X \to B$ is open, then $\underline{\mathrm{Sec}}_B(X,E) \to \underline{\mathrm{Sec}}_B(X,F)$ is a closed embedding.*

Proof. We saw earlier that $\underline{\mathrm{Sec}}_B(X,E) \to \underline{\mathrm{Sec}}_B(X,F)$ is an embedding (2.4.17). Fix a map $Z \to \underline{\mathrm{Sec}}_B(X,F)$, namely a diagram (2.4.14.1), and let us show that the locus of $z \in Z$ for which the specialization of the map $X \times_B Z \to F$ lands inside E is closed. The inverse image of $E \subseteq F$ is a closed subset K of $X \times_B Z$. Since the projection $X \times_B Z \to Z$ is open (being a pullback of $X \to B$), the locus of points $z \in Z$ whose inverse image $X \times_B z$ is contained in K is closed (being the complement of the image of the complement of K). □

2.4.20 Exercise. Fix topological spaces $E \to X \to B$. Fix a map $X_0 \to X$ for which the composition $X_0 \to B$ is open, and let $E_0 \subseteq E \times_X X_0$ be a closed substack. Conclude from (2.4.19) that the condition on $\underline{\mathrm{Sec}}_B(X,E)$ of mapping X_0 inside E_0 is closed.

2.4.21 Exercise. Show that if $E \to X$ is separated and $X \to B$ is open, then $\underline{\mathrm{Sec}}_B(X,E) \to B$ is separated.

2.4.22 Exercise. Let X be the locus $\{xy = 0\}$ (the union of the two axes in $\mathbb{R}^2$), and let $X \to B = \mathbb{R}$ be the projection to the x-coordinate. Show that $\underline{\mathrm{Hom}}_B(X,\mathbb{R}) \to B$ is not separated (a map $Z \to \underline{\mathrm{Hom}}_B(X,\mathbb{R})$ is a map $Z \to B$ and a map $X \times_B Z \to \mathbb{R}$).

2.4.23 Exercise. Argue as in (2.4.9) to show that if $E^\circ \subseteq E$ is open and $X \to B$ is universally closed, then $\underline{\mathrm{Sec}}_B(X,E^\circ) \to \underline{\mathrm{Sec}}_B(X,E)$ is an open embedding.

2.5 Stability

Recall that a topological stack is called separated when its diagonal is proper, and that this is a generalization of the Hausdorff condition to topological stacks (2.3). Many topological stacks of interest, for instance the moduli stack of compact nodal Riemann surfaces, are non-separated. Rather, they contain an open substack of 'stable' points, which is instead the object of interest for many purposes. In this section, we introduce a general structure which allows us to pick out this open 'stable locus' and deduce properties of it from properties of the ambient stack.

2.5.1 Exercise (Stable object)**.** Show that for an object X in a category C, the following are equivalent:

(2.5.1.1) Every morphism $Z \to X$ is a terminal object in the under-category $\mathsf{C}_{Z/}$.

(2.5.1.2) Every morphism $A \to B$ induces an isomorphism $\operatorname{Hom}(B, X) \to \operatorname{Hom}(A, X)$.

(2.5.1.3) For every diagram of solid arrows

$$\begin{tikzcd} A \arrow[r] \arrow[dr] & B \arrow[d, dashed, "\exists!"] \\ & X \end{tikzcd}$$

there exists a unique dotted arrow making the diagram commute.

We call an object X satisfying these conditions *stable*.

2.5.2 Exercise. Show that every morphism out of a stable object is a split monomorphism. Conclude that every morphism between stable objects is an isomorphism.

2.5.3 Exercise (Category with enough stable objects)**.** Show that for a category C, the following are equivalent:

(2.5.3.1) Every object admits a morphism to a stable object.

(2.5.3.2) Every under-category $\mathsf{C}_{Z/}$ has a terminal object, and for every morphism $Z \to Y$, the induced functor $\mathsf{C}_{Y/} \to \mathsf{C}_{Z/}$ sends terminal objects to terminal objects.

A category C satisfying these conditions is said to have *enough stable objects*.

2.5.4 Exercise (Stabilization)**.** For a category C with enough stable objects, let $i : \mathsf{C}^s \subseteq \mathsf{C}$ denote the full subcategory spanned by the stable objects (so C^s is a groupoid by (2.5.2)). Show that sending $Z \in \mathsf{C}$ to the target of a terminal object in $\mathsf{C}_{Z/}$ defines a functor $\mathrm{st} : \mathsf{C} \to \mathsf{C}^s$ with a natural transformation $\mathbf{1}_{\mathsf{C}} \to i \circ \mathrm{st}$ defining an adjunction (st, i) (hence that $\mathsf{C}^s \subseteq \mathsf{C}$ is a reflective subcategory (1.1.75)).

2.5.5 Exercise (Functor preserving stable objects)**.** Let C and D be categories with enough stable objects. Show that for a functor $F : \mathsf{C} \to \mathsf{D}$, the following are equivalent:

(2.5.5.1) F sends stable objects to stable objects.

(2.5.5.2) The induced functor $\mathsf{C}_{Z/} \to \mathsf{D}_{f(Z)/}$ sends terminal objects to terminal objects for every $Z \in \mathsf{C}$.

A functor F satisfying these conditions is said to *preserve stable objects*.

* *

We next study stable objects in the context of sheaves of categories. By a 'sheaf of categories' we mean a sheaf valued in the 2-category Cat.

2.5.6 Exercise. Show that the 'subcategory of isomorphisms' functor $\simeq\colon \mathsf{Cat} \to \mathsf{Grpd}$ is right adjoint to the inclusion $\mathsf{Grpd} \subseteq \mathsf{Cat}$, hence is continuous. Conclude that for any sheaf of categories $\vec{X}$, its subcategory of isomorphisms $X = (\vec{X})_\simeq$ is a sheaf of groupoids. Conclude that this functor

$$\simeq\colon \mathsf{Shv}(\mathsf{Top}, \mathsf{Cat}) \to \mathsf{Shv}(\mathsf{Top}, \mathsf{Grpd}) \tag{2.5.6.1}$$

is continuous.

We regard a sheaf of categories $\vec{X}$ as an 'enhancement' of its 'underlying sheaf of groupoids' $X = (\vec{X})_\simeq$ to be used to study X itself. A lift of a sheaf of groupoids X to a sheaf of categories $\vec{X}$ will be called *categorical structure* on X.

⋆ **2.5.7 Definition** (Topological stack of categories)**.** By a *topological stack of categories*, we mean a sheaf of categories on the category of topological spaces. Topological stacks of categories form the 2-category $\mathsf{Shv}(\mathsf{Top}, \mathsf{Cat})$.

2.5.8 Example. Vector bundles and linear maps is a topological stack of categories Vect, enhancing the topological stack $\mathsf{Vect}_\simeq = \bigsqcup_n */\mathrm{GL}_n\mathbb{R}$ of vector bundles and isomorphisms.

⋆ **2.5.9 Definition** (Stability structure)**.** A sheaf of categories $\vec{X}$ is called *pre-stable* iff it satisfies the following properties:

(2.5.9.1) Every $\vec{X}(Z)$ has enough stable objects.

(2.5.9.2) Every pullback $\vec{X}(Z) \to \vec{X}(Z')$ for $Z' \to Z$ preserves stable objects.

(2.5.9.3) (Isomorphism is an open condition) For every morphism $\alpha \to \beta$ in $\vec{X}(Z)$, there is an open subset $U \subseteq Z$ such that the pullback $i^*(\alpha \to \beta)$ under a map $i : Z' \to Z$ is an isomorphism iff $i(Z') \subseteq U$.

A *stability structure* on a sheaf of groupoids X is a pre-stable enhancement $\vec{X}$ of X.

⋆ **2.5.10 Exercise** (Stable locus)**.** Let $\vec{X}$ be a pre-stable categorical stack. Show that for every $\alpha \in \vec{X}(Z)$, there exists an open set $U \subseteq Z$ such that $i^*\alpha \in \vec{X}(Z')$ is stable iff $i(Z') \subseteq U$ (for any $i : Z' \to Z$). Conclude that the *stable locus* X^s defined by $X^s(Z) = \vec{X}(Z)^s$ is an open substack of X.

2.6 Smooth manifolds

We assume the reader has a foundational understanding of differential topology and smooth manifolds. The purpose of this section is to set notation and terminology and to recall arguments which will be adapted later to more novel settings.

⋆ **2.6.1 Definition** (Category of smooth manifolds Sm)**.** A *smooth manifold* is a pair (X, Φ) consisting of a topological space X and a collection Φ of pairs (U, φ) (called 'charts') where $U \subseteq \mathbb{R}^n$ is an open set and $\varphi : U \hookrightarrow X$ is an open embedding, such that for every pair $(U, \varphi), (U', \varphi') \in \Phi$, the 'transition map' $\varphi^{-1}\varphi' : (\varphi')^{-1}(\varphi(U)) \to U$ is smooth. A morphism of smooth manifolds $(X, \Phi) \to (Y, \Psi)$ is a map $X \to Y$ such that for every $(U, \varphi) \in \Phi$ and $(V, \psi) \in \Psi$, the composition $\psi^{-1} f \varphi : \varphi^{-1}(f^{-1}(\psi(V))) \to V$ is smooth. The category of smooth manifolds is denoted Sm. The underlying topological space of a smooth manifold M is denoted $|M|$.

2.6.2 Warning. The term 'smooth manifold' is usually taken to mean an object of Sm whose underlying topological space is Hausdorff and paracompact (2.6.13), since these are the objects of main interest to differential topology. As our current focus is more categorical and point set topological, it is more convenient to use the term 'smooth manifold' to refer to arbitrary objects of Sm. In later chapters, when we have a more differentiable topological focus, we will (explicitly) revert to the standard meaning of the term 'smooth manifold' (though the symbol Sm will continue to denote the category defined here). For now, it is logically clarifying to only include paracompact and Hausdorff assumptions when they are actually needed.

2.6.3 Definition (Open embedding)**.** A map $X \to Y$ in Sm is called an *open embedding* when it is an open embedding of topological spaces and sends charts $\mathbb{R}^n \supseteq U \hookrightarrow X$ to charts $U \hookrightarrow Y$. The notion of a *local isomorphism* in Sm is then defined as for topological spaces (2.1.4) with respect to this notion of open embedding. Open embeddings and local isomorphisms are preserved under pullback and closed under composition.

2.6.4 Inverse Function Theorem. *A map in* Sm *is a local isomorphism iff its derivative is an isomorphism at every point.*

2.6.5 Definition (Submersion)**.** A map in Sm is called a *submersion* (or *submersive*) when its derivative is surjective at every point (by (2.6.4), this is equivalent to being locally on the source a pullback of $\mathbb{R}^k \to *$). Submersivity is preserved under pullback, closed under composition, and local on the source.

2.6.6 Definition (Immersion)**.** A map in Sm is called an *immersion* (or *immersive*) when its derivative is injective at every point (by (2.6.4), this is equivalent to being locally on the source a submersive pullback of $* \to \mathbb{R}^k$). Immersivity is closed under composition and local on the source.

* *

The category Sm does not have all finite limits, and some finite limits which do exist are 'wrong'.

2.6.7 Example. The zero locus of a smooth function $f : \mathbb{R} \to \mathbb{R}$ is, by definition, the fiber product

$$\begin{tikzcd} f^{-1}(0) \arrow[r] \arrow[d] & * \arrow[d, "0"] \\ \mathbb{R} \arrow[r, "f"] & \mathbb{R} \end{tikzcd} \tag{2.6.7.1}$$

which we may take in either Sm or Top, resulting in two objects $f^{-1}(0)_{\mathsf{Sm}}$ and $f^{-1}(0)_{\mathsf{Top}}$ (which may or may not exist) and a comparison map $|f^{-1}(0)_{\mathsf{Sm}}| \to f^{-1}(0)_{\mathsf{Top}}$ between them.

Let us consider the smooth function $f(x) = x^n$ for a positive integer $n \geq 1$. The smooth zero locus $f^{-1}(0)_{\mathsf{Sm}}$ is a single point, with its unique structure as a smooth manifold. The topological zero locus $f^{-1}(0)_{\mathsf{Top}}$ is also a single point, with its unique topology, and the comparison map $|f^{-1}(0)_{\mathsf{Sm}}| \to f^{-1}(0)_{\mathsf{Top}}$ is an isomorphism.

Let us consider the smooth function $f(x) = e^{-1/x^2} \sin(1/x)$. The smooth zero locus $f^{-1}(0)_{\mathsf{Sm}}$ is representable: it is the zero set of f equipped with the discrete topology (which is a zero-dimensional manifold, an object of Sm). The topological zero locus $f^{-1}(0)_{\mathsf{Top}}$ is also representable, this time by the zero set of f equipped with the subspace topology inside $\mathbb{R}$. The tautological comparison map $|f^{-1}(0)_{\mathsf{Sm}}| \to f^{-1}(0)_{\mathsf{Top}}$ is evidently not an isomorphism. This difference reflects the fact that test objects in Sm cannot see how the zeroes of f converge to zero, while test objects in Top can.

Here is a class of 'good' finite limits.

2.6.8 Definition (Transverse diagram)**.** A pair of maps $M \to N \leftarrow Q$ in Sm is called *transverse* when at every point of the topological fiber product $|M| \times_{|N|} |Q|$, the map $TM \oplus TQ \to TN$ is surjective. In this case, the fiber product $M \times_N Q$ exists in Sm and has dimension $\dim M - \dim N + \dim Q$.

More generally, consider a finite diagram of smooth manifolds $D : J \to \mathsf{Sm}$ with 0-cells $(M_v)_v$, 1-cells $(f_e : M_{v(e)} \to M_{w(e)})_e$, and no 2-cells. Such a diagram is called *transverse* when at every point $p = (p_v)_v$ of its topological limit, the map

$$\bigoplus_v TM_v \xrightarrow{\bigoplus_e [\mathbf{1}_{TM_{w(e)}} - Tf_e]} \bigoplus_e TM_{w(e)} \tag{2.6.8.1}$$

is surjective. In this case, the limit of D exists in Sm and has dimension $\sum_v \dim M_v - \sum_e \dim M_{w(e)}$.

A *transverse limit* is the limit of a transverse diagram. Note that we have only defined transversality for diagrams of dimension ≤ 1. The generalization to diagrams of arbitrary dimension is given in (2.10.6).

2.6.9 Example. The zero locus $f^{-1}(0)$ (2.6.7) is a transverse limit precisely when $f(x) = 0$ implies $f'(x) \neq 0$.

* *

2.6.10 Definition (Tangent functor T)**.** The tangent functor $T : \mathsf{Sm} \to \mathsf{Sm}$ sends M to (the total space of) its tangent bundle TM and sends $f : M \to N$ to its derivative $Tf : TM \to TN$. The zero section $M \to TM$ and the projection $TM \to M$ are natural transformations $\mathbf{1} \Rightarrow T \Rightarrow \mathbf{1}$.

2.6.11 Exercise. Show that T sends vector bundles to vector bundles, in the following sense. A vector bundle is a triple $(V \to M, V \times \mathbb{R} \to V, V \times_M V \to V)$ which is locally (on M) isomorphic to the trivial family of $\mathbb{R}^k$ with its standard vector space structure. Show that applying T to such a triple yields another. Also show that T sends linear maps of vector bundles to the same.

* *

⋆ **2.6.12 Definition** (Bump function)**.** A topological space X is said to *have bump functions* when for every point $x \in X$ and every open set $U \subseteq X$ containing x, there exists a continuous 'bump' function $\varphi : X \to \mathbb{R}_{\geq 0}$ supported inside U and having nonzero value at x. This definition makes sense in contexts other than topological spaces and continuous maps. In particular, we can consider bump functions on smooth manifolds, which are, by definition, smooth. Any Hausdorff smooth manifold has bump functions: the construction is straightforward given that the function

$$\psi(x) = \begin{cases} \exp(-1/x) & x > 0 \\ 0 & x \leq 0 \end{cases} \tag{2.6.12.1}$$

is smooth.

2.6.13 Definition (Paracompact [13])**.** Let X be a topological space. A *refinement* of an open cover $X = \bigcup_i U_i$ is another open cover $X = \bigcup_j V_j$ where each V_j is contained in some U_i. An open cover $X = \bigcup_i U_i$ is called *locally finite* when every point of X has an open neighborhood which intersects at most finitely many U_i. The topological space X is called *paracompact* when every open cover has a locally finite refinement.

See [68, §29] for basic properties of locally compact Hausdorff topological spaces.

2.6.14 Exercise. Let X be a locally compact Hausdorff topological space. Show that if X is σ-compact (is a countable union of interiors of compact subspaces), then X is paracompact.

⋆ **2.6.15 Definition** (Partition of unity)**.** Let X be a topological space. A *partition of unity* on X is a collection of functions $\varphi_i : X \to \mathbb{R}_{\geq 0}$ which is locally finite (every point of X has a neighborhood over which all but finitely many φ_i are identically zero) and satisfies $\sum_i \varphi_i \equiv 1$. A partition of unity *subordinate* to an open cover $X = \bigcup_i U_i$ is a partition of unity $\sum_i \varphi_i \equiv 1$ (with the same index set) on X satisfying $\operatorname{supp} \varphi_i \subseteq U_i$. A topological space is said to *admit partitions of unity* when it has a partition of unity subordinate to every open cover.

2.6.16 Remark. For the purpose of proving the existence of a partition of unity, the condition that $\sum_i \varphi_i \equiv 1$ may be weakened to $\sum_i \varphi_i > 0$. Indeed, in the latter case, the functions $\varphi_i / \sum_j \varphi_j$ form a partition of unity in the former sense.

2.6.17 Proposition (Dieudonné [13][68, Theorem 41.7])**.** *A paracompact Hausdorff topological space admits partitions of unity.* □

2.6.18 Remark. The *numerable topology* of Dold [15] is a 'Grothendieck topology' in which a collection of open subsets $U_i \subseteq X$ counts as a covering iff it has a subordinate partition of unity. Every (ordinary) open cover of a paracompact Hausdorff space is a numerable open cover by (2.6.17). Most (all?) results about paracompact Hausdorff spaces are based on (2.6.17), hence can be viewed more generally as results about the numerable topology on arbitrary topological spaces.

2.6.19 Exercise. Let X be a paracompact Hausdorff topological space, and let V/X be a vector bundle. Show that there exists a positive definite inner product on V.

The definition of a partition of unity makes sense in contexts other than topological spaces and continuous maps. In particular, we can consider partitions of unity on smooth manifolds. Such partitions of unity consist, by definition, of smooth functions.

⋆ **2.6.20 Lemma.** *A paracompact Hausdorff smooth manifold admits (smooth) partitions of unity.*

Proof. Let $M = \bigcup_i U_i$ be an open cover of a smooth manifold. By passing to a refinement, we may assume that each U_i has compact closure in M. By passing to a further refinement, we may assume that the open cover is also locally finite.

Fix a continuous partition of unity $\varphi_i : X \to \mathbb{R}_{\geq 0}$ (2.6.17) subordinate to the open cover. Now $\operatorname{supp} \varphi_i$ is closed and contained U_i, whose closure is assumed compact, so $\operatorname{supp} \varphi_i$ is also compact. Since $\operatorname{supp} \varphi_i$ is compact, we can sum a finite number of bump functions (2.6.12) to construct a function $\psi_i : M \to \mathbb{R}_{\geq 0}$ supported inside U_i which is positive everywhere on $\operatorname{supp} \varphi_i$. Since the open cover $M = \bigcup_i U_i$ is locally finite, so is the collection of functions ψ_i. The sum $\sum_i \psi_i$ is everywhere positive since the $\operatorname{supp} \varphi_i$ cover M. □

⋆ **2.6.21 Nagata–Smirnov Metrization Theorem** ([69][82][68, Theorem 42.1])**.** *A topological space is metrizable iff it is paracompact Hausdorff and locally metrizable.* □

It follows that every open subset of a locally metrizable paracompact Hausdorff topological space is paracompact. In particular, every open subset of a paracompact Hausdorff manifold is paracompact Hausdorff.

2.6.22 Definition (Topological group)**.** A topological group is a group object (**??**) in Top.

2.6.23 Lemma. *A topological group is Hausdorff iff the identity is a closed point.*

Proof. For any group G, the diagram

$$\begin{tikzcd} G \arrow[r, "x\mapsto(x,x)"] \arrow[d] & G\times G \arrow[d, "(x,y)\mapsto xy^{-1}"] \\ * \arrow[r] & G \end{tikzcd} \tag{2.6.23.1}$$

is a pullback square. This diagram is defined for any topological group G as well, and it remains a pullback square, since applying $\operatorname{Hom}(Z,-)$ to it for any topological space Z produces the corresponding diagram for the group $\operatorname{Hom}(Z,G)$. Thus if $*\to G$ is a closed embedding, then so is the diagonal map $G\to G\times G$, and hence G is Hausdorff. □

2.6.24 Lemma. *A locally compact Hausdorff topological group is paracompact.*

Proof. Since G is locally compact Hausdorff, there exists a compact neighborhood of the identity $K\subseteq G$. Consider the infinite ascending union $K_\infty=\bigcup_i K^i\subseteq G$, which is evidently a subgroup of G. Since $K\cdot K_\infty\subseteq K_\infty$ and K contains a neighborhood of the identity, it follows that $K_\infty\subseteq G$ is open, thus also locally compact Hausdorff. Being a countable union of compact subspaces (the images of $K^i\to G$), the subgroup K_∞ is paracompact (2.6.14). Since K_∞ is open, the quotient G/K_∞ is discrete. Choosing a section of the projection $G\to G/K_\infty$ defines a homeomorphism $G=(G/K_\infty)\times K_\infty$. It is immediate that an open disjoint union of paracompact spaces is paracompact. □

⋆ **2.6.25 Definition** (Lie group)**.** A Lie group is a group object (**??**) in Sm.

2.6.26 Lemma. *Every Lie group is Hausdorff and paracompact.*

Proof. The inclusion of a point into any smooth manifold is a closed embedding, and a topological group whose identity is a closed point is Hausdorff (2.6.23). Smooth manifolds are locally compact, and a locally compact Hausdorff topological group is paracompact (2.6.24). □

2.6.27 Lemma (Local structure of $\underline{\mathrm{Sec}}(Q,M)$)**.** *Let $\pi:Q\to M$ be a submersion. If M is paracompact Hausdorff, then any section $s:M\to Q$ extends to an open embedding $(s^*T_{Q/M},0)\to(Q,s)$ over M.*

Proof. We first construct a map $f:(Q,s)\to(s^*T_{Q/M},0)$ over M whose vertical derivative is the identity along the base section. For any $p\in M$, the source-local normal form for submersions provides such a map $f_p:(Q,s)\to(s^*T_{Q/M},0)$ over an open set $U_p\subseteq Q$ containing $s(p)$. Since M is paracompact Hausdorff, there exists a partition of unity $\sum_p\varphi_p\equiv 1$ (2.6.20) subordinate to the open cover $M=\bigcup_p s^{-1}(U_p)$. Now the sum $f=\sum_p\varphi_pf_p:(Q,u)\to(u^*T_{Q/M},0)$ has the desired property and is defined over the open set $\bigcup_{I\subseteq M}(\bigcap_{p\in I}U_p\setminus\bigcup_{p\notin I}\pi^{-1}(\operatorname{supp}\varphi_p))$ (union over all finite subsets I), which contains the image of s.

Since the vertical derivative of $f:(Q,s)\to(s^*T_{Q/M},0)$ along the base section is the identity, it follows that f is a local isomorphism over a neighborhood of $s(M)\subseteq Q$. That is, for every point $p \in M$, there exists an open set $V_p \subseteq Q$ containing $s(p)$ over which f is an open embedding. It follows that f is an open embedding over the open set $\bigcup_{I\subseteq M}(\bigcap_{p\in I} V_p \setminus \bigcup_{p\notin I} \pi^{-1}(\operatorname{supp}\psi_p))$ for any choice of partition of unity $\sum_p \psi_p \equiv 1$ subordinate to the open cover $M = \bigcup_p s^{-1}(V_p)$ (indeed, it is certainly a local isomorphism over this locus, and it is also injective since injectivity can be checked fiberwise over M). Its inverse is thus an open embedding $i : (s^*T_{Q/M}, 0) \to (Q, s)$ defined in a neighborhood of the zero section.

Given an open embedding $i : (s^*T_{Q/M}, 0) \to (Q, s)$ defined in a neighborhood of the zero section, we can obtain a globally defined open embedding by pre-composing as follows. Fix a metric (positive definite inner product) on $s^*T_{Q/M}$ (sum up local metrics via a partition of unity), and find a smooth function $\varepsilon : M \to \mathbb{R}_{>0}$ (also using partition of unity) so that the fiberwise ε-balls of $s^*T_{Q/M}$ are contained in the domain of i. Pre-compose i with multiplication by ε to obtain an open embedding $(s^*T_{Q/M}, 0) \to (Q, s)$ defined on the fiberwise unit balls. Finally, pre-compose this with $\alpha(|v|) \cdot v$ for some function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{>0}$ which equals 1 in a neighborhood of the origin and for which $x \mapsto \alpha(x)x$ is a diffeomorphism $[0, \infty) \to [0, 1)$. □

2.6.28 Corollary (Local structure of $\underline{\mathrm{Sec}}(Q, M)$)**.** *Let $Q \to M$ be a submersion. If M is compact Hausdorff, then the moduli stack $\underline{\mathrm{Sec}}(M, Q)$ is covered by the open substacks $\underline{\mathrm{Sec}}(M, Q^\circ) \subseteq \underline{\mathrm{Sec}}(M, Q)$ associated to open subsets $Q^\circ \subseteq Q$ for which $Q^\circ \to M$ can be equipped with the structure of a vector bundle.*

Proof. For any open subset $Q^\circ \subseteq Q$, the induced map $\underline{\mathrm{Sec}}(M, Q^\circ) \hookrightarrow \underline{\mathrm{Sec}}(M, Q)$ is an open embedding by (2.4.9) since M is compact. Since M is paracompact Hausdorff, every section $u : M \to Q$ extends to an open embedding $(u^*T_{Q/M}, 0) \to (Q, u)$ over M (2.6.27), so every point of $\underline{\mathrm{Sec}}(M, Q)$ is in the image of $\underline{\mathrm{Sec}}(M, Q^\circ)$ for an open $Q^\circ \subseteq Q$ which is the total space of a vector bundle over M. □

* *

Many results about smooth manifolds rely on averaging of real valued functions or, more generally, sections of a vector bundle. For others, we instead need a notion of average for a collection of nearby points in a smooth manifold. We now explain how to construct this sort of non-linear averaging operation.

2.6.29 Definition (Averaging on a manifold)**.** Let M be a smooth manifold which is paracompact and Hausdorff. We consider positive measures of unit total mass on M; call this set $\mathrm{Meas}(M)$. There is a tautological inclusion $M \hookrightarrow \mathrm{Meas}(M)$ sending a point of M to the delta measure at that point. Our goal is to define a 'smooth' retraction

$$\mathrm{avg} : \mathrm{Meas}(M) \to M \tag{2.6.29.1}$$

(a 'notion of average') on the set of measures of 'small' support (meaning, there will be an open cover $M = \bigcup_i V_i$, and $\mathrm{avg}(\mu)$ will be defined when $\operatorname{supp}\mu$ is contained in some V_i).

Let $U \subseteq M$ be an open set identified with a convex open set $U \subseteq \mathbb{R}^n$. Thus any measure μ supported inside U has an average $\mathrm{avg}_U(\mu) \in U$ defined by the linear structure on $\mathbb{R}^n$. We would like to interpolate between the map avg_U for measures supported deep inside U and the identity map for measures supported away from U. Given a smooth function of compact support $\eta : U \to [0,1]$, such an interpolation can be given by

$$\operatorname{avg}_{U,\eta}(\mu) = \eta(\mu) \cdot \delta_{\operatorname{avg}_U(\mu)} + (1 - \eta(\mu))\mu, \tag{2.6.29.2}$$

where $\eta(\mu) = \int \eta \, d\mu$. This map $\operatorname{avg}_{U,\eta}$ is well behaved on the set of measures μ which are either supported inside U or supported inside $M \setminus \operatorname{supp} \eta$ (in which case $\operatorname{avg}_{U,\eta}(\mu) = \mu$). When $\operatorname{supp} \mu \subseteq \eta^{-1}(1)$, the average $\operatorname{avg}_{U,\eta}(\mu)$ is the single point $\operatorname{avg}_U(\mu)$.

We now define the averaging map avg as a composition of local averaging maps $\operatorname{avg}_{U,\eta}$. Choose an open cover $M = \bigcup_i U_i$ for open convex $U_i \subseteq \mathbb{R}^n$, and choose smooth functions of compact support $\eta_i : U_i \to [0,1]$ such that $M = \bigcup_i \eta_i^{-1}(1)^\circ$. Define avg as the ordered composition of all $\operatorname{avg}_{U_i,\eta_i}$ with respect to an arbitrarily chosen total order of the set of indices i. This map $\operatorname{avg} : \operatorname{Meas}(M) \to \operatorname{Meas}(M)$ is defined on measures of sufficiently small support, and sends measures of sufficiently small support to delta measures (hence can be viewed as having target $M \subseteq \operatorname{Meas}(M)$).

In what sense is the map avg smooth? Let us declare a map $N \to \operatorname{Meas}(M)$ from a smooth manifold N to be smooth iff for every smooth function on $N \times M$, its fiberwise integral is a smooth function on N. A map $N \to M$ is thus smooth iff it is smooth as a map to $\operatorname{Meas}(M)$ landing in the subspace of delta measures. Now the map avg is smooth in the sense that composing a smooth map $N \to \operatorname{Meas}(M)$ with it yields a smooth map $N \to \operatorname{Meas}(M)$. Indeed, it suffices to show that each map $\operatorname{avg}_{U,\eta}$ is smooth in this sense, which follows from inspection.

The next result is fundamental, and we will meet many generalizations of it. The main ingredient in its proof is the averaging operation (2.6.29) above.

⋆ **2.6.30 Ehresmann's Theorem** ([18, 19]). *A proper submersion in* Sm *is trivial locally on the target.*

Proof. Let $M \to B$ be a proper submersion, and let us show that $M \to B$ is trivial in a neighborhood of a given point $0 \in B$. We are free to shrink B at will (that is, replace B with an open neighborhood of 0 and replace M with its inverse image).

Let M_0 denote the fiber of $M \to B$ over $0 \in B$. We first construct a retraction $M \to M_0$ after possibly shrinking B. We then show that such a retraction gives a local trivialization of $M \to B$ after further shrinking.

We claim that there exists a finite covering of M by open charts $U_i \times B \subseteq M$ for open sets $U_i \subseteq \mathbb{R}^k$. The source-local normal form for submersive maps provides such a chart near any point of M_0. By universal closedness of $M \to B$, there exists a finite collection of such charts which cover the inverse image of an open neighborhood of 0. We can thus shrink B so that they cover all of M.

In a given chart $U_i \times B \subseteq M$ there is an evident retraction to the fiber over $0 \in B$, namely projection to the U_i factor. These need not agree on overlaps. We will patch them together using a partition of unity and the averaging operation (2.6.29) on M_0.

Choose smooth compactly supported functions $\varphi_i : U_i \to \mathbb{R}_{\geq 0}$ which form a partition of unity on M_0. Since $M \to B$ is separated and $(\operatorname{supp}\varphi_i) \times B \to B$ is universally closed, it follows that $(\operatorname{supp}\varphi_i) \times B \to M$ is universally closed (2.1.33). It follows that the extension by zero of $\varphi_i \pi_{U_i}$ from $U_i \times B$ to M is smooth. These functions sum to unity on M_0, but may fail to elsewhere on M. The locus where their sum is > 0 is an open neighborhood of M_0, hence can be assumed to be all of M after shrinking B. Dividing by this sum produces a smooth partition of unity $\sum_i \psi_i \equiv 1$ on M.

Now we consider the map

$$M \to \operatorname{Meas}(M_0) \tag{2.6.30.1}$$

$$m \mapsto \sum_i \psi_i(m)\delta_{\pi_{U_i}(m)} \tag{2.6.30.2}$$

where we note that if $\psi_i(m) > 0$ then m lies inside the chart $U_i \times B \subseteq M$, so $\pi_{U_i}(m) \in M_0$ is defined. Composing this map with the averaging operation (2.6.29) on M_0 produces the desired retraction $M \to M_0$ in a neighborhood of M_0 (which becomes all of M after shrinking B).

Finally, let us argue that the existence of a retraction $M \to M_0$ implies triviality of $M \to B$ near 0. The induced map $M \to M_0 \times B$ over B is a local isomorphism in a neighborhood of M_0 (which by shrinking B is wlog all of M). There is a unique section of $M \to M_0 \times B$ over $M_0 \times 0$, and this section extends to a neighborhood of M_0 by (**??**). Any section of a local isomorphism is an open embedding (2.1.32). Further shrinking B means the image of this open embedding is all of M, thus it is a diffeomorphism. □

2.6.31 Lemma (Local structure of $\underline{\operatorname{Sec}}_B(M, Q)$)**.** *Let $Q \to M \to B$ be submersions. If $M \to B$ is proper, then for any $b \in B$ and any section $s : M_b \to Q_b$, there is (after replacing B with an open subset containing b) a trivialization $M = M_b \times B$ over B covered by an open embedding $s^*T_{Q/M} \times B \hookrightarrow Q$ identifying the zero section with s.*

$$\begin{tikzcd}
 & s^*T_{Q/M} \times B \arrow[rr, hook] \arrow[dd] & & Q \arrow[dd] \\
M_b \arrow[ur, "0"] \arrow[rr, equal] \arrow[dr] \arrow[dd] & & M_b \arrow[ur, "s"] \arrow[dr] \arrow[dd] & \\
 & M_b \times B \arrow[rr, equal] \arrow[dd] & & M \arrow[dd] \\
* \arrow[rr, equal] \arrow[dr, "b"'] & & * \arrow[dr, "b"] & \\
 & B \arrow[rr, equal] & & B
\end{tikzcd} \tag{2.6.31.1}$$

Proof. This is similar to (2.6.27). Since $M \to B$ is proper, Ehresmann (2.6.30) provides a local trivialization $M = M_b \times B$. As in (2.6.27), it suffices to construct a map $(Q, s) \to (s^*T_{Q/M}, 0)$ (over this choice of local trivialization) whose vertical derivative along s is the identity map. Such a map exists locally, hence globally using a partition of unity. □

2.6.32 Corollary (Local structure of $\underline{\operatorname{Sec}}_B(M, Q)$)**.** *Let $Q \to M \to B$ be submersions. If $M \to B$ is proper, then the moduli stack $\underline{\operatorname{Sec}}_B(M, Q)$ is covered by the open substacks*

$\underline{\mathrm{Sec}}_{B^\circ}(M^\circ, Q^\circ) \subseteq \underline{\mathrm{Sec}}(M, Q)$ associated to open subsets $B^\circ \subseteq B$ (let $M^\circ = M \times_B B^\circ$) and $Q^\circ \subseteq Q \times_B B^\circ$ for which $Q^\circ \to M^\circ \to B^\circ$ is isomorphic to a product $(Q_0 \to M_0 \to *) \times B^\circ$ where $Q_0 \to M_0$ is a vector bundle.

Proof. This is similar to (2.6.28). Given an open subset $B^\circ \subseteq B$ (let $M^\circ = M \times_B B^\circ$) and an open subset $Q^\circ \subseteq Q \times_B B^\circ$, the induced map $\underline{\mathrm{Sec}}_{B^\circ}(M^\circ, Q^\circ) \to \underline{\mathrm{Sec}}_B(M, Q)$ is an open embedding since $M \to B$ is universally closed (2.4.23). That such open substacks where $Q^\circ \to M^\circ \to B^\circ$ has the form $(Q_0 \to M_0 \to *) \times B^\circ$ for $Q_0 \to M_0$ a vector bundle form a covering is the content of (2.6.31). □

* *

⋆ **2.6.33 Hadamard's Lemma.** *If $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ vanishes on $0 \times \mathbb{R}^n$, then f has the form $x \cdot g(x, y_1, \ldots, y_n)$ for some smooth function g.*

Proof #1. If $f(0) = 0$ then $f(x) = x \int_0^1 f'(xt)\, dt$. □

Proof #2. It suffices to show that $x^{-1} f(x)$ is of class C^k (k times continuously differentiable) for every $k < \infty$ under the assumption that $f(0) = 0$. By subtracting off a polynomial from $f(x)$, we may in fact assume that $f(0) = f'(0) = \cdots = f^{(N)}(0) = 0$ for some large $N < \infty$. This implies that $f^{(i)}(x) = O(x^{N+1-i})$ near $x = 0$ for $0 \le i \le N$. Now explicit differentiation shows that the ith derivative of $x^{-1} f(x)$ is $O(x^{N-i})$ near $x = 0$ for $0 \le i < N$, which implies $x^{-1} f(x)$ is of class C^{N-1}. □

2.6.34 Exercise. Conclude from Hadamard's Lemma (2.6.33) that if $f : \mathbb{R}^k \times \mathbb{R}^n \to \mathbb{R}$ vanishes on $0 \times \mathbb{R}^n$ then $f = \sum_{i=1}^k x_i g_i$ for some functions g_i.

2.6.35 Exercise. Let E be a vector bundle over a paracompact Hausdorff smooth manifold M, and let $s : M \to E$ be a smooth section transverse to zero. Show that every function $f : M \to \mathbb{R}$ vanishing over $s^{-1}(0)$ is of the form $\lambda \cdot s$ for some smooth section $\lambda : M \to E^*$ (use (2.6.34) to prove it locally, and then patch together using a partition of unity).

2.6.36 Definition (Deformation to the tangent bundle)**.** We define a functor $\mathbb{P} : \mathsf{Sm} \to \mathsf{Sm}$ which sends a smooth manifold M to (the total space of) a submersion $\mathbb{P}(M) \to \mathbb{R}$ with fiber TM over 0 and fibers $M \times M$ over $\mathbb{R} \setminus 0$.

$$\begin{tikzcd} TM \arrow[r, hook] \arrow[d] & \mathbb{P}(M) \arrow[d] & M \times M \times (\mathbb{R} \setminus 0) \arrow[l, hook'] \arrow[d] \\ 0 \arrow[r, hook] & \mathbb{R} & \mathbb{R} \setminus 0 \arrow[l, hook'] \end{tikzcd} \tag{2.6.36.1}$$

This structure is functorial in the expected way: for a smooth map $f : M \to N$, the induced map $\mathbb{P}(f) : \mathbb{P}(M) \to \mathbb{P}(N)$ is the product $f \times f \times \mathbf{1}$ over $\mathbb{R} \setminus 0$ and is the derivative Tf over 0. There is a functorial involution of $\mathbb{P}$ which swaps the two factors $M \times M$ over $\mathbb{R} \setminus 0$ and acts as negation on the fiber TM over 0.

Local coordinates for the functor $\mathbb{P}$ may be defined as follows. Fix any 'exponential' map $A : TM \to M$, meaning its vertical derivative along the zero section is the identity and its restriction to each fiber is an open embedding. Such an exponential map determines an open embedding $TM \times \mathbb{R} \hookrightarrow \mathbb{P}(M)$ in which $TM \times (\mathbb{R} \setminus 0)$ is glued to $M \times M \times (\mathbb{R} \setminus 0)$ via the map $(p, v, t) \mapsto (p, A_p(tv), t)$. To show that this recipe determines the desired functor $\mathbb{P}$, it suffices to show that for any other exponential map $B : TN \to N$ and any smooth map $f : M \to N$, the induced map $TM \times (\mathbb{R} \setminus 0) \to TN \times (\mathbb{R} \setminus 0)$ defined by conjugating $f \times f \times \mathbf{1} : M \times M \times \mathbb{R} \to N \times N \times \mathbb{R}$ by the relevant exponential maps extends smootly to $TM \times \mathbb{R} \to TN \times \mathbb{R}$.

$$\begin{tikzcd} TM \times \mathbb{R} \arrow[d, dashed] & TM \times (\mathbb{R} \setminus 0) \arrow[l, hook'] \arrow[r, hook] \arrow[d] & M \times M \times (\mathbb{R} \setminus 0) \arrow[d, "f \times f \times \mathbf{1}"] \\ TN \times \mathbb{R} & TN \times (\mathbb{R} \setminus 0) \arrow[l, hook'] \arrow[r, hook] & N \times N \times (\mathbb{R} \setminus 0) \end{tikzcd} \tag{2.6.36.2}$$

Concretely, this amounts to showing that the map $(p, v, t) \mapsto t^{-1} B^{-1}_{f(p)}(f(A_p(tv)))$ extends smoothly to $t = 0$. This follows from Hadamard's Lemma (2.6.33) since the map $B^{-1}_{f(p)}(f(A_p(tv)))$ is smooth and vanishes at $t = 0$. Existence of the claimed involution of $\mathbb{P}$ amounts to smoothness at $t = 0$ of the map $(p, v, t) \mapsto t^{-1} A^{-1}_{A_p(tv)}(p)$, which holds for the same reason.

* *

The above discussion generalizes readily to the setting of manifolds-with-corners.

2.7 Smooth stacks

In (2.3), we studied stacks on the category of topological spaces. We now turn to stacks on the category of smooth manifolds, which we will call *smooth stacks*. As before, Yoneda gives a full faithul embedding $\mathsf{Sm} \subseteq \mathsf{Shv}(\mathsf{Sm})$, and it is helpful to regard smooth stacks as 'generalized smooth manifolds'. We will be particularly interested in the class of smooth stacks which admit a submersive atlas. The theory of such stacks is essentially equivalent to the theory of 'Lie groupoids' introduced by Ehresmann [20] and studied by many others since then. References include Heinloth [30].

The category of smooth manifolds Sm is a 'topological site' in the sense of (**??**). We can thus formulate the descent property and define stacks on Sm as in (2.3). Stacks $\mathsf{Shv}(\mathsf{Sm}) \subseteq \mathsf{P}(\mathsf{Sm})$ form a reflective subcategory, and the Yoneda embedding $\mathsf{Sm} \hookrightarrow \mathsf{Shv}(\mathsf{Sm})$ is fully faithful. If $\mathsf{Sm}^- \subseteq \mathsf{Sm}$ denotes the category of open subsets of Euclidean space and smooth maps between them (a full subcategory), then the restriction functor $\mathsf{Shv}(\mathsf{Sm}) \to \mathsf{Shv}(\mathsf{Sm}^-)$ is an equivalence (**??**). Since Sm^- is essentially small, there are fewer set-theoretic complications in comparison to the case of Top discussed in (**??**).

The 2-category $\mathsf{Shv}(\mathsf{Sm})$ is complete, and the embedding $\mathsf{Sm} \hookrightarrow \mathsf{Shv}(\mathsf{Sm})$ preserves all limits which exist in Sm. The fact that the category Sm is not complete leads to some technical differences in comparison with the discussion of topological stacks in (2.3). Not all fiber products in Sm exist, so the class of all morphisms in Sm is not preserved under pullback, so we cannot define a notion of representability for general morphisms in $\mathsf{Shv}(\mathsf{Sm})$. Properties of morphisms in Sm which are preserved under pullback include submersions, local isomorphisms, and open embeddings; these notions extend to morphisms in $\mathsf{Shv}(\mathsf{Sm})$ in the usual way by pulling back to objects of Sm. The forgetful functor $\mathsf{Sm} \to \mathsf{Top}$ sends pullbacks of submersions to pullback squares in Top, so the intersection of submersion with any property of morphisms in Top preserved under pullback is a property of morphisms in Sm preserved under pullback (e.g. separated submersion, proper submersion, etc.).

2.7.1 Exercise. Show that a submersion of smooth stacks $X \to Y$ factors uniquely as a surjective submersion $X \to V$ followed by an open embedding $V \to Y$. We call the open substack $V \subseteq Y$ the *image* of the submersion $X \to Y$.

2.7.2 Exercise (Relative tangent bundle of a submersion of smooth stacks)**.** Let $X \to Y$ be a submersion of smooth stacks, and let us define a vector bundle $T_{X/Y}$ over X. For any $Z \in \mathsf{Sm}$ with a map $Z \to X$, we consider the pullback $X \times_Y Z \to Z$ with its canonical section. We declare the pullback of $T_{X/Y}$ to Z to be the pullback of $T_{X\times_Y Z/Z}$ under the canonical section $Z \to X \times_Y Z$. Show that this assignment of a vector bundle over Z to every map $Z \to X$ is compatible with pullback, hence defines a vector bundle over X. Show that the relative tangent bundle is functorial, in the sense that for submersions $X \to Y$ and $X' \to Y'$, a commutative square $(X \to Y) \to (X' \to Y')$ induces a map from $T_{X/Y}$ to the pullback of $T_{X'/Y'}$ to X. Conclude that for a composition of submersions $X \to Y \to Z$, there are induced maps $T_{X/Y} \to T_{X/Z} \to T_{Y/Z}$ (the latter pulled back to X); moreover, show that this sequence is exact.

2.7.3 Exercise. Show that a submersion of smooth stacks $X \to Y$ is a local isomorphism iff $T_{X/Y} = 0$.

* *

⋆ **2.7.4 Definition** (Submersive atlas)**.** A *submersive atlas* of a smooth stack X is a surjective representable submersion $U \to X$ from a smooth manifold U.

* *

2.7.5 Exercise (Tangent cohomology of a smooth stack with submersive atlas)**.** Let X be a smooth stack with submersive atlas. For any submersion $u : U \to X$, consider the two-term complex

$$u^*TX = [\overset{-1}{T_{U/X}} \to \overset{0}{TU}] \tag{2.7.5.1}$$

of vector bundles on U. We will eventually identify this two-term complex with the pullback of a two-term complex of vector bundles on X denoted TX, but for now the notation u^*TX is purely motivational.

Show that for any pair of submersions $V, U \to X$ with a map $V \to U$ over X, the induced map from v^*TX to the pullback of u^*TX to V is a quasi-isomorphism (pull the situation back to a third atlas $W \to X$). Conclude that the fiberwise cohomology of these two-term complexes u^*TX descends to X, in the sense that for every $p \in X$ there are well-defined vector spaces $T_p^{-1}X$ and T_p^0X (that is, functors $T^iX : \operatorname{Hom}(*, X) \to \mathsf{Vect}_{\mathbb{R}}^{\mathsf{fin}}$ for $i = -1, 0$) together with, for every submersion $U \to X$ and every point $p \in U$, an exact sequence

$$0 \to T_p^{-1}X \to (T_{U/X})_p \to T_pU \to T_p^0X \to 0 \tag{2.7.5.2}$$

compatible with maps of submersions over X. In (**??**) below, we will refine this discussion to construct a two-term complex of vector bundles TX on X with cohomology T^iX.

2.7.6 Example. If X is a smooth manifold, then $T^{-1}X = 0$ and $T_p^0X = T_pX$ is the fiber at p of the tangent bundle of X in the usual sense.

2.7.7 Exercise. Show that for a Lie group G, we have $T^0\mathbb{B}G = 0$ and $T^{-1}\mathbb{B}G = \mathfrak{g}$ is the Lie algebra of G equipped with the conjugation action.

2.7.8 Exercise. Show that for a Lie group G acting on a smooth manifold M, there is an exact sequence

$$0 \to T_p^{-1}(M/G) \to \mathfrak{g} \to T_pM \to T_p^0(M/G) \to 0 \tag{2.7.8.1}$$

at points $p \in M$.

2.7.9 Exercise. Show that for a smooth stack X with submersive atlas, the condition that $T_x^{-1}X = 0$ is open in x.

2.7.10 Corollary. *For any submersive atlas $U \to X$ and any $x \in X$, the map $x \times_X U \to U$ is, locally on the source, a submersion onto a submanifold of codimension $\dim T_x^0 X$ with fibers of dimension $\dim T_x^{-1} X$.*

Proof. The derivative of $x \times_X U \to U$ is $T_{U/X} \to TU$, whose kernel and cokernel have constant rank $\dim T_x^{-1} X$ and $\dim T_x^0 X$, respectively. □

2.7.11 Corollary. *The automorphism stack $\underline{\mathrm{Aut}}(x)$ of a point x of a smooth stack X with submersive atlas is a Lie group with Lie algebra $T_x^{-1} X$.*

Proof. The automorphism stack $\underline{\mathrm{Aut}}(x)$ is always a group object (**??**), so to show it is a Lie group, it suffices to show it is representable. Choose an atlas $U \to X$ and a lift of x to a point $u \in U$. Then $\underline{\mathrm{Aut}}(x) = x \times_X x = u \times_X u$ is the fiber of $u \times_X U \to U$ over u. By (2.7.10), this fiber is a smooth manifold whose tangent space is the kernel of $T_{U/X} \to TU$, namely $T_x^{-1} X$. □

2.7.12 Definition (Minimal submersion)**.** Given a smooth stack X, a submersion $U \to X$ from a smooth manifold U is called *minimal at $u \in U$* when the map $T_{U/X} \to TU$ vanishes at u (compare (2.7.5.2)).

2.7.13 Lemma (Proper atlas from proper diagonal)**.** *Let X be a smooth stack with proper diagonal, and let $U \to X$ be a submersion which is minimal at $p \in U$. For every sufficiently small open neighborhood $p \in V \subseteq U$, we have $p \times_X p = p \times_X V$ and the map $V \to X$ is proper over an open substack of X containing the image of p.*

Proof. Given the purely topological result (2.3.17), it suffices to show that $p \times_X p \subseteq p \times_X U$ is open, which is equivalent to minimality of $U \to X$ at p (2.7.10). □

2.7.14 Lemma (Existence of a minimal atlas)**.** *For every point x of a smooth stack X with submersive atlas, there exists an atlas $U \to X$ which is minimal at some lift $u \in U$ of x.*

Proof. Suppose $U \to X$ is a submersion and $V \to U$ is a map of smooth manifolds. We claim that $V \to X$ is a submersion iff $TV \oplus T_{U/X} \to TU$ is surjective. Submersivity of $V \to X$ can be checked after pulling back to an atlas $W \to X$, and such pullback also preserves the surjectivity condition in question. We are thus reduced to the situation that X is itself a smooth manifold, in which case the equivalence is immediate.

With this fact in hand, we can now conclude. Begin with an arbitrary atlas $U \to X$ and a lift $u \in U$ of x. Let $V \subseteq U$ be a locally closed submanifold passing through u chosen so that $TV \subseteq TU$ is a complement of the image of $T_{U/X} \to TU$ at u (and so that $TV \oplus T_{U/X} \to TU$ is everywhere surjective). Thus $V \to X$ is a submersion at u, and it remains to show that it is minimal at u.

The map $[T_{V/X} \to TV] \to [T_{U/X} \to TU]$ is a quasi-isomorphism of two-term complexes (both calculate TX (2.7.5)). Together with the fact that $TV \subseteq TU$ is a complement of the image of $T_{U/X} \to TU$ at u, this implies that the map $T_{V/X} \to TV$ vanishes at u, as desired. □

* *

We now come to the fundamental 'local linearization' result for smooth stacks with submersive atlas and proper diagonal. It was conjectured by Weinstein [89, 90] and proved by Zung [92] (see also Crainic–Struchiner [10] and Hoyo–Fernandes [11]). An analogous result in algebraic geometry was proven later by Alper–Hall–Rydh [3].

⋆ **2.7.15 Theorem** (Zung [92])**.** *A smooth stack with submersive atlas and proper diagonal is a Lie orbifold* (**??**)*.*

Proof. Let X be a smooth stack with submersive atlas and proper diagonal. Let $x \in X$, and let $G = x \times_X x$ be its automorphism Lie group (2.7.11). Since X has proper diagonal, G is compact.

Fix a submersion $U \to X$ from a smooth manifold U which is minimal at a lift $u \in U$ of x. By replacing U with an open neighborhood of u, we can ensure that $u \times_X U \to U$ has image $\{u\}$ and that $U \to X$ is proper over an open substack of X containing x (2.7.13).

We have constructed a proper submersion $U \to X$ from a smooth manifold U over a neighborhood of x. It suffices to equip it with the structure of a principal G-bundle. Indeed, this implies that $X = U/G$, which is a Lie orbifold (**??**).

A 'pseudo-principal G-bundle' structure on $U \to X$ is simply a map $\phi : U \times_X U \to G$ (2.7.16). A principal G-bundle structure on $U \to X$ is the same as a pseudo-principal G-bundle structure for which ϕ is a groupoid homomorphism (meaning the two maps $U \times_X U \times_X U \to G$ given by $(x, y, z) \mapsto \phi(x, y)\phi(y, z)$ and $\phi(x, z)$ coincide) and the restriction of ϕ to $u \times_X U \to G$ is a diffeomorphism for every $u \in U$ (2.7.17). We will first construct a pseudo-principal G-bundle structure on $U \to X$ and then correct it to a principal G-bundle structure.

Since $U \to X$ and X are separated, it follows that U is separated (Hausdorff). The map $U \times_X U \to U$ is separated (pullback of $U \to X$), so $U \times_X U$ is also Hausdorff. Now $G = u \times_X u = u \times_X U \subseteq U \times_X U$ is a smooth submanifold (it is a fiber of the submersion $U \times_X U \to U$). There is thus a retraction $U \times_X U \to G$ defined in a neighborhood of $G = u \times_X u = u \times_X U$. The complement of this neighborhood is a closed subset of $U \times_X U$, hence has closed image in X by properness of $U \to X$. This image does not contain x, since the inverse image of x is the image of $u \times_X U \to U$, which is just u. Thus after replacing X with an open substack containing x, we conclude that $U \to X$ is a pseudo-principal G-bundle.

By construction, this pseudo-principal G-bundle structure on $U \to X$ satisfies the condition for being a principal G-bundle over $x \in X$. It would thus suffice to functorially 'correct' pseudo-principal G-bundle structures to principal G-bundle structures, at least over an open subset containing the locus where they are already principal. Such a functorial correction is defined in (2.7.19) below, depending on an additional piece of data, namely that of a smooth positive fiberwise density on $U \to X$, namely a smooth positive section of $|\det T^*_{U/X}|$ (where $\det : \mathrm{GL}_n(\mathbb{R}) \to \mathrm{GL}_1(\mathbb{R})$ and $|\cdot| : \mathrm{GL}_1(\mathbb{R}) = \mathbb{R}^\times \to \mathbb{R}_{>0}$ are group homomorphisms applied to principal bundles) over U; simply choose one arbitrarily. □

2.7.16 Definition (Pseudo-principal G-bundle)**.** Let G be a compact Lie group. A *pseudo-principal G-bundle* is a proper submersion $P \to X$ together with a smooth map $\phi : P \times_X P \to G$. We denote the stack of pseudo-principal G-bundles by $\mathbb{P}G$.

2.7.17 Exercise. Every principal G-bundle is a pseudo-principal G-bundle: the map ϕ is defined by the property $\phi(x,y)y = x$; this defines a map of smooth stacks $\mathbb{B}G \to \mathbb{P}G$. Show that $\operatorname{Hom}(Z, \mathbb{B}G) \to \operatorname{Hom}(Z, \mathbb{P}G)$ is fully faithful. Show that a psuedo-principal G-bundle $P \to X$ and $\phi : P \times_X P \to G$ comes from a principal G-bundle iff ϕ is a groupoid homomorphism (meaning $\phi(x,y)\phi(y,z) = \phi(x,z)$ for $(x,y,z) \in P \times_X P \times_X P$) and its restriction to $p \times_X P \to G$ is a diffeomorphism for every $p \in P$.

2.7.18 Definition (Measured submersion)**.** A submersion $Q \to B$ equipped with a smooth fiberwise density will be called a *measured submersion*. We denote the stacks of measured principal G-bundles and measured pseudo-principal G-bundles by $\widetilde{\mathbb{B}}G$ and $\widetilde{\mathbb{P}}G$, respectively.

2.7.19 Proposition (Zung [92])**.** *Let G be a compact Lie group. The map $\widetilde{\mathbb{B}}G \to \widetilde{\mathbb{P}}G$ lands inside an open substack $\widetilde{\mathbb{P}}^\circ G \subseteq \widetilde{\mathbb{P}}G$ which has a retraction $\widetilde{\mathbb{P}}^\circ G \to \widetilde{\mathbb{B}}G$.*

$$\begin{tikzcd}
& \widetilde{\mathbb{P}}^\circ G \arrow[dl, dashed, bend right] \arrow[d, hook, dashed] \\
\widetilde{\mathbb{B}}G \arrow[r] \arrow[ur, dashed] & \widetilde{\mathbb{P}}G
\end{tikzcd} \tag{2.7.19.1}$$

Proof. Let F be a compact Hausdorff smooth manifold equipped with a positive smooth density μ. Recall that a function $\phi : F \times F \to G$ is called a groupoid homomorphism when $\phi(x,y)\phi(y,z) = \phi(x,z)$ (2.7.17). We will define, for an open locus of $(\phi,\mu) \in C^\infty(F \times F, G) \times C^\infty(F, \Omega_F^{>0})$, a groupoid homomorphism $R(\phi,\mu) : F \times F \to G$ which we call the 'rectification' of ϕ with respect to μ, so that if ϕ is a groupoid homomorphism then $R(\phi,\mu)$ is defined and equals ϕ. Applying this operation fiberwise to a measured pseudo-principal G-bundle $(Q \to B, \phi, \mu)$ produces a measured principal G-bundle $(Q^\circ \to B^\circ, R(\phi,\mu), \mu)$, where $Q^\circ = Q \times_B B^\circ$ and $B^\circ \subseteq B$ is the open subset where $R(\phi,\mu)$ is defined. This defines the desired open substack $\widetilde{\mathbb{P}}^\circ G \subseteq \widetilde{\mathbb{P}}G$ with retraction $\widetilde{\mathbb{P}}^\circ G \to \widetilde{\mathbb{B}}G$. Our goal is thus to define the rectification operation $R(\phi,\mu)$ with the aforementioned properties.

Let us begin with some general definitions and estimates. For a function $f : F \to G$ taking values in a small neighborhood of the identity, we define its expectation

$$\mathbb{E}_x[f(x)] = \exp(\mathbb{E}_x[\log f(x)]) \tag{2.7.19.2}$$

with respect to μ using the exponential map $\exp : \mathfrak{g} = T_{\mathbf{1}}G \to G$ with inverse (near the identity) denoted log. Expectation is thus conjugation invariant: $\mathbb{E}_x[af(x)a^{-1}] = a\mathbb{E}_x[f(x)]a^{-1}$. When G is non-abelian, we do not have $\mathbb{E}_x[f(x)g(x)] = \mathbb{E}_x f(x)\mathbb{E}_x g(x)$, rather we have an estimate

$$|\mathbb{E}_x[f(x)g(x)] - \mathbb{E}_x f(x)\mathbb{E}_x g(x)| \leq \text{const} \cdot \sup|f| \cdot \sup|g|, \tag{2.7.19.3}$$

where $|a|$ for $a \in G$ means $|\log a|$ for some fixed conjugation invariant norm $|\cdot| : \mathfrak{g} \to \mathbb{R}_{\geq 0}$ (which exists since G is compact). To prove this estimate, it suffices to bound both the quantities

$$|\exp \mathbb{E}_x \log f(x)g(x) - \exp \mathbb{E}_x[\log f(x) + \log g(x)]| \tag{2.7.19.4}$$

$$|\exp(\mathbb{E}_x \log f(x) + \mathbb{E}_x \log g(x)) - (\exp \mathbb{E}_x \log f(x))(\exp \mathbb{E}_x \log g(x))| \tag{2.7.19.5}$$

by $\mathrm{const} \cdot \sup |f| \cdot \sup |g|$, and these bounds follow from the estimates

$$|\log(XY) - (\log X + \log Y)| \leq \mathrm{const} \cdot |X||Y|, \tag{2.7.19.6}$$
$$|\exp(X + Y) - \exp X \exp Y| \leq \mathrm{const} \cdot |X||Y|, \tag{2.7.19.7}$$

respectively. As a special case of (2.7.19.3), we also have the estimate

$$|\mathbb{E}_x(a \cdot f(x)) - a \cdot \mathbb{E}_x f(x)| \leq \mathrm{const} \cdot |a| \cdot \sup |f| \tag{2.7.19.8}$$

(and the same with a on the right).

To measure how close a given map $\phi : F \times F \to G$ is to being a groupoid homomorphism, we consider the 'error' function $E(\phi) : F \times F \times F \to G$ given by

$$E(\phi)(a, b, c) = \phi(a, b)\phi(b, c)\phi(a, c)^{-1}. \tag{2.7.19.9}$$

Now the rectification $R(\phi, \mu)$ will be defined by iterating the 'averaging' operation $\phi \mapsto \hat{\phi}$ given by

$$\hat{\phi}(a, b) = \phi(a, b)\mathbb{E}_x[\phi(a, b)^{-1}\phi(a, x)\phi(b, x)^{-1}], \tag{2.7.19.10}$$

whose domain is, by definition, those ϕ with $\sup |E(\phi)| < \varepsilon$, for some fixed small $\varepsilon > 0$ (note that this is an open condition on ϕ since F is compact (2.4.9)). The argument of the expectation in (2.7.19.10) may be written as $\phi(a, b)^{-1}E(\phi)(a, b, x)^{-1}\phi(a, b)$, so the condition $\sup |E(\phi)| < \varepsilon$ implies that this expectation is defined (provided our fixed $\varepsilon > 0$ is chosen to be sufficiently small). Moreover this expression shows that

$$\sup |\hat{\phi} - \phi| \leq \mathrm{const} \cdot \sup |E(\phi)|. \tag{2.7.19.11}$$

The key to showing favorable asymptotic behavior of the iteration $\phi \mapsto \hat{\phi}$ is to show that the error $E(\phi)$ is rapidly decreasing.

Let us bound $E(\hat{\phi})$ in terms of $E(\phi)$ following [92, Lemma 2.12]. The product $\hat{\phi}(a, b)\hat{\phi}(b, c)$ is given by

$$\phi(a, b)\mathbb{E}_x[\phi(a, b)^{-1}\phi(a, x)\phi(b, x)^{-1}]\phi(b, c)\mathbb{E}_x[\phi(b, c)^{-1}\phi(b, x)\phi(c, x)^{-1}] \tag{2.7.19.12}$$

whereas $\hat{\phi}(a, c) = \phi(a, c)\mathbb{E}_x[\phi(a, c)^{-1}\phi(a, x)\phi(c, x)^{-1}]$. To estimate the difference between these two expressions, we appeal to the approximate homomorphism property of expectation (2.7.19.3)(2.7.19.8). The first expression $\hat{\phi}(a, b)\hat{\phi}(b, c)$ can be written, using conjugation invariance of expectation, as

$$\begin{aligned}\phi(a, b)\phi(b, c)\mathbb{E}_x[(\phi(a, b)\phi(b, c))^{-1}\phi(a, x)\phi(b, x)^{-1}\phi(b, c)]& \\ \times \mathbb{E}_x[\phi(b, c)^{-1}\phi(b, x)\phi(c, x)^{-1}].&\end{aligned} \tag{2.7.19.13}$$

We can now apply (2.7.19.3) to conclude that this expression differs by at most a constant times $(\sup |E(\phi)|)^2$ from

$$\phi(a, b)\phi(b, c)\mathbb{E}_x[(\phi(a, b)\phi(b, c))^{-1}\phi(a, x)\phi(c, x)^{-1}]. \tag{2.7.19.14}$$

This expression is in turn related to $\hat{\phi}(a,c)$ by substituting $\phi(a,c)$ for $\phi(a,b)\phi(b,c)$ (in both places at once!), which by (2.7.19.8) again incurs an error of at most a constant times $(\sup|E(\phi)|)^2$. We have thus shown the 'quadratic decay estimate'

$$\sup|E(\hat{\phi})| \leq \text{const} \cdot (\sup|E(\phi)|)^2. \tag{2.7.19.15}$$

This estimate implies that once $\sup|E(\phi)|$ is sufficiently small, it then decreases super-exponentially as we iterate the operation $\phi \mapsto \hat{\phi}$. This decay implies that the iteration converges uniformly by (2.7.19.11).

We now define the rectification R. A pair (ϕ,μ) is in the domain of R when the iteration

$$\phi_0 = \phi, \tag{2.7.19.16}$$
$$\phi_i = (\phi_{i-1})^\wedge \quad \text{for } i > 0, \tag{2.7.19.17}$$

is defined for all $i \geq 0$ and the error decays to zero

$$\sup|E(\phi_i)| \xrightarrow{i\to\infty} 0. \tag{2.7.19.18}$$

The quadratic decay estimate (2.7.19.15) implies that this is an open condition in (ϕ,μ). Combining the quadratic decay estimate with the fact that the error controls the increments of the iteration (2.7.19.11), we see that the error decay condition also implies uniform convergence of ϕ_i as $i \to \infty$. We may thus define

$$R(\phi,\mu) = \lim_{i\to\infty} \phi_i. \tag{2.7.19.19}$$

Since $\phi_i \to R(\phi,\mu)$ uniformly, the error decay property (2.7.19.18) implies that $E(R(\phi,\mu)) = 0$, which means $R(\phi,\mu)$ is a groupoid homomorphism. It is evident that $R(\phi,\mu) = \phi$ whenever ϕ is a groupoid homomorphism.

What we have shown so far is that for smooth $\phi : B \times F \times F \to G$ and $\mu : B \times F \to \Omega^{>0}_F$ (for any smooth manifold B), the rectification $R(\phi,\mu)$ is continuous on its domain of definition, which is $B^\circ \times F \times F$ for some open set $B^\circ \subseteq B$.

It remains to show that $R(\phi,\mu)$ is in fact smooth. We will show that $\lim_{i\to\infty} \phi_i$ converges in the smooth topology (of local uniform convergence of all derivatives) on the total space $B^\circ \times F \times F$. We will proceed slightly differently from Zung [92, Lemma 2.13].

First, we need to slightly generalize the basic setup. Rather than assuming that F is compact, we instead fix a compact submanifold $F_0 \subseteq F$. The function ϕ remains defined on $F \times F$, but the measure μ now lives on F_0, and the averaging (2.7.19.2) takes place over F_0. The quadratic decay estimate (2.7.19.15) holds by the same argument. Now we declare a pair (ϕ,μ) to be in the domain of R when there exists a neighborhood of F_0 inside F over which the iteration ϕ_i is defined for all i and the error decays to zero (in other words, we regard F as a germ near F_0). The domain of R is open for the same reason as before. That is, for smooth $\phi : B \times F \times F \to G$ and $\mu : B \times F_0 \to \Omega^{>0}_{F_0}$, the subset $B^\circ \subseteq B$ where $R(\phi,\mu)$ is defined is open, and $R(\phi,\mu)$ is a continuous function on an (unspecified) open subset of $B^\circ \times F \times F$ containing $B^\circ \times F_0 \times F_0$.

We now return to the question of smooth convergence of $R(\phi,\mu)=\lim_{i\to\infty}\phi_i$, now in the above generalized setup. We claim that for every $k\geq 0$, the limit $R=\lim_i\phi_i$ converges in C^k over the open set where it converges in C^0. The case $k=0$ is vacuous, and for $k\geq 1$ we will use induction via the tangent functor T (2.6.10). Given a pair $\phi:B\times F\times F\to G$ and $\mu:B\times F_0\to\Omega^{>0}_{F_0}$, we may obtain a pair $T\phi:TB\times TF\times TF\to TG$ and $\mu:TB\times F_0\to\Omega^{>0}_{F_0}$ by applying the tangent functor T to ϕ and pulling back μ under the projection $TB\to B$ (this operation $(\phi,\mu)\mapsto(T\phi,\mu)$ is what compels the generalization in the previous paragraph). Note that TG is itself a Lie group (the functor T sends group objects to group objects since it preserves finite products). Now the key point is that applying T commutes with the averaging operation, in the sense that $T\hat{\phi}=(T\phi)^\wedge$. Indeed, this holds by functoriality of T and the fact that $\exp_{TG}=T\exp_G$. Thus applying the claim at a given k to the iteration $T\phi_i=(T\phi)_i$ implies the claim at $k+1$ for the iteration ϕ_i, so the claim holds for all $k\geq 0$ by induction. $\square$

2.8 Log smooth manifolds

A log structure on a topological space is a marking which, roughly speaking, specifies how functions are 'allowed to vanish'. Log structures originated in algebraic geometry in work of Fontaine and Illusie, with further development by Kato [47]. A log smooth manifold is a log topological space (2.8.2) equipped with an atlas of charts from open subsets of real affine toric varieties $X_P = \operatorname{Hom}(P, \mathbb{R}_{\geq 0})$ for polyhedral cones P, with transition functions which are smooth in a certain sense. This key notion of 'log smoothness' arises from a certain notion of tangent bundle for the local models X_P, namely the b-tangent bundle of Melrose [61, 62, 63] or the log tangent bundle as it is called in algebraic geometry. Log smooth manifolds were introduced by Joyce [41], who also proposed their application to moduli spaces of solutions of non-linear elliptic partial differential equations on families of degenerating manifolds. There is also closely related work of Parker [73].

Our goal here is to set up basic differential topology for log smooth manifolds.

* *

Let us first recall the formalism of log topological spaces.

2.8.1 Definition (Sheaves of continuous functions)**.** For any topological space X, let C_X denote the sheaf on X of continuous maps to $\mathbb{R}$, and let $C_X^{>0} \subseteq C_X^{\geq 0} \subseteq C_X$ denote the subsheaves of functions taking values in $\mathbb{R}_{>0} \subseteq \mathbb{R}_{\geq 0} \subseteq \mathbb{R}$, respectively.

⋆ **2.8.2 Definition** (Log topological space)**.** Let X be a topological space. A *pre-log structure* on X is a sheaf of (commutative) monoids $\mathcal{O}_X^{\geq 0}$ on X together with a map of sheaves of monoids $\mathcal{O}_X^{\geq 0} \to C_X^{\geq 0}$, where the monoid operation on $C_X^{\geq 0}$ is multiplication of functions. We consider the subsheaf $\mathcal{O}_X^{>0} \subseteq \mathcal{O}_X^{\geq 0}$ defined as the pullback

$$\begin{array}{ccc} \mathcal{O}_X^{>0} & \longrightarrow & C_X^{>0} \\ \downarrow & & \downarrow \\ \mathcal{O}_X^{\geq 0} & \longrightarrow & C_X^{\geq 0} \end{array} \tag{2.8.2.1}$$

and a *log structure* is a pre-log structure for which the map $\mathcal{O}_X^{>0} \to C_X^{>0}$ is an isomorphism. A *log topological space* is a topological space equipped with a log structure. A map of log topological spaces $(f, f^\flat) : (X, \mathcal{O}_X^{\geq 0}) \to (Y, \mathcal{O}_Y^{\geq 0})$ is a continuous map $f : X \to Y$ together with a map $f^\flat : f^*\mathcal{O}_Y^{\geq 0} \to \mathcal{O}_X^{\geq 0}$ such that the following diagram commutes.

$$\begin{array}{ccc} f^*\mathcal{O}_Y^{\geq 0} & \xrightarrow{f^\flat} & \mathcal{O}_X^{\geq 0} \\ \downarrow & & \downarrow \\ f^*C_Y^{\geq 0} & \xrightarrow{f^*} & C_X^{\geq 0} \end{array} \tag{2.8.2.2}$$

It is sometimes helpful to think in terms of 'log coordinates' $\log : \mathbb{R}_{\geq 0} \xrightarrow{\sim} \mathbb{R}_{\geq -\infty}$. In these coordinates, the sheaf $\mathcal{O}_X^{\geq 0}$ becomes an extension of the sheaf of real-valued functions by allowing some functions taking the value $-\infty$ at some points. The category of log topological spaces is denoted LogTop.

2.8.3 Remark. If $\mathcal{O}_X^{\geq 0} \to C_X^{\geq 0}$ is injective, then there is at most one log map $X \to Y$ lifting a given continuous map $X \to Y$.

2.8.4 Example (Trivial log structure)**.** Every topological space X has a 'trivial' log structure $\mathcal{O}_X^{\geq 0} = C_X^{>0}$, which is the default way to view X as a log topological space. This defines a full faithful embedding $\mathsf{Top} \to \mathsf{LogTop}$ which is right adjoint to the forgetful functor $\mathsf{LogTop} \to \mathsf{Top}$. In practice, one is usually interested in log structures which are 'finite extensions' of the trivial log structure.

The left adjoint to the forgetful functor $\mathsf{LogTop} \to \mathsf{Top}$ equips a topological space X with the log structure $\mathcal{O}_X^{\geq 0} = C_X^{\geq 0}$. This log structure is too 'wild' to be of much use.

2.8.5 Exercise (Log structure associated to a pre-log structure)**.** Show that the inclusion of log structures on X into pre-log structures on X has a left adjoint given by sending $\mathcal{O}_X^{\geq 0}$ to the colimit of $\mathcal{O}_X^{\geq 0} \leftarrow \mathcal{O}_X^{>0} \to C_X^{>0}$. Describe concretely a section of this colimit over an open set $U \subseteq X$, and verify that it is indeed a sheaf.

2.8.6 Exercise (Log structure from a function)**.** Let X be a topological space, and let $f : X \to \mathbb{R}_{\geq 0}$ be a continuous function with $Z := f^{-1}(0)$. There is an induced pre-log structure $\mathbb{Z}_{\geq 0} \to C_X^{\geq 0}$ given by $n \mapsto f^n$ for $n > 0$ and $0 \mapsto 1$; denote by $\mathcal{O}_X^{\geq 0}$ the associated log structure. Show that a global section of $\mathcal{O}_X^{\geq 0}$ consists of a function $g : X \to \mathbb{R}_{\geq 0}$, a locally constant function $n : Z \to \mathbb{Z}_{\geq 0}$, and functions $h_k : (X \setminus Z) \cup n^{-1}(k) \to \mathbb{R}_{>0}$ such that $f^k h_k = g|_{(X \setminus Z) \cup n^{-1}(k)}$. Show that $\mathcal{O}_X^{\geq 0} \to C_X^{\geq 0}$ is injective iff $Z^\circ = \varnothing$. Show that there are maps

$$0 \to \mathcal{O}_X^{>0} \to \mathcal{O}_X^{\geq 0} \to (i_Z)_* \mathbb{Z}_{\geq 0} \to 0 \tag{2.8.6.1}$$

which form a 'short exact sequence', in the sense that $\mathcal{O}_X^{\geq 0} \to (i_Z)_* \mathbb{Z}_{\geq 0}$ is an epimorphism of underlying sheaves of sets (i.e. every section of $(i_Z)_* \mathbb{Z}_{\geq 0}$ is locally the image of a section of $\mathcal{O}_X^{\geq 0}$; compare (**??**)) and its fibers are $\mathcal{O}_X^{>0}$-torsors (i.e. any two sections of $\mathcal{O}_X^{\geq 0}$ with the same image in $(i_Z)_* \mathbb{Z}_{\geq 0}$ are related by a unique section of $\mathcal{O}_X^{\geq 0}$).

Given a pair (X, Z) consisting of a topological space X and a closed subset $Z \subseteq X$, one might also attempt to consider the log structure given by those non-negative functions on X whose zero locus is contained in Z. Like the log structure $\mathcal{O}_X^{\geq 0} = C_X^{\geq 0}$ (2.8.4), this log structure is too wild to be of much use.

2.8.7 Remark (Log structure from a Cartier divisor)**.** The construction above (2.8.6) defines a map from $C_X^{\geq 0}$ to the sheaf of log structures on open subsets of X. Since multiplication by a positive function determines an isomorphism of the associated log structures, this map descends to the groupoid quotient $C_X^{\geq 0}/C_X^{>0}$. A section of $C_X^{\geq 0}/C_X^{>0}$ is called a *Cartier divisor*.

2.8.8 Exercise (Standard log structure $'\mathbb{R}_{\geq 0}$)**.** We denote by $'\mathbb{R}_{\geq 0}$ the topological space $\mathbb{R}_{\geq 0}$ equipped with the log structure associated to the identity function by the construction in

(2.8.6). Show that $'\mathbb{R}_{\geq 0}$ has the following universal property: maps $X \to {'\mathbb{R}_{\geq 0}}$ are in natural bijection with global sections of $\mathcal{O}_X^{\geq 0}$ for log topological spaces X. What are the global sections of this log structure on $'\mathbb{R}_{\geq 0}$? (Equivalently, what are the log maps $'\mathbb{R}_{\geq 0} \to {'\mathbb{R}_{\geq 0}}$?)

2.8.9 Definition (Pullback log structure)**.** Given a map of topological spaces $X \to Y$ and a log structure $\mathcal{O}_Y^{\geq 0}$ on Y, the pullback log structure $f^\# \mathcal{O}_Y^{\geq 0}$ on X is the log structure associated to the pre-log structure $f^* \mathcal{O}_Y^{\geq 0}$ (ordinary sheaf pullback). That is, $f^\# \mathcal{O}_Y^{\geq 0}$ is the sheaf pushout (i.e. the sheafification of the presheaf pushout) of $f^* \mathcal{O}_Y^{\geq 0} \leftarrow f^* \mathcal{O}_Y^{>0} \to \mathcal{O}_X^{>0}$ (compare (2.8.5)). A map of log topological spaces $(X, \mathcal{O}_X^{\geq 0}) \to (Y, \mathcal{O}_Y^{\geq 0})$ can be equivalently defined as a map of topological spaces $X \to Y$ together with a map $f^\# \mathcal{O}_Y^{\geq 0} \to \mathcal{O}_X^{\geq 0}$ of log structures on X.

2.8.10 Example. The log structure associated to a continuous function $f : X \to \mathbb{R}_{\geq 0}$ by the construction (2.8.6) is precisely $f^\#$ of the log structure on $'\mathbb{R}_{\geq 0}$.

2.8.11 Exercise. Let $f : X \to Y$ be continuous, and let $\mathcal{O}_Y^{\geq 0}$ be a log structure on Y. Show that its pullback $f^\# \mathcal{O}_Y^{\geq 0}$ satisfies the following universal property. For any log map $g : (Z, \mathcal{O}_Z^{\geq 0}) \to (Y, \mathcal{O}_Y^{\geq 0})$ and continuous map $h : Z \to X$ satisfying $g = f \circ h$, there exists a unique refinement of h to a log map $(Z, \mathcal{O}_Z^{\geq 0}) \to (X, f^\# \mathcal{O}_Y^{\geq 0})$ such that $g = f \circ h$ as log maps.

2.8.12 Exercise (Limits of log topological spaces)**.** Show that the limit of a diagram of log topological spaces $(X_\alpha, \mathcal{O}_{X_\alpha}^{\geq 0})$ is given by the limit of underlying topological spaces X_α equipped with the colimit of the pullbacks of $\mathcal{O}_{X_\alpha}^{\geq 0}$.

2.8.13 Definition (Strict log map)**.** A map of log topological spaces $(X, \mathcal{O}_X^{\geq 0}) \to (Y, \mathcal{O}_Y^{\geq 0})$ is called *strict* when the map $f^\# \mathcal{O}_Y^{\geq 0} \to \mathcal{O}_X^{\geq 0}$ is an isomorphism.

2.8.14 Definition (Embedding of log topological spaces)**.** An *embedding* of log topological spaces is a strict log map which is an embedding of underlying topological spaces.

2.8.15 Exercise. Show that strictness is preserved under pullback, hence so is the property of being an embedding.

2.8.16 Definition (Ghost sheaf)**.** For a log topological space X, the sheaf of monoids $\mathcal{Z}_X = \mathcal{O}_X^{\geq 0} / \mathcal{O}_X^{>0}$ is called the *ghost sheaf* of X.

2.8.17 Example. Let X be equipped with the log structure (2.8.6) associated to a function $f : X \to \mathbb{R}_{\geq 0}$ with zero set Z. The short exact sequence (2.8.6.1) shows that $\mathcal{Z}_X = (i_Z)_* \mathbb{Z}_{\geq 0}$.

2.8.18 Exercise. Show that two sections $f, g \in \mathcal{O}_X^{\geq 0}$ have the same image in $\mathcal{Z}_X$ iff there exist local expressions $f = ug$ for various local $u \in \mathcal{O}_X^{>0}$.

2.8.19 Exercise. Show that the only invertible section of the ghost sheaf $\mathcal{Z}_X$ is the identity.

2.8.20 Exercise. Show that a log map $f : X \to Y$ induces a map $f^{\flat\flat} : f^* \mathcal{Z}_Y \to \mathcal{Z}_X$ as a quotient of $f^\flat$. Show that if f is strict then $f^{\flat\flat}$ is an isomorphism (note that $\mathcal{Z} = \mathcal{O}^{\geq 0} / \mathcal{O}^{>0}$ is the pushout of $0 \leftarrow \mathcal{O}^{>0} \to \mathcal{O}^{\geq 0}$ and use the fact that f^* preserves colimits).

2.8.21 Definition (Quasi-integral)**.** A log topological space $(X, \mathcal{O}_X^{\geq 0})$ *quasi-integral* when the action of $\mathcal{O}_X^{>0}(U)$ on $\mathcal{O}_X^{\geq 0}(U)$ is free for every open $U \subseteq X$.

2.8.22 Exercise. Show that $(X, \mathcal{O}_X^{\geq 0})$ is quasi-integral iff the sequence

$$0 \to \mathcal{O}_X^{>0} \to \mathcal{O}_X^{\geq 0} \to \mathcal{Z}_X \to 0 \tag{2.8.22.1}$$

is exact in the sense that any two sections of $\mathcal{O}_X^{\geq 0}$ with the same image in $\mathcal{Z}_X$ differ by a unique section of $\mathcal{O}_X^{>0}$.

2.8.23 Exercise (Checking strictness via ghost sheaves)**.** For a log map $f : X \to Y$, show that if $f^{\flat\flat} : f^*\mathcal{Z}_Y \to \mathcal{Z}_X$ is an isomorphism and X is quasi-integral, then f is strict.

2.8.24 Definition (Cancellative)**.** A monoid M is called *cancellative* when $x + a = y + a$ implies $x = y$ for all elements $x, y, a \in M$. A sheaf of monoids is called cancellative when its monoid of sections over any open set is cancellative. A log topological space (or log structure) $(X, \mathcal{O}_X^{\geq 0})$ is called cancellative when $\mathcal{O}_X^{\geq 0}$ is cancellative.

2.8.25 Exercise. Show that for a sheaf of monoids M on a topological space X, the following are equivalent:
(2.8.25.1) M is cancellative.
(2.8.25.2) $M|_{U_i}$ is cancellative for every i for some open cover $X = \bigcup_i U_i$
(2.8.25.3) M_x is cancellative for every $x \in X$.

2.8.26 Exercise. Show that a cancellative log structure is quasi-integral.

* *

We now begin our discussion of log smooth manifolds.

⋆ **2.8.27 Definition** (Polyhedral cone)**.** A (real) polyhedral cone $P \subseteq \mathbb{R}^n$ is a subset defined by finitely many inequalities of the form $\sum_i a_i x_i \geq 0$. A map of polyhedral cones $P \to Q$ is the restriction of a linear map (the embedding into $\mathbb{R}^n$ is thus irrelevant).

2.8.28 Remark (Integral vs real polyhedral cones)**.** One could work with integral polyhedral cones (subsets $P \subseteq \mathbb{Z}^n$ defined by finitely many inequalities of the form $\sum_i a_i x_i \geq 0$ for $a_i \in \mathbb{Z}$) instead of real polyhedral cones. The resulting geometric theory would be very similar, but somewhat more rigid. The additional flexibility afforded by real polyhedral cones is needed to describe the elliptic partial differential equations, solutions thereof, and moduli spaces of solutions, which we will study later. It is for this reason that we choose to work here with real polyhedral cones.

2.8.29 Exercise. Let P be a polyhedral cone. Show that the groupification P^{gp} is a finite-dimensional real vector space, and that the map $P \to P^{\mathrm{gp}}$ identifies P with a polyhedral cone in P^{gp}.

2.8.30 Exercise. Show that every polyhedral cone P admits a surjection from some $\mathbb{R}^n_{\geq 0}$.

2.8.31 Exercise. Show that the category of polyhedral cones is an additive category (**??**).

2.8.32 Definition (Face)**.** A *face* $F \subseteq P$ of a polyhedral cone P is a subset of the form $F = P \cap \ell^{-1}(0)$ for some linear functional $\ell \in (P^{\mathrm{gp}})^*$ with the property that $\ell(p) \geq 0$ for all $p \in P$. For any map of polyhedral cones $P \to Q$, the inverse image of a face of Q is evidently a face of P.

⋆ **2.8.33 Definition** (Real affine toric varieties X_P)**.** Let P be a polyhedral cone. We consider the real affine toric variety

$$X_P = \operatorname{Hom}((P, +), (\mathbb{R}_{\geq 0}, \times)), \tag{2.8.33.1}$$

which we equip with the compact-open topology (2.4.12) and with the log structure associated to the pre-log structure $P \to C^{\geq 0}_{X_P}$ (the tautological 'evaluation' map). This log structure $\mathcal{O}^{\geq 0}_{X_P} \subseteq C^{\geq 0}_{X_P}$ consists of those functions on X_P which locally take the form $x(p)g(x)$ for some $p \in P$ and $g \in C^{>0}_{X_P}$. A *log topological manifold* is a log topological space locally isomorphic to open subsets of various X_P.

2.8.34 Example. If $P = \mathbb{R}$, then $X_P = \mathbb{R}_{>0}$ with the trivial log structure (2.8.4). If $P = \mathbb{R}_{\geq 0}$, then $X_P = {}'\mathbb{R}_{\geq 0}$ is the half-line $\mathbb{R}_{\geq 0}$ with its the standard log structure (2.8.8), namely the sheaf of continuous functions which locally have the form $f(x)x^a$ for some real number $a \geq 0$ and a continuous positive function f. If $P = \mathbb{R}^n_{\geq 0}$, then $X_P = {}'\mathbb{R}^n_{\geq 0}$, namely $\mathbb{R}^n_{\geq 0}$ equipped with the sheaf of continuous functions locally of the form $f(x)\prod_{i=1}^n x_i^{a_i}$ for real $a_i \geq 0$ and continuous positive f.

2.8.35 Exercise (Monomial maps $X_P \to X_Q$)**.** Show that a map of polyhedral cones $Q \to P$ induces a map $X_P \to X_Q$ (such maps are called *monomial*). Show that if $Q \twoheadrightarrow P$ is surjective then $X_P \hookrightarrow X_Q$ is a closed embedding.

2.8.36 Exercise (Presentation of X_P)**.** Show that a surjection $\varphi : \mathbb{R}^n_{\geq 0} \twoheadrightarrow P$ (which always exists (2.8.30)) presents $X_P \subseteq {}'\mathbb{R}^n_{\geq 0}$ as the subset cut out by finitely many conditions of the form $\prod_{i=1}^n x_i^{a_i} = 1$ for real $a_i \geq 0$ (corresponding to a finite set of generators of $\ker \varphi$). For instance, presenting $P = \mathbb{R}$ as $\mathbb{R}^2_{\geq 0}/(1,1)$ corresponds to realizing $X_P = \mathbb{R}_{>0}$ as the locus $\{xy = 1\} \subseteq {}'\mathbb{R}^2_{\geq 0}$.

2.8.37 Definition ($\mathbb{R}_{\geq 0}$-linear log structure)**.** An $\mathbb{R}_{\geq 0}$-linear monoid is a monoid M equipped with a bilinear operation $\mathbb{R}_{\geq 0} \times M \to M$; maps of $\mathbb{R}_{\geq 0}$-linear monoids are monoid maps respecting the $\mathbb{R}_{\geq 0}$-linear structure. For example, a real polyhedral cone has a canonical $\mathbb{R}_{\geq 0}$-linear structure, and in this way real polyhedral cones form a full subcategory of $\mathbb{R}_{\geq 0}$-linear monoids.

For X any topological space, $C^{\geq 0}_X$ is a sheaf of $\mathbb{R}_{\geq 0}$-linear monoids. By taking the definition of a log structure and replacing the category of monoids with that of $\mathbb{R}_{\geq 0}$-linear monoids, we obtain the notion of an $\mathbb{R}_{\geq 0}$-linear log structure. The foundations of log topological spaces expressed in (**??**) carry over as written. The map $P \to C^{\geq 0}_{X_P}$ is $\mathbb{R}_{\geq 0}$-linear, so it determines an $\mathbb{R}_{\geq 0}$-linear log structure on X_P. (This discussion of $\mathbb{R}_{\geq 0}$-linearity is due to our use of real polyhedral cones as opposed to integral polyhedral cones (2.8.28).)

2.8.38 Exercise (Universal properties of X_P)**.** Show that:

(2.8.38.1) Maps $Z \to X_P$ from topological spaces Z are in natural bijection with $\mathbb{R}_{\geq 0}$-linear maps of monoids $P \to C_Z^{\geq 0}$.

(2.8.38.2) Maps $Z \to X_P$ from $\mathbb{R}_{\geq 0}$-linear log topological spaces Z are in natural bijection with $\mathbb{R}_{\geq 0}$-maps of monoids $P \to \mathcal{O}_Z^{\geq 0}$.

Conclude that the natural map $X_{P \oplus Q} \to X_P \times X_Q$ (induced by the embeddings $P \to P \oplus Q \leftarrow Q$) is an isomorphism of $\mathbb{R}_{\geq 0}$-linear log topological spaces (that is, the contravariant functor $P \mapsto X_P$ sends coproducts to products).

⋆ **2.8.39 Example** (Log coordinates)**.** The map $P \to P^{\mathrm{gp}}$ to the groupification induces $X_{P^{\mathrm{gp}}} \to X_P$, which is a dense open embedding denoted $X_P^\circ \subseteq X_P$ called the 'interior', whose complement $X_P^{\mathrm{id}} = X_P \setminus X_P^\circ$ is called the 'ideal locus'. The interior $X_P^\circ \subseteq X_P$ is also the set $\operatorname{Hom}((P,+),(\mathbb{R}_{>0},\times))$. Applying the logarithm map $\log : (\mathbb{R}_{>0},\times) \xrightarrow{\sim} (\mathbb{R},+)$ yields an isomorphism

$$(P^{\mathrm{gp}})^* = X_P^\circ \subseteq X_P \tag{2.8.39.1}$$

referred to as *log coordinates* on X_P°. In log coordinates, monomial maps are linear.

2.8.40 Exercise. Show that $X_P^\circ \subseteq X_P$ is dense. Conclude that the map $\mathcal{O}_{X_P}^{\geq 0} \to C_{X_P}^{\geq 0}$ is injective and that X_P is cancellative (2.8.24). It was observed in (2.8.3) that this implies that for any log topological space Z, log map $X_P \to Z$ is a continuous map with a *property*. Show that a log map from X_P to an $\mathbb{R}_{\geq 0}$-linear log topological space is automatically $\mathbb{R}_{\geq 0}$-linear.

⋆ **2.8.41 Exercise** (Asymptotically cylindrical structures as log structures)**.** Let U and V be open subsets of $\mathbb{R}^k$, and consider maps $U \times {}'\mathbb{R}_{\geq 0} \to V \times {}'\mathbb{R}_{\geq 0}$. Show that such a map necessarily takes the interior $U \times \mathbb{R}_{>0}$ to the interior $V \times \mathbb{R}_{>0}$. Show that near the ideal locus $U \times 0$, such a map locally takes the form (using log coordinates $x = e^s$)

$$(u,s) \mapsto (f(u) + o(1), a \cdot s + b(u) + o(1)) \tag{2.8.41.1}$$

where $f : U \to V$, $a \geq 0$, $b : U \to \mathbb{R}$, and $o(1)$ indicates a quantity approaching zero as $s \to -\infty$, uniformly over compact subsets of U.

⋆ **2.8.42 Definition** (Stratification of X_P)**.** Each space X_P is stratified by the set of faces of P. Namely, to a monoid homomorphism $x : P \to \mathbb{R}_{\geq 0}$ we associate the face $F_x = x^{-1}(\mathbb{R}_{>0}) \subseteq P$. Given a face $F \subseteq P$, there is an embedding of topological spaces $X_F \subseteq X_P$ given by extension by zero on $P \setminus F$ (but note this is not an embedding of log topological spaces). The stratum of X_P associated to F is X_F°. Restriction along the inclusion $F \subseteq P$ defines a morphism of log topological spaces $X_P \to X_F$ which is a topological retraction.

2.8.43 Remark. It is known that X_P and P are homeomorphic as stratified spaces (a reference is [70, Theorem 1.4]). We will only ever use elementary special cases of this result, such as for $P = \mathbb{R}^n_{\geq 0}$.

2.8.44 Exercise (Rational maps $X_P \to X_Q$)**.** Let $Q \to P^{\mathrm{gp}}$ be a map of polyhedral cones, and consider the union of the strata $X_F^\circ \subseteq X_P$ for the faces $F \subseteq P$ for which $Q \to P^{\mathrm{gp}}$ lands inside $P + F^{\mathrm{gp}} \subseteq P^{\mathrm{gp}}$. Show that this is an open subset of X_P, and that $Q \to P^{\mathrm{gp}}$ defines a map $X_P \to X_Q$ on this open subset (such maps are called *rational*).

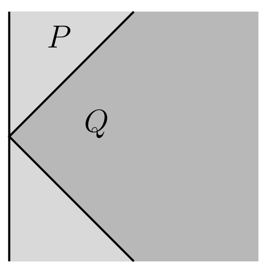
2.8.45 Definition (Sharp)**.** A polyhedral cone P is called *sharp* when its minimal stratum is $\{0\}$ (that is, when P contains no nonzero invertible elements). The quotient of a polyhedral cone P by its minimal stratum $P_0 \subseteq P$ is denoted $P^\#$, which is always sharp. The functor $P \mapsto P^\#$ is left adjoint to the inclusion of sharp polyhedral cones into all polyhedral cones.

2.8.46 Example (Local structure of X_P)**.** Let $x \in X_P$, and let $F_x \subseteq P$ index the stratum $X^\circ_{F_x} \subseteq X_P$ containing x, namely $F_x = x^{-1}(\mathbb{R}_{>0})$. Then x lies in the open subset $X_{P+F_x^{\mathrm{gp}}} \subseteq X_P$, in which it lies on the minimal stratum, namely $X_{F_x^{\mathrm{gp}}} = X^\circ_{F_x}$. We define $P_x = P/F_x^{\mathrm{gp}}$, so there is a short exact sequence

$$0 \to F_x^{\mathrm{gp}} \to P + F_x^{\mathrm{gp}} \to P_x \to 0. \tag{2.8.46.1}$$

The polyhedral cone P_x is sharp and is the stalk $\mathcal{Z}_{X_P,x}$ of the ghost sheaf $\mathcal{Z}_{X_P} = \mathcal{O}^{\geq 0}_{X_P}/\mathcal{O}^{>0}_{X_P}$ (2.8.16) at x. The polyhedral cone P_x controls the local structure of X_P near x: a choice of splitting of (2.8.46.1) induces an isomorphism $X_{P_x} \times X_{F_x^{\mathrm{gp}}} = X_{P+F_x^{\mathrm{gp}}} \subseteq X_P$.

2.8.47 Exercise. Consider $P = \mathbb{R}_{\geq 0} \times \mathbb{R} = \{x \geq 0\} \times \{y\}$ and $Q = \{y \leq |x|\} \subseteq P$.



(2.8.47.1)

Let $f : X_P \to X_Q$ denote the restriction map, and let $p = 0 \in X_P$ be the basepoint $p(0,y) = 1$ and $p(x,y) = 0$ for $x > 0$. The short exact sequence (2.8.46.1) for $f(p) \in X_Q$ maps naturally to that for $p \in X_P$. Show that this results in the following diagram:

$$\begin{tikzcd} 0 \arrow[r] & 0 \arrow[r] \arrow[d] & Q \arrow[r] \arrow[d] & Q \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathbb{R} \arrow[r] & P \arrow[r] & \mathbb{R}_{\geq 0} \arrow[r] & 0 \end{tikzcd} \tag{2.8.47.2}$$

Describe the map f geometrically.

2.8.48 Exercise. Let $f : X_P \to X_Q$ and let $x \in X_P$. Show that $f = f_{\mathrm{rat}} \cdot g$ in a neighborhood of x for some rational map $f_{\mathrm{rat}} : X_P \to X_Q$ and some map $g : X_P \to X^\circ_Q$, where $\cdot$ denotes multiplication in X_Q. Moreover, show that this pair (f_{rat}, g) is unique up to a natural action of $\mathrm{Hom}(Q, F_x^{\mathrm{gp}})$.

2.8.49 Example (Recovering local coordinates on a log topological manifold)**.** Let M be a log topological manifold, and let $p \in M$. The action of $\mathcal{O}^{>0}_M$ on $\mathcal{O}^{\geq 0}_M$ is free since $X^\circ_P \subseteq X_P$ is dense (2.8.40). It follows that we can construct a section of the forgetful map $\mathcal{O}^{\geq 0}_p \to \mathcal{Z}_p$

(2.8.16) by choosing $z_1, \ldots, z_n \in \mathcal{Z}_p$ which form a basis for the groupification $\mathcal{Z}_p^{\mathrm{gp}}$ and lifting each to $\mathcal{O}_p^{\geq 0}$. Such a choice of section is equivalently the data of a germ

$$(M, p) \to (X_{\mathcal{Z}_p}, 0) \tag{2.8.49.1}$$

(where $0 \in X_{\mathcal{Z}_p}$ is the map $\mathcal{Z}_p \to \mathbb{R}_{\geq 0}$ sending everything other than the identity to zero) whose action on ghost sheaves (2.8.16) at the basepoint is the identity map of $\mathcal{Z}_p$. The target is stratified by the faces of $\mathcal{Z}_p$ (2.8.42), which determines by pullback a germ of stratification of M near p. This stratification is described more intrinsically as follows: a point q near p is sent to the face of $\mathcal{Z}_p$ consisting of those functions which do not vanish at q. It thus recovers the canonical local stratification of M by the face poset of $\mathcal{Z}_p$ (2.8.46). It hence induces an isomorphism on ghost sheaves, so is a strict log map (2.8.23).

Now let $f : M \to N$ be a map of log manifolds, and let $x \in M$. The map of sharp polyhedral cones $f_x^{\flat\flat} : \mathcal{Z}_{N,f(x)} \to \mathcal{Z}_{M,x}$ satisfies $(f_x^{\flat\flat})^{-1}(0) = 0$ (if $f_x^{\flat\flat}(z) = 0$, then $z(f(x)) = (f_x^{\flat\flat} z)(x) > 0$, implying $z = 0$) but need not be injective (2.8.47). For any point y in a neighborhood of x, we can consider the associated faces $F_y \subseteq \mathcal{Z}_{M,x}$ and $F_{f(y)} \subseteq \mathcal{Z}_{N,f(x)}$ (the functions which do not vanish at y and $f(y)$, respectively), which satisfy $(f_x^{\flat\flat})^{-1}(F_y) = F_{f(y)}$. The map $f_x^{\flat\flat}$ induces a map $\mathcal{Z}_{N,f(y)} = \mathcal{Z}_{N,f(x)}/F_{f(y)}^{\mathrm{gp}} \to \mathcal{Z}_{M,x}/F_x^{\mathrm{gp}} = \mathcal{Z}_{M,y}$ which is precisely $f_y^{\flat\flat}$ (compare (2.8.46)).

If $f_x^{\flat\flat}$ is injective, then in the diagram

$$\begin{tikzcd} \mathcal{O}_{M,x}^{\geq 0} \arrow[r] & \mathcal{Z}_{M,x} \\ \mathcal{O}_{N,f(x)}^{\geq 0} \arrow[r] \arrow[u, "f_x^{\flat}"] & \mathcal{Z}_{N,f(x)} \arrow[u, "f_x^{\flat\flat}"'] \end{tikzcd} \tag{2.8.49.2}$$

we can choose compatible sections of the horizontal maps by first lifting $z_1, \ldots, z_n \in \mathcal{Z}_{N,f(x)}$ which are a basis for the groupification, and then extending their images in $\mathcal{Z}_{M,x}$ to a basis of its groupification and lifting these. The result is a diagram of germs

$$\begin{tikzcd} (M, x) \arrow[r] \arrow[d, "f"'] & (X_{\mathcal{Z}_{M,x}}, 0) \arrow[d, "f_x^{\flat\flat}"] \\ (N, f(x)) \arrow[r] & (X_{\mathcal{Z}_{N,f(x)}}, 0) \end{tikzcd} \tag{2.8.49.3}$$

in which the horizontal maps are strict.

2.8.50 Exercise. Show that for a map of log smooth manifolds $f : X \to Y$ and a point $x \in X$, the following are equivalent:
(2.8.50.1) $f_x^{\flat\flat}$ is an isomorphism.
(2.8.50.2) $f_{x'}^{\flat\flat}$ is an isomorphism for all x' in a neighborhood of x.
(2.8.50.3) f is strict in a neighborhood of x.

⋆ **2.8.51 Definition** (Depth)**.** The *depth* of a log topological manifold M at a point x is the dimension of the sharp polyhedral cone $\mathcal{Z}_{M,x}$ (2.8.46). The depth of M itself is the maximum depth over all its points.

2.8.52 Example. The depth of X_P is the dimension of $P^\# = P/P_0$ where $P_0 \subseteq P$ denotes the minimal stratum.

2.8.53 Example.

(2.8.53.1) Depth 0 is equivalent to being locally modelled on open subsets of $\mathbb{R}^k$.
(2.8.53.2) Depth ≤ 1 is equivalent to being locally modelled on open subsets of $\mathbb{R}^k \times {}'\mathbb{R}_{\geq 0}$.
(2.8.53.3) Depth ≤ 2 is equivalent to being locally modelled on open subsets of $\mathbb{R}^k \times {}'\mathbb{R}^2_{\geq 0}$.
(2.8.53.4) In depth 3, there are infinitely many local models (indeed, there infinitely many isomorphism classes of sharp polyhedral cones of dimension three).

∗ ∗

Our next major topic is differentiability on log manifolds. This discussion is a reformulation of Melrose's notions of b-tangent bundle, b-differential operators, etc.

⋆ **2.8.54 Definition** (Tangent space of X_P)**.** The tangent bundle $TX_P \to X_P$ is the trivial vector bundle with fiber $(P^{\mathrm{gp}})^*$. Over the interior of X_P (which is a smooth manifold), we identify this with the tangent bundle in the usual sense using log coordinates $X_P^\circ = (P^{\mathrm{gp}})^*$.

2.8.55 Example. The vector field $x\partial_x$ is an everywhere non-vanishing section of $T'\mathbb{R}_{\geq 0}$. More generally, $x_1\partial_{x_1}, \ldots, x_n\partial_{x_n}$ is a basis for $T'\mathbb{R}^n_{\geq 0}$. A basis for the tangent space of $U \times {}'\mathbb{R}_{\geq 0}$ ($U \subseteq \mathbb{R}^n$ open) is given, in log coordinates $x = e^s$ on ${}'\mathbb{R}_{\geq 0}$, by $\partial_{u_1}, \ldots, \partial_{u_n}, \partial_s$.

⋆ **2.8.56 Definition** (Cotangent cone of X_P)**.** The *cotangent cone* of X_P at a point x is the polyhedral cone $T^\circledast_x X_P = P + F_x^{\mathrm{gp}}$, whose groupification is the *cotangent space* $T^*_x X_P$.

2.8.57 Example. A general section of the cotangent cone of $T^\circledast{}'\mathbb{R}^n_{\geq 0}$ takes the form $\sum_i a_i(x)\frac{dx_i}{x_i}$ where $a_i(x) \geq 0$ over the locus where $x_i = 0$.

⋆ **2.8.58 Definition** (Log (co)tangent short exact sequence)**.** The short exact sequence (2.8.46.1) associated to a point $x \in X_P$ can be viewed as a sequence of cotangent cones

$$0 \to T^*_x X_{F_x} = T^\circledast_x X_{F_x} \to T^\circledast_x X_P \to \mathcal{Z}_x \to 0. \tag{2.8.58.1}$$

Dualizing gives a short exact sequence of tangent spaces

$$0 \to (\mathcal{Z}_x^{\mathrm{gp}})^* \to T_x X_P \to T_x X_{F_x} \to 0. \tag{2.8.58.2}$$

Note that the direction of these maps is opposite to the situation of a manifold-with-boundary: a tangent vector to X_P at x determines a tangent vector to the stratum $X_{F_x} \subseteq X_P$ containing x, rather than the other way around.

2.8.59 Definition (Differentiability of maps $X_P \to X_Q$)**.** A map $f : X_P \to \mathbb{R}_{>0}$ is said to be differentiable at $x \in X_P$ when its restriction to $X^\circ_{F_x} = (F_x^{\mathrm{gp}})^*$ is differentiable at x. The derivative of f at x is then the composition of the map $T_xX_P \to T_xX^\circ_{F_x}$ (namely $(P^{\mathrm{gp}})^* \to (F_x^{\mathrm{gp}})^*$) with the derivative of f restricted to $X^\circ_{F_x}$ at x; dually, this is an element of $T^*_xF_x = F_x^{\mathrm{gp}} \subseteq P + F_x^{\mathrm{gp}} = T^{\circledast}_xX_P$. We say f is continuously differentiable when its derivative $df : X_P \to T^{\circledast}X_P$ is continuous.

A map $f : X_P \to {}'\mathbb{R}_{\geq 0}$ is said to be differentiable at $x \in X_P$ when $f = p \cdot g$ (near x) for $p \in P$ and $g : X_P \to \mathbb{R}_{>0}$ differentiable at x. The derivative of f at x is the sum of $p \in P \subseteq P + F_x^{\mathrm{gp}} = T^{\circledast}_xX_P$ and the derivative of g at x; this is independent of the choice of decomposition $f = p \cdot g$.

A map $f : X_P \to X_Q$ is said to be differentiable at $x \in X_P$ iff for every $q \in Q$, the composite $q \circ f : X_P \to {}'\mathbb{R}_{\geq 0}$ is differentiable at x. In this case, the induced map $Q \to T^{\circledast}_xX_P$ sends $F_{f(x)}$ to invertible elements $F_x^{\mathrm{gp}} \subseteq P + F_x^{\mathrm{gp}} = T^{\circledast}_xX_P$ since for $q \in F_{f(x)}$ the composite $q \circ f$ is nonzero at x. The map $Q \to T^{\circledast}_xX_P$ thus extends to $Q + F^{\mathrm{gp}}_{f(x)} = T^{\circledast}_{f(x)}X_Q$, and the resulting map $T^{\circledast}_xf : T^{\circledast}_{f(x)}X_Q \to T^{\circledast}_xX_P$ is called the derivative of f at x. This derivative respects the short exact sequences (2.8.58), namely the following diagram commutes.

$$\begin{tikzcd} 0 \arrow[r] & T^*_xX_{F_x} \arrow[r] & T^{\circledast}_xX_P \arrow[r] & \mathcal{Z}_x \arrow[r] & 0 \\ 0 \arrow[r] & T^*_{f(x)}X_{F_{f(x)}} \arrow[r] \arrow[u, "T^*_x(f|_{X_{F_x}})"] & T^{\circledast}_{f(x)}X_Q \arrow[r] \arrow[u, "T^{\circledast}_xf"] & \mathcal{Z}_{f(x)} \arrow[r] \arrow[u, "f^{\flat\flat}_x"] & 0 \end{tikzcd} \tag{2.8.59.1}$$

A map $f : X_P \to X_Q$ is called *continuously differentiable* iff it is differentiable at every point and the map $Tf : TX_P \to TX_Q$ given on each fiber by the derivative is continuous. In this case, Tf is a log map since $TX_P \to X_P$ is strict. The adjectives 'k times continuously differentiable' ('class C^k') and 'smooth' ('class C^∞') are now defined by iterating T.

2.8.60 Example. The derivative of a monomial map $X_P \to X_Q$ is the map $(P^{\mathrm{gp}})^* \to (Q^{\mathrm{gp}})^*$ induced by $Q \to P$. In particular, the derivative of a monomial map is again a monomial map; hence monomial maps are smooth.

2.8.61 Example. The diagonal map $\mathbb{R}_{\geq 0} \to \mathbb{R}^2_{\geq 0}$ of polyhedral cones induces the monomial map

$$f : {}'\mathbb{R}^2_{\geq 0} \to {}'\mathbb{R}_{\geq 0} \tag{2.8.61.1}$$
$$(x, y) \mapsto xy = \lambda \tag{2.8.61.2}$$

which is thus a log smooth map. The vector fields $\{x\partial_x, y\partial_y\}$ and $\lambda\partial_\lambda$ form bases of the tangent spaces of the source and target, respectively (note that, in particular, these vector fields are *nonzero* even where $x = 0$, $y = 0$, or $\lambda = 0$). The derivative of f sends $x\partial_x \mapsto \lambda\partial_\lambda$ and $y\partial_y \mapsto \lambda\partial_\lambda$. Its kernel (i.e. the vertical tangent bundle) is thus of constant rank one, spanned by $x\partial_x - y\partial_y$. Lifting $\lambda\partial_\lambda$ to $\frac{1}{2}(x\partial_x + y\partial_y)$ may be viewed as defining a 'connection'.

2.8.62 Example. Write $f : X_P \to X_Q$ near $x \in X_P$ as $f = f_{\mathrm{rat}} \cdot g$ as in (2.8.48). There is a unique such decomposition for which the derivative of $g|_{X^\circ_{F_x}}$ at x vanishes. For this decomposition, f_{rat} is the rational map associated to the derivative $T^{\circledast}_x f : Q + F^{\mathrm{gp}}_{f(x)} \to P + F^{\mathrm{gp}}_x$ at x.

2.8.63 Lemma (Chain Rule)**.** *If $f : X_P \to X_Q$ is differentiable at x and $g : X_Q \to X_R$ is differentiable at $f(x)$, then their composition $X_P \to X_R$ is differentiable at x, and its derivative is the composition of the derivatives of f and g.*

Proof. By the definition of differentiability of maps with target X_R, it suffices to treat the case $X_R = {}'\mathbb{R}_{\geq 0}$. We are thus in the situation of a pair of maps $f : X_P \to X_Q$ and $g : X_Q \to {}'\mathbb{R}_{\geq 0}$. By the definition of $T^{\circledast}_{f(x)} g$ as a sum, we are reduced to two cases, namely $g(f(x)) > 0$ or $g \in Q$. When $g \in Q$, the relation $T^{\circledast}(g \circ f) = T^{\circledast} g \circ T^{\circledast} f$ is the definition of $T^{\circledast} f$. When $g(f(x)) > 0$, we are reduced to the chain rule for the restrictions $f|_{X_{F_x}} : X_{F_x} \to X_{F_{f(x)}}$ and $g|_{X_{F_{f(x)}}}$. □

⋆ **2.8.64 Definition** (Log smooth manifold)**.** A *log smooth manifold* is a log topological space equipped with an atlas of charts from open subsets of various X_P whose transition functions are smooth. The category of log smooth manifolds and smooth maps is denoted LogSm.

2.8.65 Exercise (Log smooth manifolds via the structure sheaf)**.** For any log smooth manifold M, let $\mathcal{A}^{\geq 0}_M \subseteq \mathcal{O}^{\geq 0}_M$ denote the subsheaf of functions to ${}'\mathbb{R}_{\geq 0}$ which are smooth. Prove that $X_P \to X_Q$ is smooth iff it pulls back functions in $\mathcal{A}^{\geq 0}_{X_Q}$ to functions in $\mathcal{A}^{\geq 0}_{X_P}$. Conclude that a log smooth manifold is equivalently a log topological space M with a subsheaf $\mathcal{A}^{\geq 0}_M \subseteq \mathcal{O}^{\geq 0}_M$ which is locally isomorphic to $(X_P, \mathcal{A}^{\geq 0}_{X_P})$. Conclude that a log smooth manifold is also a topological space M with a subsheaf $\mathcal{A}^{\geq 0}_M \subseteq C^{\geq 0}_M$ which is locally isomorphic to $(X_P, \mathcal{A}^{\geq 0}_{X_P})$.

2.8.66 Exercise (Strata functor)**.** For a log smooth manifold X, define a topological space $S(X)$ ('strata') mapping to X by taking $S(X_P) = \bigsqcup_{F \subseteq P} X_F$ (mapping to X_P by 'extension by zero' (2.8.42)) and gluing. Equip $S(X)$ with the log structure given by the subsheaf $\mathcal{O}^{\geq 0}_{S(X)} \subseteq \operatorname{im}(\mathcal{O}^{\geq 0}_X \to C^{\geq 0}_{S(X)})$ of functions whose zero set is nowhere dense, and show that $S(X_P) = \bigsqcup_{F \subseteq P} X_F$ as log topological spaces (note that the map $S(X) \to X$ is *not* a log map). Make the same definition with log smooth functions (2.8.65), and show that $S(X_P) = \bigsqcup_{F \subseteq P} X_F$ as log smooth manifolds, hence $S(X)$ is a log smooth manifold for all X. Finally, show that S is a functor from the category of log smooth manifolds to itself (first argue it is a functor to topological spaces, and then show that the maps are log smooth by inspecting the rings of log smooth functions).

⋆ **2.8.67 Definition** (Log (co)tangent short exact sequence)**.** In view of the functoriality of the log (co)tangent short exact sequences (2.8.59.1), they make sense on log smooth manifolds. That is, for any point x of a log smooth manifold M, there is a short exact sequence

$$0 \to T^*_x M_x \to T^{\circledast}_x M \to \mathcal{Z}_{M,x} \to 0, \tag{2.8.67.1}$$

where $M_x \subseteq M$ denotes the local stratum containing x. A smooth map of log smooth manifolds $f : M \to N$ induces a map of such short exact sequences.

$$\begin{tikzcd} 0 \arrow[r] & T_x^* M_x \arrow[r] & T_x^{\circledast} M \arrow[r] & \mathcal{Z}_{M,x} \arrow[r] & 0 \\ 0 \arrow[r] & T_{f(x)}^* N_{f(x)} \arrow[r] \arrow[u, "T_x^*(f|_{M_x})"] & T_{f(x)}^{\circledast} N \arrow[r] \arrow[u, "T_x^{\circledast} f"] & \mathcal{Z}_{N,f(x)} \arrow[r] \arrow[u, "f_x^{\flat\flat}"] & 0 \end{tikzcd} \tag{2.8.67.2}$$

Recall that the map $f_x^{\flat\flat}$ determines $f_y^{\flat\flat}$ for all y in a neighborhood of x (2.8.49).

2.8.68 Definition. There is an evident short exact sequence

$$0 \to \mathcal{A}_M^{>0} \to \mathcal{A}_M^{\geq 0} \to \mathcal{Z}_M \to 0 \tag{2.8.68.1}$$

for log smooth manifolds M, which is functorial under log smooth maps $f : M \to N$. By taking logarithmic derivatives, this sequence maps to the log cotangent short exact sequence (2.8.67), resulting in a diagram with exact rows and columns.

$$\begin{tikzcd} \ker(\mathcal{A}_{M,x}^{>0} \to T_x^* M_x) \arrow[r, equal] \arrow[d] & \ker(\mathcal{A}_{M,x}^{\geq 0} \to T_x^{\circledast} M) \arrow[d] & \\ \mathcal{A}_{M,x}^{>0} \arrow[r] \arrow[d] & \mathcal{A}_{M,x}^{\geq 0} \arrow[r] \arrow[d] & \mathcal{Z}_{M,x} \arrow[d, equal] \\ T_x^* M_x \arrow[r] & T_x^{\circledast} M \arrow[r] & \mathcal{Z}_{M,x} \end{tikzcd} \tag{2.8.68.2}$$

In particular, we have a short exact sequence

$$0 \to \ker(\mathcal{A}_{M,x}^{>0} \to T_x^* M_x) \to \mathcal{A}_{M,x}^{\geq 0} \to T_x^{\circledast} M \to 0 \tag{2.8.68.3}$$

expressing $T_x^{\circledast} M$ as germs of smooth functions to ${}'\mathbb{R}_{\geq 0}$ modulo those functions with vanishing derivative at x.

⋆ **2.8.69 Exercise** (Asymptotically cylindrical structures as log structures)**.** As a continuation of (2.8.41), show that a log map $U \times {}'\mathbb{R}_{\geq 0} \to V \times {}'\mathbb{R}_{\geq 0}$ of class C^k takes the form

$$(u, s) \mapsto (f(u), a \cdot s + b(u)) + o(1)_{C^k} \tag{2.8.69.1}$$

for $f \in C^k$, $b \in C^k$, and $o(1)_{C^k}$ indicating a function of class C^k whose derivatives of order up to k approach zero as $s \to -\infty$, uniformly over compact subsets of U.

2.8.70 Exercise. Show that the map ${}'\mathbb{R}_{\geq 0} \to \mathbb{R}$ given by $x \mapsto x$ is smooth.

2.8.71 Exercise (Smooth functions ${}'\mathbb{R}^2_{\geq 0} \to \mathbb{R}$)**.** Show that a function ${}'\mathbb{R}^2_{\geq 0} \to \mathbb{R}$ is smooth iff it is given in log coordinates $(x, y) = (e^s, e^t)$ by $(s, t) \mapsto a + b(s) + c(t) + o(1)_{C^\infty}$ as $\min(s, t) \to -\infty$ (uniformly over sets on which $\max(s, t)$ is bounded) for $a \in \mathbb{R}$, $b(s) = o(1)_{C^\infty}$ as $s \to -\infty$, and similarly for $c(t)$.

2.8.72 Lemma. *A Hausdorff log smooth manifold has bump functions, hence partitions of unity (if also paracompact).*

Proof. It suffices to consider the case of a local model X_P. A surjection $\mathbb{R}^n_{\geq 0} \to P$ determines a closed embedding $X_P \hookrightarrow {}'\mathbb{R}^n_{\geq 0}$ (2.8.35) which is smooth (2.8.60), so it suffices to exhibit bump functions on ${}'\mathbb{R}^n_{\geq 0}$. The tautological inclusion map ${}'\mathbb{R}^n_{\geq 0} \to \mathbb{R}^n$ is also smooth (2.8.70), so we reduce further to the case of $\mathbb{R}^n$, which was treated earlier (2.6.12). The construction of partitions of unity from bump functions (2.6.20) is a purely topological argument, so applies without change. □

⋆ **2.8.73 Definition** (Averaging on a log manifold)**.** Let M be a paracompact Hausdorff log smooth manifold, and let $\operatorname{Meas}(M)$ denote the set of positive measures on M of unit total mass. As in (2.6.29), we seek to construct an 'averaging' operation

$$\operatorname{avg} : \operatorname{Meas}(M) \to M \tag{2.8.73.1}$$

on the set of measures of 'sufficiently small' support.

Let us call an open set $U \subseteq X_P$ *convex* when its intersection with every stratum $X_F^\circ \subseteq X_P$ (for faces $F \subseteq P$) is convex in log coordinates $X_F^\circ = (F^{\mathrm{gp}})^*$. For a measure μ on U supported inside $U \cap X_F^\circ$ for some face $F \subseteq P$, the average $\operatorname{avg}_U(\mu) \in U$ is defined via the linear structure on $(F^{\mathrm{gp}})^* = X_F^\circ$. Given a smooth function of compact support $\eta : U \to [0,1]$, we may define the cutoff average $\operatorname{avg}_{U,\eta}$ as in (2.6.29.2).

Now consider a cover $M = \bigcup_i U_i$ by open sets identified with convex open sets $U_i \subseteq X_P$. To see that such an open cover exists, it suffices to show that every point of X_P has arbitrarily small convex open neighborhoods. By embedding $X_P \hookrightarrow {}'\mathbb{R}^n_{\geq 0}$ via a surjection $\mathbb{R}^n_{\geq 0} \twoheadrightarrow P$ (2.8.36), we may reduce to the case of points of ${}'\mathbb{R}^n_{\geq 0}$ and hence, since convexity is preserved by products, to the case of ${}'\mathbb{R}_{\geq 0}$ where the result is obvious. We may now follow the manifold case (2.6.29) and define the global average (2.8.73.1) as a composition of local averages $\operatorname{avg}_{U_i,\eta_i}$. This composition is defined on measures of 'sufficiently small' support, which now means, for some fine open cover $M = \bigcup_i U_i$ by convex open sets $U_i \subseteq X_{P_i}$, that the support of μ is contained in $U_i \cap X_F^\circ$ for some i and some face $F \subseteq P_i$.

The averaging map avg is smooth in the following sense. Consider families of measures $N \to \operatorname{Meas}(M)$ parameterized by a log smooth manifold N which, locally on N, are of the form of a finite sum $\sum_i w_i \delta_{p_i}$ for some smooth functions $w_i : N \to [0,1]$ and $p_i : N \to M$. Now each local averaging operation, hence also the global averaging operation, preserves families of this form (inspection).

* *

We now discuss the inverse function theorem for log smooth manifolds. Recall that each point of a log smooth manifold has a cotangent polyhedral cone (2.8.56), which carries more information than its groupification the cotangent space. For this reason, the correct statement of the inverse function theorem involves cotangent cones rather than (co)tangent spaces.

2.8.74 Exercise (Failure of naive log inverse function theorem)**.** Consider the map $\mathbb{R}^2_{\geq 0} \to \mathbb{R}^2_{\geq 0}$ given by $f(x,y) = (x^2y, xy^2)$. Show that f induces an isomorphism on tangent spaces yet is not a local homeomorphism (in fact, it is not even open).

2.8.75 Exercise (Log inverse function theorem hypothesis)**.** Use the log cotangent short exact sequence (2.8.67) to show that for $f : X \to Y$ a smooth map of log smooth manifolds, the following are equivalent:

(2.8.75.1) $T^{\circledast}_x f : T^{\circledast}_{f(x)} Y \to T^{\circledast}_x X$ is an isomorphism (of polyhedral cones).

(2.8.75.2) $f_x^{\flat\flat} : \mathcal{Z}_{Y,f(x)} \to \mathcal{Z}_{X,x}$ and $T_x f : T_x X \to T_{f(x)} Y$ are isomorphisms.

(2.8.75.3) $f_x^{\flat\flat} : \mathcal{Z}_{Y,f(x)} \to \mathcal{Z}_{X,x}$ and $T_x f|_{X_x} : T_x X_x \to T_{f(x)} Y_{f(x)}$ are isomorphisms.

Use (2.8.50) to observe that these conditions are open.

⋆ **2.8.76 Log Inverse Function Theorem.** *Let $f : M \to N$ be C^k for $k \geq 1$. If $T^{\circledast}_x f$ is an isomorphism of polyhedral cones, then f is a local log homeomorphism at x, its local inverse is also C^k, and $T(f^{-1}) = (Tf)^{-1}$.*

Proof. We first treat the case $k = 1$ and then deduce the general case by induction.

The assertion is local, so we may consider a C^1 map $f : (X_P, p) \to (X_Q, f(p))$ of real affine toric varieties for which $T^{\circledast}_p f$ is an isomorphism of polyhedral cones. By replacing P with $P + F_p^{\mathrm{gp}}$ and replacing Q with $Q + F_{f(p)}^{\mathrm{gp}}$, we may assume wlog that p and $f(p)$ are on the minimal strata of X_P and X_Q, respectively. We thus have $T^{\circledast}_p X_P = P$ and $T^{\circledast}_{f(p)} X_Q = Q$, so the derivative of f at p is a map $Q \to P$, which we have assumed to be an isomorphism. Identifying Q with P via this isomorphism and translating so that $p = 0 \in X_P$ and $f(p) = 0 \in X_Q$, our map now takes the form

$$f : (X_P, 0) \to (X_P, 0) \tag{2.8.76.1}$$

$$x \mapsto u(x)x \tag{2.8.76.2}$$

for some C^1 map $u : X_P \to X_P^\circ$ whose derivative vanishes at $0 \in X_P$ (compare (2.8.62)).

Consider $u : X_P \to X_P^\circ$ in log coordinates $X_P^\circ = (P^{\mathrm{gp}})^*$ (2.8.39) on (the interior of) the source and target. The first derivative of u in such coordinates approaches zero as $x \to 0 \in X_P$; that is, we have

$$u(x) = \mathrm{const} + o(1)_{C^1} \tag{2.8.76.3}$$

in log coordinates in the limit $x \to 0 \in X_P$. Now the key point is simply that every map of the form $\mathbf{1} + \mathrm{const} + o(1)_{C^1}$ on $\mathbb{R}^n$ has an inverse of the same form. Thus f is a diffeomorphism from each stratum $X_F^\circ \subseteq X_P$ (faces $F \subseteq P$) to itself, in a neighborhood of $0 \in X_P$. In particular, f is a continuous bijection in a neighborhood of zero, which implies it is a local homeomorphism there since X_P is locally compact Hausdorff. Now $f^{\flat\flat}$ is an isomorphism in a neighborhood of 0 (2.8.75), which implies f is strict (2.8.23), hence is an isomorphism of log topological spaces.

Now let us show that the inverse f^{-1} is continuously differentiable with derivative $T(f^{-1}) = (Tf)^{-1}$. Note that Tf is an isomorphism of vector bundles covering an isomorphism of log topological spaces, so it is itself an isomorphism of log topological spaces, hence has an inverse $(Tf)^{-1}$. It thus suffices to show that f^{-1} is differentiable with derivative $(Tf)^{-1}$ at any given point. We may wlog just treat the case of the basepoint $0 \in X_P$ itself, where the desired assertion follows from the fact that f^{-1} has the form $\mathbf{1} + \text{const} + o(1)_{C^1}$ on each stratum in log coordinates. We have thus proven the case $k = 1$.

We may now derive the case of general $k \geq 1$ from the case $k = 1$ using induction. Suppose f is C^k and $T^{\circledast}_p f$ is an isomorphism of polyhedral cones. Since $k \geq 1$, the inverse f^{-1} exists and is C^1 with derivative $T(f^{-1}) = (Tf)^{-1}$. We wish to show that f^{-1} is C^k, equivalently that $T(f^{-1}) = (Tf)^{-1}$ is C^{k-1}. This follows from the induction hypothesis and the fact that Tf is C^{k-1}, provided we show that the derivative of Tf is an isomorphism of polyhedral cones. The cotangent cone of TM along the zero section $M \subseteq TM$ is the direct sum of $T^{\circledast}M$ (cotangent to the zero section) and T^*M (cotangent to the fibers). The derivative of Tf respects this decomposition and is an isomorphism on each piece, hence is an isomorphism over the zero section of TM. It is thus an isomorphism over a neighborhood of the zero section, hence is so everywhere by scaling equivariance. □

2.8.77 Example (Recovering local coordinates on a log smooth manifold)**.** Let $p \in M$ be a point of a log smooth manifold, and let us show how to construct a germ of diffeomorphism

$$(M, p) \to X_{T^{\circledast}_p M} \tag{2.8.77.1}$$

using the log inverse function theorem (2.8.76). The map $\mathcal{A}^{\geq 0}_{M,p} \to T^{\circledast}_p M$ is a torsor for the subspace of $\mathcal{A}^{>0}_{M,p}$ consisting of those functions whose first derivative at p vanishes (2.8.68.3). We may thus construct a section $T^{\circledast}_p M \to \mathcal{A}^{\geq 0}_{M,p}$ using the procedure from (2.8.49): pick any set in $T^{\circledast}_p M$ which is a basis of the groupification, lift each of its elements to $\mathcal{A}^{\geq 0}_{M,p}$, and note that the torsor property implies that this extends to a unique section defined on all of $T^{\circledast}_p M$. Such a section is equivalently a germ (2.8.77.1) which induces the 'identity' map on cotangent cones. It is thus a local diffeomorphism by the log inverse function theorem (2.8.76).

We will see a relative version of this arugment in (2.8.81).

* *

We now study submersions of log smooth manifolds. As can be expected from the form of the inverse function theorem for log smooth manifolds (2.8.76), arbitrary submersions of log smooth manifolds are not so well behaved. Instead, we will see that a more useful notion ('exact submersion') is obtained by imposing additional conditions on cotangent cones.

⋆ **2.8.78 Definition** (Submersion)**.** A map of log smooth manifolds $f : M \to N$ is called a *submersion* at $p \in M$ when its derivative $T_p M \to T_{f(p)} N$ is surjective (equivalently, $T^{\circledast}_{f(p)} N \to T^{\circledast}_p M$ is injective). The locus of points $p \in M$ where f is a submersion is evidently an open set.

Geometrically speaking, a map $f : X \to Y$ is a submersion when it is a submersion on interiors $f^\circ : X^\circ \to Y^\circ$ and the inverse to Tf° is 'uniformly bounded' in log coordinates as one approaches the ideal locus. No condition, however, is imposed on how f interacts with the compactifications $X^\circ \subseteq X$ and $Y^\circ \subseteq Y$. Any injective map of polyhedral cones $Q \to P$ gives a submersion $X_P \to X_Q$, and such maps are in general very far from being topologically locally trivial (2.1.3).

2.8.79 Exercise. Use the log cotangent short exact sequence (2.8.67) to show that f is a submersion at p iff its restriction $f : M_p \to N_{f(p)}$ to the strata containing p and $f(p)$, respectively, is a submersion of smooth manifolds and the snake map $\ker f_p^{\flat\flat} \to \operatorname{coker} T_p^* f|_{M_p}$ is injective.

2.8.80 Exercise. Show that the strata functor (2.8.66) preserves submersions (show that for $f : M \to N$, a point $p \in M$, and a face $F \subseteq T_p^\circledast M$, the restriction of $T_p^\circledast f : T_{f(p)}^\circledast N \to T_p^\circledast M$ to the inverse image of F coincides with the derivative of $S(f) : S(M) \to S(N)$ at the lift of p corresponding to F).

2.8.81 Lemma (Local normal form of a submersion)**.** *A map of log smooth manifolds is a submersion iff it is locally (on the source) modelled on monomial maps associated to injective maps of polyhedral cones.*

Proof. This will be a relative version of (2.8.77).

Let $f : M \to N$ be a submersion at $p \in M$. We seek to construct a diagram

$$\begin{tikzcd} (M,p) \arrow[r] \arrow[d] & X_{T_p^\circledast M} \arrow[d, "T_p^\circledast f"] \\ (N,f(p)) \arrow[r] & X_{T_{f(p)}^\circledast N} \end{tikzcd} \tag{2.8.81.1}$$

in which the horizontal maps induce the identity on cotangent cones, hence are local diffeomorphisms (2.8.76). Such a diagram is equivalent to the data of compatible sections in the following diagram.

$$\begin{tikzcd} \mathcal{A}_{M,p}^{\geq 0} \arrow[r] & T_p^\circledast M \arrow[l, bend right, dashed] \\ \mathcal{A}_{N,f(p)}^{\geq 0} \arrow[r] \arrow[u] & T_{f(p)}^\circledast N \arrow[l, bend right, dashed] \arrow[u, "T_p^\circledast f"'] \end{tikzcd} \tag{2.8.81.2}$$

For any subset $S_M \subseteq T_p^\circledast M$ (resp. $S_N \subseteq T_{f(p)}^\circledast N$) which is a basis of the groupification, a choice of section $S_M \to \mathcal{A}_{M,p}^{\geq 0}$ (resp. $S_N \to \mathcal{A}_{N,f(p)}^{\geq 0}$) extends uniquely to a section on $T_p^\circledast M$ (resp. $T_{f(p)}^\circledast N$) (2.8.77). Since $T_p^\circledast f$ is injective, we may produce a compatible pair of sections by first choosing S_N and its section and then choosing $S_M \supseteq S_N$ and its section to be an extension of that of S_N. $\square$

* *

We now introduce exact submersions of log smooth manifolds, which have better local behavior than submersions. Exact submersions are also preserved under pullback, which is crucial for a great number of applications. It is not surprising that the relevant notion of 'exactness' (introduced by Kato [47, Definition (4.6)] and since recognized as a key notion in log geometry) is a condition on the derivative on cotangent cones.

2.8.82 Definition (Local)**.** Let $f : Q \to P$ be a map of polyhedral cones, and let $P_0 \subseteq P$ and $Q_0 \subseteq Q$ denote the minimal strata (equivalently, the subgroups of invertible elements). It is always the case that $Q_0 \subseteq f^{-1}(P_0)$ (the image of an invertible element is always invertible). The map f is called *local* when this inclusion is an equality, that is $Q_0 = f^{-1}(P_0)$.

2.8.83 Exercise. Show that $Q \to P$ is local iff the associated monomial map $X_P \to X_Q$ sends the minimal stratum of X_P to the minimal stratum of X_Q. Show that the monomial map $X_P \to X_Q$ associated to $f : Q \to P$ is locally modelled near $x \in X_P$ on the monomial map associated to $Q + f^{-1}(F_x)^{\mathrm{gp}} \to P + F_x^{\mathrm{gp}}$ (compare (2.8.46)). Show that every map $Q + f^{-1}(F)^{\mathrm{gp}} \to P + F^{\mathrm{gp}}$ is local.

2.8.84 Exercise. Show that $Q \to P$ is local iff $Q^\# = Q/Q_0 \to P/P_0 = P^\#$ is local. Conclude that the derivative $T_x^{\circledast} f : T_{f(x)}^{\circledast} N \to T_x^{\circledast} M$ of a log smooth map $f : M \to N$ is always local (recall that $f_x^{\flat\flat} : \mathcal{Z}_{N,f(x)} \to \mathcal{Z}_{M,x}$ is always local (2.8.49)).

2.8.85 Definition (Exact; Kato [47, Definition (4.6)])**.** A map of real polyhedral cones $f : Q \to P$ is called *exact* when $(f^{\mathrm{gp}})^{-1}(P) = Q$.

2.8.86 Exercise. Show that $Q \to P$ is exact iff $Q^\# \to P^\#$ is exact. Conclude that $T_x^{\circledast} f$ is exact iff $f_x^{\flat\flat}$ is exact.

2.8.87 Exercise. Show that if $f : Q \to P$ is exact, then so is $f^{-1}(F) \to F$ for every face $F \subseteq P$.

2.8.88 Definition (Locally exact; Illusie–Kato–Nakayama [34, (A.3.2)(iii)] and Nakayama–Ogus [70, Definition 2.1(3)])**.** A map of real polyhedral cones $f : Q \to P$ is called *locally exact* when for every face $F \subseteq P$, the localized map

$$Q + f^{-1}(F)^{\mathrm{gp}} \to P + F^{\mathrm{gp}} \tag{2.8.88.1}$$

is exact.

2.8.89 Exercise. Show that $Q \to P$ is locally exact iff $Q^\# \to P^\#$ is locally exact. Conclude that $T_x^{\circledast} f$ is locally exact iff $f_x^{\flat\flat}$ is locally exact.

2.8.90 Exercise. Let $f : M \to N$ be a map of log smooth manifolds. Show that $f_x^{\flat\flat}$ is locally exact iff $f_y^{\flat\flat}$ is exact for all y in a neighborhood of x (use (2.8.49)).

2.8.91 Definition (Exact; Kato [47, Definition (4.6)])**.** Let f be a map of log smooth manifolds. We say that f is *exact* at x when the following equivalent conditions hold:

(2.8.91.1) $T_y^{\circledast} f$ (equivalently $f_y^{\flat\flat}$ (2.8.86)) is exact for all y in a neighborhood of x.
(2.8.91.2) $T_x^{\circledast} f$ (equivalently $f_x^{\flat\flat}$ (2.8.89)) is locally exact.
Exactness is evidently an open condition.

2.8.92 Exercise. Show that an exact local map of sharp polyhedral cones is injective. Conclude that if f is exact then $f_x^{\flat\flat}$ is injective for every x.

2.8.93 Exercise (Relative depth)**.** The *(relative) depth* of an *exact* map of log smooth manifolds $f : M \to N$ at a point $x \in M$ is the difference

$$\begin{aligned} \operatorname{depth}_x(f) &= \dim \mathcal{Z}_{M,x} - \dim \mathcal{Z}_{N,f(x)} && (2.8.93.1) \\ &= (\dim M - \dim N) - (\dim M_x - \dim N_{f(x)}). && (2.8.93.2) \end{aligned}$$

Use (2.8.92) to show that the relative depth is non-negative and upper semicontinuous on M. Show that an exact map has depth zero iff it is strict (use (2.8.23)). What is the depth (as a function on the source) of the multiplication map $'\mathbb{R}^2_{\geq 0} \to {}'\mathbb{R}_{\geq 0}$?

2.8.94 Exercise. Conclude from (2.8.87) that the strata functor (2.8.66) preserves exactness.

The significance of the notions of exactness and local exactness, at least for us, comes from the fact that they behave well under pushout (corresponding to pullback of log smooth manifolds).

2.8.95 Lemma. *Locally exact (resp. exact and locally exact) morphisms of polyhedral cones are preserved under pushout (in the category of $\mathbb{R}_{\geq 0}$-linear monoids).*

Proof. We begin with a criterion for the existence of the pushout $P \sqcup_Q R$ of a diagram of polyhedral cones $R \leftarrow Q \to P$. Let I^{gp} denote the pushout of groupifications (a finite-dimensional real vector space).

$$\begin{array}{ccc} Q^{\mathrm{gp}} & \longrightarrow & P^{\mathrm{gp}} \\ \downarrow & & \downarrow \\ R^{\mathrm{gp}} & \longrightarrow & I^{\mathrm{gp}} \end{array} \qquad (2.8.95.1)$$

Now define the polyhedral cone $I \subseteq I^{\mathrm{gp}}$ to be the image of $P \oplus R \to P^{\mathrm{gp}} \oplus R^{\mathrm{gp}} \twoheadrightarrow I^{\mathrm{gp}}$. The notation is consistent: the groupification of I is indeed I^{gp}. We now wish to formulate a condition under which the resulting diagram

$$\begin{array}{ccc} Q & \longrightarrow & P \\ \downarrow & & \downarrow \\ R & \longrightarrow & I \end{array} \qquad (2.8.95.2)$$

is a pushout.

Consider the following two equivalence relations $\sim_Q$ and $\sim_{Q^{\mathrm{gp}}}$ on the set $P \oplus R$. For pairs $(p,r),(p',r') \in P \oplus R$, we declare that $(p,r) \sim_{Q^{\mathrm{gp}}} (p',r')$ iff

$$p' = p + q \tag{2.8.95.3}$$
$$r = r' + q \tag{2.8.95.4}$$

for some $q \in Q^{\mathrm{gp}}$. We set $(p,r) \sim_Q^{\mathrm{pre}} (p',r')$ when (2.8.95.3)–(2.8.95.4) hold for some $q \in Q$ (as opposed to Q^{gp}), and we take $\sim_Q$ to be the equivalence relation closure of $\sim_Q^{\mathrm{pre}}$. Now a map out of $P \oplus R$ comes from a (necessarily unique) map out of I iff Q^{gp}-equivalent pairs have the same image, while it comes from a (necessarily unique) map out of $R \leftarrow Q \to P$ iff Q-equivalent pairs have the same image. We thus conclude that if $\sim_Q$ and $\sim_{Q^{\mathrm{gp}}}$ coincide, then (2.8.95.2) is a pushout.

Now let us argue that if $f : Q \to P$ is locally exact, then this criterion is satisfied, namely Q^{gp}-equivalence implies Q-equivalence. Consider a point $(p,r) \in P \oplus R$, and let $F \subseteq P$ denote the minimal face containing p. Local exactness of $Q \to P$ means that $(f^{\mathrm{gp}})^{-1}(P + F^{\mathrm{gp}}) = Q + f^{-1}(F)^{\mathrm{gp}}$, and $Q + f^{-1}(F)^{\mathrm{gp}}$ is equivalently the set of differences $Q - f^{-1}(F)$, so

$$(f^{\mathrm{gp}})^{-1}(P + F^{\mathrm{gp}}) = Q - f^{-1}(F). \tag{2.8.95.5}$$

For any point $(p',r') \in P \oplus R$, the difference $p' - p$ lies in $P + F^{\mathrm{gp}}$. A lift of this difference to Q^{gp} is thus an element of $Q - f^{-1}(F)$. Now if (p',r') is Q^{gp}-equivalent and sufficiently close to (p,r), then the element of Q^{gp} lifting $p' - p$ realizing this equivalence can be taken arbitrarily small. It is thus an element of Q minus an arbitrarily small element of $f^{-1}(F)$. Now $p \in F^\circ$, so p minus a sufficiently small element of F lies in P, and we thus conclude that (p',r') is Q-equivalent to (p,r). To conclude that Q^{gp}-equivalence implies Q-equivalence in general, it suffices to note that Q^{gp}-equivalence classes in $P \oplus R$ are convex (being the intersection of the convex set $P \oplus R \subseteq P^{\mathrm{gp}} \oplus R^{\mathrm{gp}}$ with the inverse image of a point of $P^{\mathrm{gp}} \sqcup_{Q^{\mathrm{gp}}} R^{\mathrm{gp}}$) hence connected.

Now let us show that if $Q \to P$ is exact, then so is $R \to I$. Fix an element $r \in R^{\mathrm{gp}}$, and suppose that its image in I^{gp} is contained in I. This means $(0,r)$ is Q^{gp}-equivalent to some $(p',r') \in P \oplus R$, namely there is $q \in Q^{\mathrm{gp}}$ lifting $p' \in P$ and $r - r' \in R^{\mathrm{gp}}$. Exactness of $Q \to P$ means $q \in Q$, so $r = r' + q \in R$ as desired.

Finally, we should show that $R \to I$ is locally exact. A face $A \subseteq I$ pulls back to faces $F \subseteq P$, $G \subseteq Q$, and $H \subseteq R$, and we have a resulting localized diagram.

$$\begin{tikzcd} Q + G^{\mathrm{gp}} \arrow[r] \arrow[d] & P + F^{\mathrm{gp}} \arrow[d] \\ R + H^{\mathrm{gp}} \arrow[r] & I + A^{\mathrm{gp}} \end{tikzcd} \tag{2.8.95.6}$$

Now the pullback of A to $P \oplus R$ is the face $F \oplus H \subseteq P \oplus R$ (indeed, every face of $P \oplus R$ is a product, hence is the direct sum of its pullbacks to P and R). Thus A^{gp} is the image of $F^{\mathrm{gp}} \oplus H^{\mathrm{gp}}$, which implies that $I + A^{\mathrm{gp}}$ is the image of $(P + F^{\mathrm{gp}}) \oplus (R + H^{\mathrm{gp}})$. The top map $Q + G^{\mathrm{gp}} \to P + F^{\mathrm{gp}}$ is locally exact, so the localized diagram remains a pushout. Exactness of the top map thus implies exactness of the bottom map $R + H^{\mathrm{gp}} \to I + A^{\mathrm{gp}}$ as desired. $\square$

2.8.96 Example. The map ${}'\mathbb{R}^2_{\geq 0} \to {}'\mathbb{R}_{\geq 0}$ given by $(x,y) \mapsto xy = \lambda$ (i.e. corresponding to the diagonal embedding $\mathbb{R}_{\geq 0} \to \mathbb{R}^2_{\geq 0}$) is an exact submersion. Although the fibers of this map over points $\lambda \in {}'\mathbb{R}_{\geq 0}$ develop a singularity as $\lambda \to 0$, the family is at least topologically locally trivial on the source. In fact, all exact submersions are topologically locally trivial on the source by a result of Nakayama–Ogus [70, Theorem 0.2] (though we will not appeal to this result).

⋆ **2.8.97 Proposition.** *Exact submersions are preserved under pullback.*

Proof. Since submersions are locally monomial (2.8.81), it suffices to consider monomial maps $X_P \to X_Q$ associated to injective and locally exact maps $Q \to P$.

Now let $Z \to X_Q$ be an arbitrary log smooth map, and let us show that $Z \times_{X_Q} X_P$ exists and maps exactly submersively to Z. The map $Z \to X_Q$ need not be locally monomial, but it is at least expressible in local coordinates $Z = X_R$ as the product ug of a monomial map $g : X_R \to X_Q$ and a log smooth map $u : X_R \to X_Q^\circ$. Let us argue that the pullbacks of $X_P \to X_Q$ under the two maps g and ug are identified. It suffices to consider the 'universal' case of $Z = X_R = X_Q \times X_Q^\circ$ where g and u are the first and second projections, respectively. To make the desired identification, it is enough to lift the action of X_Q° on X_Q to X_P. Now X_P° acts on X_P, and the map $X_P \to X_Q$ is equivariant for the map $X_P^\circ \to X_Q^\circ$. It thus suffices to fix a section of $X_P^\circ \to X_Q^\circ$, which is just the map of vector spaces $(P^{\mathrm{gp}})^* \to (Q^{\mathrm{gp}})^*$, hence has a section since it is surjective (since $Q \to P$ is injective).

We have thus reduced our problem to showing existence of the pullback $X_R \times_{X_Q} X_P$ of log smooth manifolds for maps of polyhedral cones $R \leftarrow Q \to P$ in which $Q \to P$ is injective and locally exact. Now log smooth maps $Z \to X_P$ are in natural bijection with $\mathbb{R}_{\geq 0}$-linear maps $P \to \mathcal{A}_Z^{\geq 0}$. It follows that $X_R \times_{X_Q} X_P = X_{R \sqcup_Q P}$. Local exactness of $Q \to P$ implies the pushout $R \sqcup_Q P$ exists and that the map $R \to R \sqcup_Q P$ is locally exact (2.8.95), so $X_{R \sqcup_Q P} \to X_R$ is an exact submersion. □

The fibers of an exact submersion over *interior* points of the base are log smooth manifolds by stability of exact submersions under pullback (2.8.97). Stability under pullback says nothing about fibers over ideal points of the base (note that a map of log smooth manifolds $* \to M$ must land inside the interior M°). Such fibers may be 'singular' as in (2.8.96), and may be called 'broken' log smooth manifolds (2.8.96). While not log smooth manifolds, such fibers are objects in a certain 'hybrid' category (2.12). Here is another way to make sense of the fibers of an exact submersion:

2.8.98 Definition (Normalized fiber)**.** Let $f : M \to N$ be an exact submersion of log smooth manifolds, and let $n \in N$ be a point (not necessarily in the interior of N). Apply the strata functor (2.8.66) to obtain a map $S(f) : S(M) \to S(N)$, which remains an exact submersion (2.8.80)(2.8.94). Now a point $n \in N$ has a unique inverse image in $S(N)$ lying in the interior $S(N)^\circ$, so the fiber of $S(f)$ over this point is a log smooth manifold (2.8.97) which we call the *normalized fiber* of f over n.

2.8.99 Exercise. Compute the fiber and the normalized fiber of the multiplication map ${}'\mathbb{R}^2_{\geq 0} \to {}'\mathbb{R}_{\geq 0}$ (which is an exact submersion) over the point $0 \in {}'\mathbb{R}_{\geq 0}$.

Here are two special classes of exact submersions of interest.

2.8.100 Example (Strict submersion)**.** A map f of log smooth manifolds is strict precisely when $f^{\flat\flat}$ is an isomorphism at every point (2.8.49). In particular, a strict submersion is exact. A strict submersion is locally on the source a pullback of $\mathbb{R}^k \to *$ (2.8.81).

2.8.101 Definition (Simply-broken submersion)**.** A *simply-broken submersion* of log smooth manifolds is a map which is locally a pullback of $\mathbb{R}^k \to *$ or its product with the multiplication map ${}'\mathbb{R}^2_{\geq 0} \to {}'\mathbb{R}_{\geq 0}$.

* *

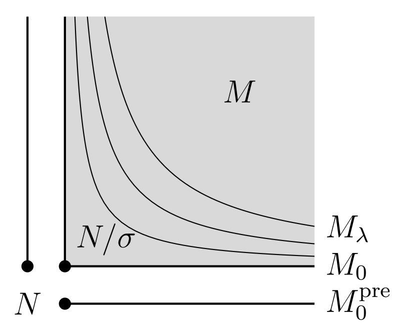
We now explore generalizations of Ehresmann's Theorem (2.6.30) (proper submersions of smooth manifolds are trivial locally on the target) to log smooth manifolds.

2.8.102 Proposition. *A proper submersion of log smooth manifolds which is trivial locally on the source is trivial locally on the target.*

Proof. We generalize the proof for smooth manifolds (2.6.30) as follows. Let $M \to B$ be a proper submersion which is trivial locally on the source, and let us show that $M \to B$ is trivial in a neighborhood of a given point $0 \in B$. Since $M \to B$ is trivial locally on the source, the fiber $M_0 = M \times_B 0$ exists.

We now seek to construct a local retraction $M \to M_0$. Such a retraction may be defined locally near any point of M_0 using the fact that $M \to B$ is trivial locally on the source (2.8.100). To patch together these local retractions, we appeal to the averaging operation for measures on log smooth manifolds (2.8.73). The resulting map $M \to M_0 \times B$ is an isomorphism on cotangent cones (inspection) so the log inverse function theorem (2.8.76) applies to show that it is a local isomorphism. $\square$

⋆ **2.8.103 Definition** (Gluing coordinates)**.** Let M_0^{pre} be a log smooth manifold, let $i : N \times {}'\mathbb{R}_{\geq 0} \hookrightarrow M_0^{\mathrm{pre}}$ be an open embedding covering all points of positive depth (thus M_0^{pre} has depth one), and let $\sigma : N \to N$ be a free involution. Associated to this data $(M_0^{\mathrm{pre}}, N, i, \sigma)$ is a standard 'gluing coordinates' family $M \to {}'\mathbb{R}_{\geq 0}$ as we now recall.



(2.8.103.1)

The fiber M_0 over $0 \in {}'\mathbb{R}_{\geq 0}$ is the quotient of M_0^{pre} by the involution σ acting on $N \times 0 \subseteq N \times {}'\mathbb{R}_{\geq 0} \subseteq M_0^{\mathrm{pre}}$. The fiber M_λ for $\lambda > 0$ is the quotient of $(M_0^{\mathrm{pre}})^\circ$ by the relation

$i(n,x) = i(\sigma(n),y)$ whenever $xy = \lambda$. Thus M_λ is a 'gluing' of M_0 with 'gluing parameter' $\lambda \in {}'\mathbb{R}_{\geq 0}$.

The map $M \to {}'\mathbb{R}_{\geq 0}$ is defined as follows. Consider the two maps

$$\begin{tikzcd}[column sep=small] (M_0^{\mathrm{pre}})^\circ \times {}'\mathbb{R}_{\geq 0} \arrow[dr] & & (N \times {}'\mathbb{R}^2_{\geq 0})/(\sigma \times s) \arrow[dl] \\ & {}'\mathbb{R}_{\geq 0} & \end{tikzcd} \tag{2.8.103.2}$$

given by projection to ${}'\mathbb{R}_{\geq 0}$ and projection to ${}'\mathbb{R}^2_{\geq 0}$ followed by multiplication, respectively, where s denotes the involution $(x,y) \mapsto (y,x)$ of ${}'\mathbb{R}^2_{\geq 0}$. We glue these total spaces together by identifying $(n,x,y) = (\sigma(n),y,x)$ with $(i(n,x),xy)$ and $(i(\sigma(n),y),xy)$ to obtain M.

$$\begin{tikzcd} M \arrow[d] \arrow[r, phantom, "="] & \Big((M_0^{\mathrm{pre}})^\circ \times {}'\mathbb{R}_{\geq 0}\Big) \displaystyle\bigcup_{(N\times({}'\mathbb{R}^2_{\geq 0}\setminus(0,0)))/(\sigma\times s)} \Big((N \times {}'\mathbb{R}^2_{\geq 0})/(\sigma \times s)\Big) \arrow[dl] \\ {}'\mathbb{R}^2_{\geq 0} & \end{tikzcd} \tag{2.8.103.3}$$

When M_0^{pre} is compact Hausdorff, the resulting family $M \to {}'\mathbb{R}_{\geq 0}$ is proper.

Finally, let us note that we may also take a separate gluing parameter for every connected component of N/σ to produce a family $M \to {}'\mathbb{R}^{\pi_0 N/\sigma}_{\geq 0}$.

2.8.104 Proposition. *Every proper simply-broken submersion is, locally on the base, a pullback of a standard gluing family* (2.8.103)*.*

Proof. This is a generalization of (2.6.30).

Let $Q \to B$ be a proper simply-broken submersion. Fix a basepoint $b \in B$, and let Q_b denote the fiber over b. The map $Q \to B$ is covered by local pullback diagrams.

$$\begin{tikzcd} Q \arrow[r] \arrow[d] & {}'\mathbb{R}^2_{\geq 0} \times \mathbb{R}^k \arrow[d, "(x,y,t)\mapsto xy"] \\ B \arrow[r, "b\mapsto 0"] & {}'\mathbb{R}_{\geq 0} \end{tikzcd} \tag{2.8.104.1}$$

Our task is to patch together these local charts into gluing coordinates (2.8.103) near the basepoint $b \in B$. In fact, we will not do exactly this, rather we will show how to recover such charts intrinsically from the map $Q \to B$, and we will then globalize this intrinsic construction.

Let us call a point $q \in Q$ *non-singular* when the map $\pi : Q \to B$ is strict in a neighborhood of q, equivalently when $\mathcal{Z}_{B,\pi(q)} \to \mathcal{Z}_{Q,q}$ is an isomorphism (2.8.50). At a non-singular point $q \in Q$, the map $Q \to B$ is a strict submersion, hence is locally of the form $Q = B \times \mathbb{R}^k \to B$ (2.8.100). Thus the stratum $Q_q \subseteq Q$ of q is the unique local stratum lying over the stratum

$B_b \subseteq B$ of b, and the restriction $Q_q \to B_b$ is a submersion. Thus the (open) non-singular locus of Q_b is canonically a smooth manifold. In a neighborhood of any non-singular point of Q_b, a local trivialization $Q = Q_b \times B$ may be constructed intrinsically as follows: construct a retraction $Q \to Q_b$ simply by extension of smooth functions from strata, and note that the induced map $Q \to Q_b \times B$ is a local isomorphism by the inverse function theorem (2.8.76).

Let us now investigate what happens near the singular points of Q. Singular points of Q are precisely those lying over $0 \times 0 \times \mathbb{R}^k$ in the local charts (2.8.104.1) (points not lying over $0 \times 0 \times \mathbb{R}^k$ are non-singular since the map $'\mathbb{R}^2_{\geq 0} \to {'\mathbb{R}_{\geq 0}}$ is strict away from $(0,0)$, and it will be clear from the present discussion that conversely all points lying over $0 \times 0 \times \mathbb{R}^k$ are in fact singular). At a singular point $q \in Q_b$, a choice of local chart (2.8.104.1) induces a pushout square of cotangent cones (2.8.97).

$$\begin{tikzcd} T^{\circledast}_q Q & \mathbb{R}^2_{\geq 0} \times \mathbb{R}^k \arrow[l] \\ T^{\circledast}_b B \arrow[u] & \mathbb{R}_{\geq 0} \arrow[l] \arrow[u, "a \mapsto (a,a,0)"'] \end{tikzcd} \tag{2.8.104.2}$$

It is somewhat more convenient to quotient by the invertible elements to obtain the following pushout square.

$$\begin{tikzcd} \mathfrak{Z}_{Q,q} & \mathbb{R}^2_{\geq 0} \arrow[l] \\ \mathfrak{Z}_{B,b} \arrow[u] & \mathbb{R}_{\geq 0} \arrow[l] \arrow[u, "a \mapsto (a,a)"'] \end{tikzcd} \tag{2.8.104.3}$$

By the construction of such pushouts (2.8.95), this means that $\mathfrak{Z}_{Q,q}$ is the image of $\mathfrak{Z}_{B,b} \oplus \mathbb{R}^2_{\geq 0}$ inside the pushout of vector spaces $\mathfrak{Z}^{\mathrm{gp}}_{B,b} \sqcup_{\mathbb{R}} \mathbb{R}^2$. We now make some deductions by inspecting this description of $\mathfrak{Z}_{Q,q}$. There exist precisely two non-zero faces of $\mathfrak{Z}_{Q,q}$ whose intersection with $\mathfrak{Z}_{B,b}$ is zero. Both are rays $\mathbb{R}_{\geq 0} \subseteq \mathfrak{Z}_{Q,q}$, and the quadrant $\mathbb{R}^2_{\geq 0} \subseteq \mathfrak{Z}_{Q,q}$ they span (namely the upper map in (2.8.104.3)) intersects $\mathfrak{Z}_{B,b}$ in a ray $\mathbb{R}_{\geq 0} \subseteq \mathfrak{Z}_{B,b}$ (namely the lower map in (2.8.104.3), not necessarily a face). We conclude that the pushout diagram (2.8.104.3) is actually determined uniquely by the map $\mathfrak{Z}_{B,b} \to \mathfrak{Z}_{Q,q}$, up to simultaneous scaling of $\mathbb{R}_{\geq 0} \to \mathbb{R}^2_{\geq 0}$. Recall that the strata of B (resp. Q) near b (resp. q) are indexed by the faces of $\mathfrak{Z}_{B,b}$ (resp. $\mathfrak{Z}_{Q,q}$) and that the stratum of Q corresponding to a face $F \subseteq \mathfrak{Z}_{Q,q}$ maps to the stratum of B corresponding to $F \cap \mathfrak{Z}_{B,b} \subseteq \mathfrak{Z}_{B,b}$. There are thus precisely three strata of Q lying over the stratum of b, namely those corresponding to zero and to the two distinguished rays inside $\mathfrak{Z}_{Q,q}$. The stratum Q_q of Q corresponding to the zero face of $\mathfrak{Z}_{Q,q}$ is precisely the singular locus near q. The stalks of $\mathfrak{Z}_Q$ at nearby singular points are identified canonically, so they have 'the same' distinguished rays. In particular, every component of the singular locus of Q_b determines a ray in $\mathfrak{Z}_{B,b}$.

Given this knowledge of the structure of the map $\mathfrak{Z}_{B,b} \to \mathfrak{Z}_{Q,q}$ for singular points $q \in Q_b$, we can now give an 'intrinsic' construction of local charts (2.8.104.1) near such q. Suppose $x, y : (Q,q) \to ({'\mathbb{R}_{\geq 0}}, 0)$ and $\lambda : (B,b) \to ({'\mathbb{R}_{\geq 0}}, 0)$ are maps whose classes in $\mathfrak{Z}_{Q,q}$ and $\mathfrak{Z}_{B,b}$ generate the distinguished rays in these polyhedral cones. We therefore have $\lambda = e^f x^a y^b$

for some smooth $f : M \to \mathbb{R}$ and some real numbers $a, b > 0$. By replacing (x, y) with $(e^{f/2}x^a, e^{f/2}y^b)$, we may achieve that $\lambda = xy$ on M, and hence that we have a diagram of the following form.

$$\begin{tikzcd} Q \arrow[r] \arrow[d] & {}'\mathbb{R}^2_{\geq 0} \arrow[d] \\ B \arrow[r] & {}'\mathbb{R}_{\geq 0} \end{tikzcd} \tag{2.8.104.4}$$

Now fix in addition a function on the singular stratum of Q to $\mathbb{R}^k$ which is a local diffeomorphism, and extend it to a smooth function on a neighborhood of q. The resulting diagram (with ${}'\mathbb{R}^2_{\geq 0} \times \mathbb{R}^k$ in the upper right corner) is a pullback square: the pullback $B \times_{{}'\mathbb{R}_{\geq 0}} {}'\mathbb{R}^2_{\geq 0} \times \mathbb{R}^k$ exists since the map being pulled back is an exact submersion, and the inverse function theorem (2.8.76) guarantees that the map from Q to this pullback is an isomorphism (its derivative at q is an isomorphism by construction).

Now let us extract from the fiber Q_b the data necessary to define a standard gluing chart (2.8.103). The map $b : * \to B$ is a map of topological spaces, but not of log smooth manifolds unless b lies in the interior B°, so the fiber $Q_b = Q \times_B *$ is merely a topological space. We can, however, refine the topological fiber Q_b to a log smooth manifold $\tilde{Q}_b$ (the 'normalized fiber') mapping to Q_b (2.8.98). To do this, we use the strata functor (2.8.66). There is a unique point $\tilde{b} \in S(B)^\circ$ lying over $b \in B$ (namely, it is the inverse image of b corresponding to the local stratum of B containing b). The normalized fiber $\tilde{Q}_b$ is the fiber of $S(Q) \to S(B)$ over $\tilde{b}$, which is a log smooth manifold since it is a pullback of the exact submersion $S(Q) \to S(B)$ (2.8.94)(2.8.80).

There is an evident map of topological spaces $\tilde{Q}_b \to Q_b$ (induced by $S(Q) \to Q$ and $S(B) \to B$), and we can describe it concretely as follows (by inspection). Near a non-singular point of Q_b, the map $\tilde{Q}_b \to Q_b$ is a homeomorphism and $\tilde{Q}_b$ is a smooth manifold (and this coincides with the smooth manifold structure on Q_b defined above). A singular point of Q_b has three inverse images in $\tilde{Q}_b$, corresponding to the three local strata of Q lying over the stratum of $b \in B$. This decomposes $\tilde{Q}_b$ into the union of a smooth manifold N/σ (in bijection with the singular points of Q_b) and a log smooth manifold M_0^{pre} of depth one whose ideal locus $N = (M_0^{\mathrm{pre}})^{\mathrm{id}}$ has a free involution σ with quotient N/σ.

Now let $M \to {}'\mathbb{R}^{\pi_0 N/\sigma}_{\geq 0}$ denote the gluing family (2.8.103) associated to the data $(M_0^{\mathrm{pre}}, \sigma)$ defined above, and let us construct a pullback diagram of the desired shape.

$$\begin{tikzcd} Q \arrow[r] \arrow[d] & M \arrow[d] \\ B \arrow[r, "\lambda"] & {}'\mathbb{R}^{\pi_0 N/\sigma}_{\geq 0} \end{tikzcd} \tag{2.8.104.5}$$

We have already seen how to construct such pullback diagrams locally on Q, so we just need to globalize. Each component of $\pi_0 N/\sigma$ (i.e. the singular locus of Q_b) gives a distinguished ray in $\mathcal{Z}_{B,b}$, and we fix any bottom map λ inducing the same rays. To define a lift $Q \to M$ in a neighborhood of the singular locus of Q_b, we should first construct functions $x, y : Q \to {}'\mathbb{R}_{\geq 0}$

satisfying $\lambda = xy$ (of course, really (x, y) is an *unordered* pair of functions indexed by the two local branches of $M_0^{\mathrm{pre}} \subseteq \tilde{Q}_b$, but we will stick with the abuse of notation for simplicity). We can certainly fix functions $x, y : Q \to {}'\mathbb{R}_{\geq 0}$ in a neighborhood of the singular locus of Q_b whose classes lie in the two relevant distinguished rays in $\mathcal{Z}_{Q,q}$ for singular points q, and as before we have $\lambda = e^f x^a y^b$ for locally constant $a, b : Q \to \mathbb{R}_{>0}$ and smooth $f : Q \to \mathbb{R}$, so replacing (x, y) with $(e^{f/2}x^a, e^{f/2}y^b)$ we may achieve $\lambda = xy$ on Q. A choice of local retraction $Q \to N/\sigma$ (construct it locally and then average (2.6.29)) completes the data of a lift $Q \to M$ near the singular locus of Q_b. Finally, to extend the lift $Q \to M$ to a neighborhood the rest of Q_b, we patch together local retractions $Q \to M_0^{\mathrm{pre}}$ as in the proof of Ehresmann for smooth manifolds (2.6.30). As we already saw above, the resulting diagram is a pullback square by the inverse function theorem. □

2.8.105 Exercise. Let $Q \to B$ be a proper simply-broken submersion, and let $B \to X_P$ be strict. Show that $Q \to B$ is, locally on the target, the pullback of a standard gluing family along a monomial map $X_P \to {}'\mathbb{R}^n_{\geq 0}$ (note that in the above construction of such pullbacks (2.8.104.5), the map $\lambda : B \to {}'\mathbb{R}^{\pi_0 N/\sigma}_{\geq 0}$ just needs to induce the correct rays in $\mathcal{Z}_{B,b}$).

2.8.106 Definition (Gluing coordinates with vector bundles)**.** The gluing construction (2.8.103) may be enhanced to carry along vector bundles. Recall that the input to the gluing construction is a log smooth manifold M_0^{pre} of depth one (let $N = (M_0^{\mathrm{pre}})^{\mathrm{id}}$ denote its ideal locus, in this case a smooth manifold), a collar $N \times {}'\mathbb{R}_{\geq 0} \hookrightarrow M_0^{\mathrm{pre}}$, and a free involution $\sigma : N \to N$. The output is a family $M \to {}'\mathbb{R}^{\pi_0 N/\sigma}_{\geq 0}$. Now enhance everything with a vector bundle: fix a vector bundle $V_0^{\mathrm{pre}} \to M_0^{\mathrm{pre}}$ (its restriction to the ideal locus denoted $W \to N$), an isomorphism $W = V_0^{\mathrm{pre}}$ over the collar, and a lift of the involution σ to W. Such data evidently gives rise to a vector bundle $V \to M$.

2.9 Topological sites

In (2.3), we studied sheaves on the category of topological spaces. Later, we will want to consider sheaves on other similar categories (e.g. smooth manifolds) which share a common fundamental structure: their objects are topological spaces X equipped with some extra structure $\mathcal{S}_X$ of a local nature, and their morphisms are maps of underlying topological spaces $f : X \to Y$ together with some correspondence between $\mathcal{S}_X$ and $\mathcal{S}_Y$ over f, also of a local nature. We formulate the notion of a *perfect topological site* which is a precise axiomatization of this idea. It makes sense to consider sheaves on any such category, and we show how to carry over elements of the theory of topological stacks (2.3) to this setting. The reader who desires even more abstraction is referred to the notion of a Grothendieck site from [1, Exposé II] and the notion of a 'geometry' from Lurie [57] (neither of which we will need here).

* *

Any functor $|\cdot| : \mathsf{C} \to \mathsf{Top}$ may be regarded as specifying an 'underlying topological space' for each object of C (and an 'underlying continuous map' for every morphism). Such data is not particularly useful without additional axioms. We introduce the relevant axioms one by one.

⋆ **2.9.1 Definition** (Open embedding)**.** Let C be an ∞-category equipped with a functor $|\cdot| : \mathsf{C} \to \mathsf{Top}$. An *open embedding* in C is a morphism which is cartesian (**??**) over an open embedding in Top. An *open covering* of $X \in \mathsf{C}$ is a collection of open embeddings into X which after applying $|\cdot|$ becomes an open covering of $|X|$.

In other words, a morphism $U \to X$ in C is an open embedding when $|U| \to |X|$ is an open embedding and U represents the functor of maps to X which upon applying $|\cdot|$ factor through $|U| \subseteq |X|$. Open embeddings are closed under composition (since cartesian morphisms are closed under composition (**??**) and open embeddings in Top are closed under composition).

⋆ **2.9.2 Definition** (Topological site)**.** A *topological site* is a pair $(\mathsf{C}, |\cdot| : \mathsf{C} \to \mathsf{Top})$ which *has enough open embeddings*, meaning that for every $X \in \mathsf{C}$, every open subset of $|X|$ is realized by an open embedding $U \to X$ in C (in other words, every map $(\Delta^1, 1) \to (\mathsf{Top}, \mathsf{C})$ whose underlying morphism in Top is an open embedding has a cartesian lift).

In any topological ∞-site C, the functor

$$(\mathsf{C} \downarrow^{\mathsf{opemb}} X) \to (\mathsf{Top} \downarrow^{\mathsf{opemb}} |X|) = \mathsf{Open}(|X|) \tag{2.9.2.1}$$

is an equivalence (**??**) since $\mathsf{C}^{\mathsf{opemb}} \to \mathsf{Top}^{\mathsf{opemb}}$ is cartesian.

2.9.3 Exercise. Show that the following are topological sites.

(2.9.3.1) The category Top of topological spaces with $|\cdot|$ the identity functor.

(2.9.3.2) The category of open subsets of any fixed topological space.

(2.9.3.3) The category Sm of smooth manifolds with the underlying topological space functor.

(2.9.3.4) The category of pairs (X, ω_X) where $X \in \mathsf{Sm}$ and ω_X a closed 3-form on X and morphisms $(X, \omega_X) \to (Y, \omega_Y)$ given by smooth maps $f : X \to Y$ satisfying $f^*\omega_Y = \omega_X$.

(2.9.3.5) The category $\mathsf{Vect} \rtimes \mathsf{Top}$ whose objects are pairs (X, V) where X is a topological space and $V \to X$ is a vector bundle, and in which a morphism $(X, V) \to (Y, W)$ is a continuous map $X \to Y$ covered by a map of vector bundles $V \to W$ (with underlying topological space functor $(X, V) \mapsto X$).

(2.9.3.6) The arrow category $\mathsf{Fun}(\Delta^1, \mathsf{Top})$ with the functor $\mathsf{Fun}(\Delta^1, \mathsf{Top}) \to \mathsf{Top}$ sending an arrow to its target (that is, $(X \to Y) \mapsto Y$).

(2.9.3.7) The category $\mathsf{Top}_{/X}$ of topological spaces over a fixed topological space X.

(2.9.3.8) The category of sets.

(2.9.3.9) The category whose objects are pairs $(I, \{X_i\}_{i \in I})$ where I is a set and X_i is a pointed topological space for every $i \in I$ and whose morphisms $(I, \{X_i\}_{i \in I}) \to (J, \{Y_j\}_{j \in J})$ are maps $f : I \to J$ with finite fibers together with pointed maps $\prod_{f(i)=j} X_i \to Y_j$ for every $j \in J$ (with underlying topological space functor $(I, \{X_i\}_{i \in I}) \mapsto I$).

(2.9.3.10) The category of schemes.

(2.9.3.11) The ∞-category $\mathsf{C}^{\mathsf{opemb}}$ for any topological ∞-site C.

(2.9.3.12) The ∞-category $\mathsf{C}^{\triangleleft}$ for any ∞-category C with underlying topological space functor sending the cone point to $\varnothing$ and sending all objects of C to $*$.

(2.9.3.13) Any full subcategory $\mathsf{C}^- \subseteq \mathsf{C}$ of a topological ∞-site C with the property that if $X \in \mathsf{C}^-$ and $U \to X$ is an open embedding in C, then $U \in \mathsf{C}^-$.

(2.9.3.14) An ∞-category E with a cartesian fibration $\mathsf{E} \to \mathsf{C}$ where C is a topological ∞-site (more generally, it is enough to assume that every map $(\Delta^1, 1) \to (\mathsf{C}, \mathsf{E})$ whose underlying morphism in C is an open embedding has a cartesian lift). For example, this applies to $\mathsf{P}(-)^{\mathsf{op}} \rtimes \mathsf{Top}$ and $\mathsf{Shv}(-)^{\mathsf{op}} \rtimes \mathsf{Top}$ (2.2.15). Which of the above examples are special cases of this?

2.9.4 Lemma. *Let C be an ∞-category with a functor $|\cdot| : \mathsf{C} \to \mathsf{Top}$. Consider a square*

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & Y' \arrow[d] \\ X \arrow[r] & Y \end{tikzcd} \tag{2.9.4.1}$$

in C whose bottom arrow is an open embedding and whose image in Top is a pullback. In this case, the diagram (2.9.4.1) *is a pullback iff $X' \to Y'$ is an open embedding.*

Proof. This is simply a special case of (1.5.85). □

2.9.5 Lemma. *Open embeddings in a topological ∞-site are preserved under pullback, and $|\cdot|$ sends pullbacks of open embeddings in C to pullbacks of open embeddings in Top.*

Proof. While (1.5.86) does not apply directly since the functor $|\cdot| : \mathsf{C} \to \mathsf{Top}$ is not cartesian, its proof applies without change (the only cartesian lifting problems encountered are over open embeddings in Top). □

2.9.6 Exercise. Conclude from the description of pullbacks of open embeddings (2.9.5) that open coverings are preserved under pullback in any topological ∞-site.

2.9.7 Exercise. Conclude from (2.9.5) that a fiber product of open embeddings $U, V \to X$ in a topological ∞-site is the open embedding corresponding to the intersection $|U| \cap |V| \subseteq |X|$.

2.9.8 Exercise. Consider the cartesian fibration $\mathsf{Open} \rtimes \mathsf{Top} \to \mathsf{Top}$ where $\mathsf{Open} \rtimes \mathsf{Top}$ is the full subcategory of $\mathsf{Fun}(\Delta^1, \mathsf{Top})$ spanned by open embeddings and the map to Top is evaluation at $1 \in \Delta^1$. This cartesian fibration encodes the functor $\mathsf{Open} : \mathsf{Top}^{\mathsf{op}} \to \mathsf{Po} \subseteq \mathsf{Cat}$ (where Po is partially ordered sets (1.1.31)).

Now let C be a topological ∞-site, and consider the pullback $\mathsf{Open}(|-|) \rtimes \mathsf{C} = (\mathsf{Open} \rtimes \mathsf{Top}) \times_{\mathsf{Top}} \mathsf{C}$. There is an evident forgetful functor from the full subcategory of $\mathsf{Fun}(\Delta^1, \mathsf{C})$ spanned by open embeddings to this pullback. Show that this forgetful functor is a trivial Kan fibration by lifting $(\Delta^1, 1)^{\#} \wedge (\Delta^k, \partial\Delta^k)$ against $(\mathsf{C}, \mathsf{opemb}) \to (\mathsf{Top}, \mathsf{opemb})$. Conclude that $\mathsf{Fun}_{\mathsf{opemb}}(\Delta^1, \mathsf{C}) \to \mathsf{C}$ (evaluate at $1 \in \Delta^1$) is a cartesian fibration encoding the functor $\mathsf{Open}(|\cdot|) : \mathsf{C}^{\mathsf{op}} \to \mathsf{Po}$.

Many notions and constructions in the context of topological spaces depend only on the notions of open embeddings and open coverings, hence make sense in any topological ∞-site. For example, a morphism $X \to Y$ in a topological ∞-site is called a local isomorphism (2.1.4) when there exists an open cover $X = \bigcup_i U_i$ such that each composition $U_i \to X \to Y$ is an open embedding. Of central importance is the notion of a sheaf: a presheaf on a topological ∞-site C is called a sheaf when it sends open coverings to limits (equivalently, when its pullback to $\mathsf{Open}(|X|) = (\mathsf{C} \downarrow^{\mathsf{opemb}} X)$ (2.9.2.1) is a sheaf on $|X|$ for every $X \in \mathsf{C}$).

On the other hand, the axioms of a topological ∞-site *do not* guarantee that morphisms are of a local nature. Rather, this is an additional (very important) property called being 'subcanonical'. While most topological ∞-sites of interest are subcanonical, various key foundational constructions will involve non-subcanonical topological ∞-sites in an important way.

⋆ **2.9.9 Definition** (Subcanonical)**.** A topological site C is called *subcanonical* when every Yoneda presheaf $\mathsf{C}(-, X) \in \mathsf{P}(\mathsf{C})$ is a sheaf (equivalently, when open coverings are colimits (**??**)).

2.9.10 Exercise. Which of the topological sites in (2.9.3) are subcanonical?

2.9.11 Exercise. Show that a morphism $X \to Y$ in a subcanonical topological ∞-site is an isomorphism iff it is an isomorphism locally on the target.

2.9.12 Exercise (Coproducts in a subcanonical topological ∞-site)**.** Let X be an object of a subcanonical topological ∞-site C. Let $X = \bigcup_i U_i$ be a cover by open embeddings. Show that if the $|U_i|$ are disjoint, then $X = \bigsqcup_i U_i$ is their coproduct in C. In particular, if $|X| = \varnothing$, then X is an initial object of C.

2.9.13 Exercise. Show that every subcanonical topological site for which the essential image of $|\cdot|$ is $\{\varnothing, *\} \subseteq \mathsf{Top}$ is of the form (2.9.3.12).

2.9.14 Exercise. Let $D : K \to \mathsf{C}$ be a diagram in a subcanonical topological ∞-site C and suppose that $\lim_K |D| = \varnothing$. Show that the limit $\lim_K D = \varnothing$ (the initial object).

* *

⋆ **2.9.15 Definition** (Topological functor)**.** Let $(\mathsf{C}, |\cdot|_{\mathsf{C}})$ and $(\mathsf{D}, |\cdot|_{\mathsf{D}})$ be topological sites. A *topological functor* $(\mathsf{C}, |\cdot|_{\mathsf{C}}) \to (\mathsf{D}, |\cdot|_{\mathsf{D}})$ is a functor $f : \mathsf{C} \to \mathsf{D}$ preserving open embeddings, together with a natural transformation $\pi : |f(\cdot)|_{\mathsf{D}} \to |\cdot|_{\mathsf{C}}$ which sends open embeddings to pullbacks.

$$\begin{tikzcd} \mathsf{C} \arrow[r, "\times 0"] \arrow[d, "f"'] & \mathsf{C} \times \Delta^1 \arrow[d, "\pi"'] & \mathsf{C} \arrow[l, "\times 1"'] \arrow[dl, "|\cdot|_{\mathsf{C}}"] \\ \mathsf{D} \arrow[r, "|\cdot|_{\mathsf{D}}"] & \mathsf{Top} & \end{tikzcd} \tag{2.9.15.1}$$

A topological functor (f, π) is called *strict* when π is a natural isomorphism. Topological functors from C to D form an ∞-category denoted $\mathsf{Top}(\mathsf{C}, \mathsf{D})$, namely the full subcategory $\mathsf{Fun}(\mathsf{C}, \mathsf{D})_{(|\cdot|_{\mathsf{D}} \circ -)/|\cdot|_{\mathsf{C}}}$ spanned by those pairs (f, π) for which f preserves open embeddings and π sends open embeddings to pullbacks.

2.9.16 Exercise. Show that the following are topological functors.

(2.9.16.1) The forgetful functor $\mathsf{Sm} \to \mathsf{Top}$.

(2.9.16.2) The functor $|\cdot| : \mathsf{C} \to \mathsf{Top}$ for any topological site C.

(2.9.16.3) The functor $\mathsf{Vect} \rtimes \mathsf{Top} \to \mathsf{Top}$ sending (X, V) to the total space of V.

(2.9.16.4) The forgetful functor $\mathsf{Sm}^n \to \mathsf{Sm}^m$ for $n \geq m$, where Sm^k denotes the category of C^k-manifolds (i.e. with transition maps of class C^k rather than smooth).

(2.9.16.5) The inverse image functor $f^{-1} = \mathsf{Open}(f) : \mathsf{Open}(Y) \to \mathsf{Open}(X)$ associated to a continuous map of topological spaces $f : X \to Y$.

(2.9.16.6) The functor $\mathsf{Top} \to \mathsf{Top}$ given by sending a topological space to its underlying set equipped with the discrete topology.

(2.9.16.7) The functor $\mathsf{Top} \to \mathsf{Top}$ given by $X \mapsto X \times A$ (any fixed topological space A).

(2.9.16.8) The functor $\mathsf{Sm} \to \mathsf{Sm}$ given by $X \mapsto TX$ (the tangent bundle).

(2.9.16.9) The functor $\mathsf{Sm}^{\mathsf{lociso}} \to \mathsf{Sm}^{\mathsf{lociso}}$ given by sending a smooth manifold to its frame bundle.

2.9.17 Exercise. Show that a topological functor preserves pullbacks of open embeddings.

2.9.18 Exercise. Show that a natural transformation of topological functors $f \to g$ sends open embeddings to pullbacks.

2.9.19 Exercise. Show that a topological functor preserves open coverings, and hence that presheaf pullback along a topological functor sends sheaves to sheaves.

2.9.20 Exercise. Let $f : \mathsf{C} \to \mathsf{D}$ be a strict topological functor. Show that the essential image of f is a topological site and that the functors $\mathsf{C} \to \mathrm{im}(f) \to \mathsf{D}$ are both strict topological.

2.9.21 Exercise. Let $(f,\pi):(\mathsf{C},|\cdot|_{\mathsf{C}})\to(\mathsf{D},|\cdot|_{\mathsf{D}})$ be a topological functor. Since f preserves open embeddings, it restricts to a functor $(\mathsf{C}\downarrow^{\mathsf{opemb}}X)\to(\mathsf{D}\downarrow^{\mathsf{opemb}}f(X))$. Show that under the identifications $(\mathsf{C}\downarrow^{\mathsf{opemb}}X)=\mathsf{Open}(|X|)$ and $(\mathsf{D}\downarrow^{\mathsf{opemb}}f(X))=\mathsf{Open}(|f(X)|)$, this functor is canonically identifed with $\mathsf{Open}(\pi_X)$.

2.9.22 Lemma (Deducing a universal property in the realm of topological ∞-sites from a universal property in the realm of ∞-categories)**.** *Let $i:\mathsf{C}\to\overline{\mathsf{C}}$ be a strict topological functor, and let E be a topological ∞-site. Let α and β (resp. $\overline{\alpha}$ and $\overline{\beta}$) be conditions on functors from C (resp. $\overline{\mathsf{C}}$) to E and Top (respectively). Suppose (as summarized in* (2.9.22.1)*) that composition with $|\cdot|_{\mathsf{E}}$ sends α (resp. $\overline{\alpha}$) functors to β (resp. $\overline{\beta}$) functors, the functors $|\cdot|_{\mathsf{C}}$ and $|\cdot|_{\overline{\mathsf{C}}}$ satisfy β and $\overline{\beta}$ (respectively), that $i^*:\mathsf{Fun}_{\overline{\alpha}}(\overline{\mathsf{C}},E)\to\mathsf{Fun}_{\alpha}(\mathsf{C},\mathsf{E})$ is an equivalence, and that $i^*:\mathsf{Fun}_{\overline{\alpha}}(\overline{\mathsf{C}},\mathsf{Top})\to\mathsf{Fun}_{\alpha}(\mathsf{C},\mathsf{Top})$ has a right adjoint i_* for which the unit map $|\cdot|_{\overline{\mathsf{C}}}\to i_*i^*|\cdot|_{\overline{\mathsf{C}}}=i_*|\cdot|_{\mathsf{C}}$ is an isomorphism.*

$$\begin{tikzcd}
\mathsf{Fun}_{\overline{\alpha}}(\overline{\mathsf{C}},\mathsf{E}) \arrow[r, "|\cdot|_{\mathsf{E}}\circ-"] \arrow[d, "i^*", "\sim"'] & \mathsf{Fun}_{\overline{\beta}}(\overline{\mathsf{C}},\mathsf{Top}) \ni |\cdot|_{\overline{\mathsf{C}}} \arrow[d, "i^*"] \\
\mathsf{Fun}_{\alpha}(\mathsf{C},\mathsf{E}) \arrow[r, "|\cdot|_{\mathsf{E}}\circ-"] & \mathsf{Fun}_{\beta}(\mathsf{C},\mathsf{Top}) \ni |\cdot|_{\mathsf{C}} \arrow[u, "i_*"', shift right=6]
\end{tikzcd} \tag{2.9.22.1}$$

In this case, the induced pullback map

$$i^*:\mathsf{Fun}_{\overline{\alpha}}(\overline{\mathsf{C}},\mathsf{E})_{(|\cdot|_{\mathsf{E}}\circ-)/|\cdot|_{\overline{\mathsf{C}}}}\xrightarrow{\sim}\mathsf{Fun}_{\alpha}(\mathsf{C},\mathsf{E})_{(|\cdot|_{\mathsf{E}}\circ-)/|\cdot|_{\mathsf{C}}} \tag{2.9.22.2}$$

is an equivalence. Now suppose the following additional condition is satisfied:

(2.9.22.3) *For all $f:\overline{\mathsf{C}}\to\mathsf{E}$ satisfying $\overline{\alpha}$ and all $\pi:|\cdot|_{\mathsf{E}}\circ f\to|\cdot|_{\overline{\mathsf{C}}}$, if f and π send open embeddings in C to open embeddings and pullbacks (respectively), then they do the same for open embeddings in $\overline{\mathsf{C}}$.*

In this case, pullback under i induces an equivalence on ∞-categories of topological functors $\mathsf{Top}_{\overline{\alpha}}(\overline{\mathsf{C}},\mathsf{E})\to\mathsf{Top}_{\alpha}(\mathsf{C},\mathsf{E})$ satisfying $\overline{\alpha}$ and α.

Proof. To upgrade the equivalence $i^*:\mathsf{Fun}_{\overline{\alpha}}(\overline{\mathsf{C}},E)\to\mathsf{Fun}_{\alpha}(\mathsf{C},\mathsf{E})$ to the equivalence of slice categories (2.9.22.2), it suffices to show that for $f:\overline{\mathsf{C}}\to\mathsf{E}$ satisfying $\overline{\alpha}$, the tautological map

$$\operatorname{Hom}_{\mathsf{Fun}_{\overline{\beta}}(\overline{\mathsf{C}},\mathsf{Top})}(|f(\cdot)|_{\mathsf{E}},|\cdot|_{\overline{\mathsf{C}}})\to\operatorname{Hom}_{\mathsf{Fun}_{\beta}(\mathsf{C},\mathsf{Top})}(|f(\cdot)|_{\mathsf{E}},|\cdot|_{\mathsf{C}}) \tag{2.9.22.4}$$

is a homotopy equivalence. By the adjunction (i^*,i_*), the right side may also be written as $\operatorname{Hom}_{\mathsf{Fun}_{\overline{\beta}}(\overline{\mathsf{C}},\mathsf{Top})}(|f(\cdot)|_{\mathsf{E}},i_*|\cdot|_{\mathsf{C}})$, and the map is then identified with composition with the unit map $|\cdot|_{\overline{\mathsf{C}}}\to i_*i^*|\cdot|_{\overline{\mathsf{C}}}=i_*|\cdot|_{\mathsf{C}}$, which is an isomorphism by hypothesis.

To conclude the equivalence of ∞-categories of topological functors $\mathsf{Top}_{\overline{\alpha}}(\overline{\mathsf{C}},\mathsf{E})\to\mathsf{Top}_{\alpha}(\mathsf{C},\mathsf{E})$, we note that these are full subcategories of the domain and target of (2.9.22.2), and the assertion that they coincide under i^* is precisely the condition (2.9.22.3). $\square$

* *

We have already seen how the category of presheaves $\mathsf{P}(\mathsf{C})$ is a useful enlargement of a category C. We now explore this construction in the case that C is a topological site.

2.9.23 Definition. Let C be a topological ∞-site. We equip the ∞-category of presheaves $\mathsf{P}(\mathsf{C})$ with the functor $|\cdot|_{\mathsf{P}(\mathsf{C})}$ defined as the composition

$$\mathsf{P}(\mathsf{C}) \xrightarrow{|\cdot|_{\mathsf{C}!}} \mathsf{P}(\mathsf{Top}) \xrightarrow{\mathrm{colim}} \mathsf{Top} \tag{2.9.23.1}$$

of left Kan extension along $|\cdot|_{\mathsf{C}}$ and the colimit functor. Thus $|\cdot|_{\mathsf{P}(\mathsf{C})} : \mathsf{P}(\mathsf{C}) \to \mathsf{Top}$ is left adjoint to the composition $|\cdot|_{\mathsf{C}}^* \circ \mathcal{Y}_{\mathsf{Top}} : \mathsf{Top} \to \mathsf{P}(\mathsf{C})$ which we will usually abbreviate as $|\cdot|_{\mathsf{C}}^*$.

The following technical characterization of open embeddings in $\mathsf{P}(\mathsf{C})$ will be very useful.

2.9.24 Lemma. *Let C be a topological ∞-site. For a morphism $F \to G$ in $\mathsf{P}(\mathsf{C})$, the following are equivalent:*

(2.9.24.1) *$F \to G$ is an open embedding with respect to $|\cdot|_{\mathsf{P}(\mathsf{C})}$ in the sense of* (2.9.1).

(2.9.24.2) *$|F|_{\mathsf{P}(\mathsf{C})} \to |G|_{\mathsf{P}(\mathsf{C})}$ is an open embedding and the unit map from $F \to G$ to $|\cdot|_{\mathsf{C}}^*(|F|_{\mathsf{P}(\mathsf{C})} \to |G|_{\mathsf{P}(\mathsf{C})})$ is a pullback.*

(2.9.24.3) *$F \to G$ is a pullback of $|\cdot|_{\mathsf{C}}^*$ of an open embedding in* Top.

(2.9.24.4) *$F \to G$ is an open embedding in the sense of* (1.1.92) *(the pullback $F \times_G c \to c$ is an open embedding in C for every $c \in \mathsf{C}$ and every map $c \to G$).*

(2.9.24.5) *$F \to G$ is a colimit in $\mathsf{Fun}(\Delta^1, \mathsf{P}(\mathsf{C}))$ of a diagram $K \to \mathsf{Fun}(\Delta^1, \mathsf{C})$ which sends vertices to open embeddings and sends edges to pullbacks.*

Proof. The equivalence (2.9.24.1)⇔(2.9.24.2) is a general categorical fact (1.5.84) (using the fact that $|\cdot|_{\mathsf{C}}^* : \mathsf{Top} \to \mathsf{P}(\mathsf{C})$ is right adjoint to $|\cdot|_{\mathsf{P}(\mathsf{C})}$). Certainly (2.9.24.2)⇒(2.9.24.3).

Let us show (2.9.24.3) $\Longrightarrow$ (2.9.24.4). Property (2.9.24.3) is certainly preserved under pullback, so it suffices to show that if G is representable and $F \to G$ is a pullback of $|\cdot|_{\mathsf{C}}^*$ of an open embedding in Top, then $F \to G$ is an open embedding in C. This holds since C has enough open embeddings.

Let us show (2.9.24.4) $\Longrightarrow$ (2.9.24.5). Write G as the colimit $G = \mathrm{colim}_K\, p$ in $\mathsf{P}(\mathsf{C})$ of a diagram $p : K \to \mathsf{C}$. Since presheaf pullback is cocontinuous (**??**), we have $F = \mathrm{colim}_K\,(p \times_G F)$. Thus $(F \to G)$ is the colimit in $\mathsf{Fun}(\Delta^1, \mathsf{P}(\mathsf{C}))$ of the diagram $p \times_G (F \to G) : K \to \mathsf{Fun}(\Delta^1, \mathsf{C})$. This diagram sends vertices in K to open embeddings in C by hypothesis, and it sends edges in K to pullbacks by construction.

Let us show (2.9.24.5) $\Longrightarrow$ (2.9.24.2). Suppose $F \to G$ is a colimit in $\mathsf{Fun}(\Delta^1, \mathsf{P}(\mathsf{C}))$ of a diagram $p : K \to \mathsf{Fun}(\Delta^1, \mathsf{C})$ which sends vertices to open embeddings and sends edges to pullbacks. The functor $|\cdot|_{\mathsf{C}}$ preserves open embeddings and pullbacks of open embeddings (2.9.5), so the diagram $|p| : K \to \mathsf{Fun}(\Delta^1, \mathsf{Top})$ has the same property. It follows from the explicit description of colimits of topological spaces that $(|F| \to |G|) = \mathrm{colim}_K\, |p|_{\mathsf{C}}$ is an open embedding of topological spaces and that the map from $|p(k)|$ to this open embedding is a pullback for every vertex $k \in K$. It follows that the map from $p(k)$ to $|\cdot|_{\mathsf{C}}^*|F| \to |\cdot|_{\mathsf{C}}^*|G|$ is a pullback for every $k \in K$. Since presheaf pullback is cocontinuous (**??**), we conclude that the map from $F \to G$ to $|\cdot|_{\mathsf{C}}^*|F| \to |\cdot|_{\mathsf{C}}^*|G|$ is a pullback. □

2.9.25 Corollary. *The ∞-category of presheaves $\mathsf{P}(\mathsf{C})$ with the functor $|\cdot|_{\mathsf{P}(\mathsf{C})}$ is a topological ∞-site.*

Proof. We must show that $\mathsf{P}(\mathsf{C})$ has enough open embeddings. Let $G \in \mathsf{P}(\mathsf{C})$, and write G as the colimit $G = \operatorname{colim}_K p$ in $\mathsf{P}(\mathsf{C})$ of a diagram $p : K \to \mathsf{C}$. By the explicit description of colimits of topological spaces, an open subset of $|G| = \operatorname{colim}_K |p|$ is the same as a choice of open subset of $|p(k)|$ for every $k \in K$, compatible with pullback along every edge of K. Since C has enough open embeddings, we can promote such a collection to a diagram $p : K \to \mathsf{Fun}(\Delta^1, \mathsf{C})$ satisfying (2.9.24.5). The resulting open subset of $|G|$ is the one we started with by construction. $\square$

2.9.26 Exercise (Coarse local isomorphism)**.** The notion of a local isomorphism in C induces a notion of a local isomorphism in $\mathsf{P}(\mathsf{C})$ via pullback (1.1.92). The topological ∞-site structure on $\mathsf{P}(\mathsf{C})$ also gives rise to a notion when a morphism in $\mathsf{P}(\mathsf{C})$ is to be called a local isomorphism (2.1.4), which to distinguish from the former notion we will a *coarse local isomorphism.* Show that a coarse local isomorphism is a local isomorphism, but that the converse need not hold.

2.9.27 Lemma. *The Yoneda functor $\mathsf{C} \to \mathsf{P}(\mathsf{C})$ of any topological ∞-site C is a strict topological functor.*

Proof. There is a tautological isomorphism $|\text{よ}_{\mathsf{C}}(\cdot)|_{\mathsf{P}(\mathsf{C})} = |\cdot|_{\mathsf{C}}$. An open embedding in C remains an open embedding in $\mathsf{P}(\mathsf{C})$ by (2.9.24.5). $\square$

Recall that for any functor $f : \mathsf{C} \to \mathsf{D}$, the presheaf pullback $f^* : \mathsf{P}(\mathsf{D}) \to \mathsf{P}(\mathsf{D})$ has a left adjoint called *left Kan extension* $f_! : \mathsf{P}(\mathsf{C}) \to \mathsf{P}(\mathsf{D})$ (**??**).

2.9.28 Lemma. *Let $f : \mathsf{C} \to \mathsf{D}$ be a topological functor. The left Kan extension functor $f_! : \mathsf{P}(\mathsf{C}) \to \mathsf{P}(\mathsf{D})$ is topological when equipped with the unique transformation $|f_!(\cdot)|_{\mathsf{P}(\mathsf{D})} \to |\cdot|_{\mathsf{P}(\mathsf{C})}$ restricting to the given transformation $|f(\cdot)|_{\mathsf{D}} \to |\cdot|_{\mathsf{C}}$. If f is strict then so is $f_!$.*

Proof. The functors $|\cdot|_{\mathsf{P}(\mathsf{C})}$, $|\cdot|_{\mathsf{P}(\mathsf{D})}$, and $f_!$ are cocontinuous, so by the universal property of presheaf categories (**??**), natural transformations $|f_!(\cdot)|_{\mathsf{P}(\mathsf{D})} \to |\cdot|_{\mathsf{P}(\mathsf{C})}$ are the same as natural transformations $|f(\cdot)|_{\mathsf{D}} \to |\cdot|_{\mathsf{C}}$ (and moreover this correspondence respects isomorphisms).

Now let us show that $f_!$ sends open embeddings to open embeddings and $|f_!(\cdot)|_{\mathsf{P}(\mathsf{D})} \to |\cdot|_{\mathsf{P}(\mathsf{C})}$ sends open embeddings to pullbacks. Write an open embedding $F \to G$ in $\mathsf{P}(\mathsf{C})$ as a colimit of a diagram $p : K \to \mathsf{Fun}(\Delta^1, \mathsf{C})$ of open embeddings in C as in (2.9.24.5). The pushforward $f_!F \to f_!G$ is the colimit of $f(p)$, from which the result follows by inspection. $\square$

2.9.29 Lemma. *Let $f : \mathsf{C} \to \mathsf{D}$ be a strict topological functor. The pulback functor $f^* : \mathsf{P}(\mathsf{D}) \to \mathsf{P}(\mathsf{C})$ is topological when equipped with the natural transformation $|f^*(\cdot)|_{\mathsf{P}(\mathsf{C})} = |f_!f^*(\cdot)|_{\mathsf{P}(\mathsf{D})} \to |\cdot|_{\mathsf{P}(\mathsf{D})}$ induced by the adjunction $(f^*, f_!)$.*

Proof. Let $F \to G$ be an open embedding in $\mathsf{P}(\mathsf{D})$. Every such open embedding is a pullback of $|\cdot|^*_{\mathsf{D}}(U \to X)$ for some open embedding of topological spaces $U \to X$ (2.9.24.3). We have $f^*|\cdot|^*_{\mathsf{D}} = |\cdot|^*_{\mathsf{C}}$ since f is strict, so we may apply f^* (which is continuous) to see that $f^*F \to f^*G$ is a pullback of $|\cdot|^*_{\mathsf{C}}(U \to X)$, hence is an open embedding. Now we saw in (2.9.24) that moreover $|F|_{\mathsf{P}(\mathsf{D})} \to |G|_{\mathsf{P}(\mathsf{D})}$ and $|f^*F|_{\mathsf{P}(\mathsf{C})} \to |f^*G|_{\mathsf{P}(\mathsf{D})}$ are both pullbacks of $U \to X$. By cancellation (1.1.52), this implies $|f^*(\cdot)|_{\mathsf{P}(\mathsf{C})} \to |\cdot|_{\mathsf{P}(\mathsf{D})}$ sends $F \to G$ to a pullback. $\square$

The ∞-category of presheaves $\mathsf{P}(\mathsf{C})$ on an ∞-category C satisfies a purely categorical universal property (**??**). When C is a topological ∞-site, so is $\mathsf{P}(\mathsf{C})$, and it is natural to ask whether the topological functor $\mathsf{C} \hookrightarrow \mathsf{P}(\mathsf{C})$ satisfies a universal property which characterizes $\mathsf{P}(\mathsf{C})$ uniquely as a topological ∞-site. Let us now deduce such a universal property.

2.9.30 Exercise. Let C be a topological ∞-site. Show that the colimit of any diagram $K \to \mathsf{Fun}(\Delta^1, \mathsf{P}(\mathsf{C}))$ sending vertices to open embeddings and edges to pullbacks is an open embedding.

2.9.31 Proposition (Universal property of presheaves on a topological ∞-site)**.** *Let C be a topological ∞-site. Let E be a cocomplete topological ∞-site for which $|\cdot|_{\mathsf{E}}$ is cocontinuous and for which the colimit of any diagram $K \to \mathsf{Fun}(\Delta^1, \mathsf{E})$ sending vertices to open embeddings and edges to pullbacks is an open embedding. Pullback along the strict topological functor $\mathsf{C} \to \mathsf{P}(\mathsf{C})$ induces an equivalence between the ∞-categories of cocontinuous topological functors $\mathsf{P}(\mathsf{C}) \to \mathsf{E}$ and topological functors $\mathsf{C} \to \mathsf{E}$.*

Proof. We apply (2.9.22).

Properties α and β are vacuous, and properties $\overline{\alpha}$ and $\overline{\beta}$ are cocontinuous. Pullback i^* is an equivalence by the category theoretic universal property of $\mathsf{C} \to \mathsf{P}(\mathsf{C})$ (**??**). The functor $|\cdot|_{\mathsf{P}(\mathsf{C})}$ is cocontinuous by definition, and composition with $|\cdot|_{\mathsf{E}}$ preserves cocontinuity since $|\cdot|_{\mathsf{E}}$ is assumed cocontinuous. Since i^* is an equivalence, it has the required right adjoint and the unit map is an isomorphism.

It remains to verify (2.9.22.3). Fix $f : \mathsf{P}(\mathsf{C}) \to \mathsf{E}$ cocontinuous and $\pi : |\cdot|_{\mathsf{E}} \circ f \to |\cdot|_{\overline{\mathsf{C}}}$, and suppose f and π send open embeddings in C to open embeddings and pullbacks (respectively). Express an open embedding $F \to G$ in $\mathsf{P}(\mathsf{C})$ as the colimit of a diagram $K \to \mathsf{Fun}(\Delta^1, \mathsf{C})$ sending vertices to open embeddings and edges to pullbacks (2.9.24.5). Since f is cocontinuous, the map $f(F \to G)$ is the colimit of the diagram $K \to \mathsf{Fun}(\Delta^1, \mathsf{E})$ obtained by composing with f. Since $(f, \pi)|_{\mathsf{C}}$ is a topological functor, this composed diagram also sends vertices to open embeddings and edges to pullbacks, hence its colimit is an open embedding by hypothesis on E. The square $\pi(F \to G)$ is a pullback by inspection (using the fact that π sends open embeddings in C to pullbacks and the explicit description of colimits of topological spaces). □

* *

We now establish some basic properties of the ∞-category of sheaves $\mathsf{Shv}(\mathsf{C}) \subseteq \mathsf{P}(\mathsf{C})$ on a topological ∞-site C.

⋆ **2.9.32 Proposition** (Universal property of sheaves on a topological ∞-site)**.** *Let C be a topological ∞-site. For any cocomplete ∞-category E, pullback along the functors $\mathsf{C} \xrightarrow{\text{よ}_{\mathsf{C}}} \mathsf{P}(\mathsf{C}) \xrightarrow{\#} \mathsf{Shv}(\mathsf{C})$ defines equivalences between the following ∞-categories of functors:*

(2.9.32.1) *Cocontinuous functors $\mathsf{Shv}(\mathsf{C}) \to \mathsf{E}$.*

(2.9.32.2) *Cocontinuous functors $\mathsf{P}(\mathsf{C}) \to \mathsf{E}$ which send sheafifications to isomorphisms.*

(2.9.32.3) *Cocontinuous functors $\mathsf{P}(\mathsf{C}) \to \mathsf{E}$ which send Čech nerves $N(X, \{U_i\}_i) \to X$ to isomorphisms.*

(2.9.32.4) *Cosheaves* $\mathsf{C} \to \mathsf{E}$.

Proof. The reasoning given for the case $\mathsf{C} = \mathsf{Top}$ (2.3.2) applies without change. $\square$

* *

2.9.33 Definition. Let C be a topological ∞-site. We equip the ∞-category of sheaves $\mathsf{Shv}(\mathsf{C})$ with the functor $|\cdot|_{\mathsf{Shv}(\mathsf{C})}$ defined as the restriction of $|\cdot|_{\mathsf{P}(\mathsf{C})}$ to the full subcategory $\mathsf{Shv}(\mathsf{C}) \subseteq \mathsf{P}(\mathsf{C})$. Note that $|\cdot|_{\mathsf{Shv}(\mathsf{C})}$ is cocontinuous since $|\cdot|_{\mathsf{P}(\mathsf{C})}$ is cocontinuous and $|\cdot|_{\mathsf{C}}$ sends open coverings to colimits (2.9.32).

⋆ **2.9.34 Lemma.** *Let* C *be a topological* ∞*-site. The* ∞*-category of sheaves* $\mathsf{Shv}(\mathsf{C})$ *with the functor* $|\cdot|_{\mathsf{Shv}(\mathsf{C})}$ *is a topological* ∞*-site, and the adjoint pair* $i : \mathsf{Shv}(\mathsf{C}) \rightleftarrows \mathsf{P}(\mathsf{C}) : \#$ *are strict topological functors.*

Proof. If $G \in \mathsf{Shv}(\mathsf{C})$ and $F \to G$ is an open embedding in $\mathsf{P}(\mathsf{C})$, then F is a sheaf. Indeed, we saw in (2.9.25) that such an open embedding is a pullback of $|\cdot|_{\mathsf{C}}^*(U \to X)$ for some open embedding of topological spaces $U \to X$, and this expresses F as a fiber product of sheaves $|\cdot|_{\mathsf{C}}^* U \times_{|\cdot|_{\mathsf{C}}^* X} G$ which is thus itself a sheaf (**??**). It follows that $\mathsf{Shv}(\mathsf{C}) \subseteq \mathsf{P}(\mathsf{C})$ is a topological ∞-site and that its inclusion functor i is a strict topological functor.

Now let us show that sheafification $\#$ is topological. If $F \to G$ is an open embedding in $\mathsf{P}(\mathsf{C})$, then it is a pullback of $|\cdot|_{\mathsf{C}}^*(U \to X)$ for some open embedding of topological spaces $U \to X$. Sheafification preserves finite limits (**??**), so $F^\# \to G^\#$ is also a pullback of $|\cdot|_{\mathsf{C}}^*(U \to X)$, hence is also an open embedding. Thus sheafification preserves open embeddings. Finally, we should show that the canonical map $|\cdot|_{\mathsf{P}(\mathsf{C})} \to |\#(\cdot)|_{\mathsf{Shv}(\mathsf{C})}$ arising from the sheafification adjunction $(\#, i)$ is an isomorphism. That is, we should show that $|\cdot|_{\mathsf{P}(\mathsf{C})}$ sends sheafifications to isomorphisms. By (2.9.32), this is equivalent to $|\cdot|_{\mathsf{C}} : \mathsf{C} \to \mathsf{Top}$ being a cosheaf. Now $|\cdot|_{\mathsf{C}}$ sends open coverings to open coverings, and open coverings in Top are colimits. $\square$

⋆ **2.9.35 Definition** (Sheaf left Kan extension)**.** Let $f : \mathsf{C} \to \mathsf{D}$ be a topological functor. Since presheaf pullback f^* sends sheaves to sheaves (2.9.19) and sheaves are a reflective subcategory of presheaves (2.2.13), it follows that the adjunction $(f_!, f^*)$ of functors $f_! : \mathsf{P}(\mathsf{C}) \rightleftarrows \mathsf{P}(\mathsf{D}) : f^*$ descends to the reflective subcategories of sheaves (1.1.87), producing an adjunction $(f_!, f^*)$ of functors $f_! : \mathsf{Shv}(\mathsf{C}) \rightleftarrows \mathsf{Shv}(\mathsf{D}) : f^*$. Explicitly, sheaf pullback f^* is simply presheaf pullback restricted to sheaves, and sheaf left Kan extension $f_!$ is presheaf left Kan extension followed by sheafification (note that this notation is somewhat hazardous, as sheaf pushforward $f_!$ does not coincide with the restriction of presheaf pushforward $f_!$ to sheaves). Sheaf pushforward and presheaf pushforward are related by the following commuting diagram.

$$\begin{tikzcd} \mathsf{C} \arrow[r, "\mathcal{Y}_{\mathsf{C}}"] \arrow[d, "f"'] & \mathsf{P}(\mathsf{C}) \arrow[r, "\#"] \arrow[d, "f_!"] & \mathsf{Shv}(\mathsf{C}) \arrow[d, "f_!"] \\ \mathsf{C} \arrow[r, "\mathcal{Y}_{\mathsf{D}}"] & \mathsf{P}(\mathsf{D}) \arrow[r, "\#"] & \mathsf{Shv}(\mathsf{D}) \end{tikzcd} \tag{2.9.35.1}$$

Sheaf pushforward $f_!$ is a topological functor by (2.9.28)(2.9.34). When f is strict, sheaf pushforward $f_!$ is strict and sheaf pullback f^* is also a topological functor (2.9.29).

2.9.36 Exercise. Explain why f^* sending sheaves to sheaves implies that the right square in (2.9.35.1) commutes.

2.9.37 Lemma. *If $f : \mathsf{C} \to \mathsf{D}$ is strict, then presheaf pullback f^* commutes with sheafification (i.e. sends sheafifications to sheafifications).*

Proof. For any $X \in \mathsf{C}$, consider the functor $\mathsf{Open}(|X|) \to \mathsf{C}$ and its composition with $\mathsf{C} \to \mathsf{D}$. Pullback under $\mathsf{Open}(|X|) \to \mathsf{C}$ and the composition $\mathsf{Open}(|X|) \to \mathsf{D}$ both commute with sheafification by (**??**) (using, in the latter case, the fact that f is strict). Since the joint pullback under the functors $\mathsf{Open}(|X|) \to \mathsf{C}$ for all $X \in \mathsf{C}$ together reflect isomorphisms, this implies pullback under $\mathsf{C} \to \mathsf{D}$ also commutes with sheafification. □

2.9.38 Exercise. Conclude from (2.9.37) that if $f : \mathsf{C} \to \mathsf{D}$ is strict then sheaf pullback $f^* : \mathsf{Shv}(\mathsf{D}) \to \mathsf{Shv}(\mathsf{C})$ is cocontinuous.

2.9.39 Lemma. *The topological ∞-site $\mathsf{Shv}(\mathsf{C})$ is subcanonical.*

Proof. We should show that for every open cover of a C-stack $X = \bigcup_i U_i$, the map

$$\operatorname*{colim}_{\mathbf{\Delta}^{\mathsf{op}}}^{\mathsf{Shv}(\mathsf{C})} N(X, \{U_i\}_i) \to X \tag{2.9.39.1}$$

is an isomorphism, where we have used the notation

$$N(X, \{U_i\}_i) = \Bigl(\cdots \substack{\rightarrow\\\rightarrow\\\rightarrow\\\rightarrow} \coprod_{i,j,k} U_i \times_X U_j \times_X U_k \substack{\rightarrow\\\rightarrow\\\rightarrow} \coprod_{i,j} U_i \times_X U_j \rightrightarrows \coprod_i U_i \Bigr) \tag{2.9.39.2}$$

for the Čech simplicial object (2.2.14). Recall that $U_i = X \times_{|\cdot|^*|X|} |\cdot|^*|U_i|$, and observe that

$$N(X, X \times_{|\cdot|^*|X|} |\cdot|^*|U_i|\}_i) = X \times_{|\cdot|^*|X|} |\cdot|^* N(|X|, \{|U_i|\}_i). \tag{2.9.39.3}$$

Now the operation $X \times_{|\cdot|^*|X|}$ commutes with colimits of spaces (**??**), hence with colimits of presheaves since these are computed pointwise, hence with colimits of sheaves since sheafification preserves finite limits (**??**). We are thus reduced to showing that the map

$$\operatorname*{colim}_{\mathbf{\Delta}^{\mathsf{op}}}^{\mathsf{Shv}(\mathsf{C})} |\cdot|^* N(|X|, \{|U_i|\}_i) \to |\cdot|^*|X| \tag{2.9.39.4}$$

is an isomorphism. The sheaf pullback functor $|\cdot|^* : \mathsf{Shv}(\mathsf{Top}) \to \mathsf{Shv}(\mathsf{C})$ is cocontinuous (2.9.38), so it suffices to show that the map

$$\operatorname*{colim}_{\mathbf{\Delta}^{\mathsf{op}}}^{\mathsf{Shv}(\mathsf{Top})} N(|X|, \{|U_i|\}_i) \to |X| \tag{2.9.39.5}$$

is an isomorphism, which is a special case of (1.5.103)(**??**). □

2.9.40 Lemma. *If a topological functor $f : \mathsf{C} \to \mathsf{D}$ preserves finite products, then $f_! : \mathsf{Shv}(\mathsf{C}) \to \mathsf{Shv}(\mathsf{D})$ does as well.*

Proof. Recall that if $f : \mathsf{C} \to \mathsf{D}$ preserves finite products then $f_! : \mathsf{P}(\mathsf{C}) \to \mathsf{P}(\mathsf{D})$ preserves finite products (**??**). Now write sheaf left Kan extension $f_!$ as the composition $\# \circ f_! \circ i$ (where $f_!$ is presheaf left Kan extension), and note that i preserves all limits (**??**) and $\#$ preserves finite limits (**??**). ☐

2.9.41 Lemma. *If $f : \mathsf{C} \to \mathsf{D}$ is fully faithful and strict, then $f_! : \mathsf{Shv}(\mathsf{C}) \to \mathsf{Shv}(\mathsf{D})$ is fully faithful.*

Proof. This is a special case of (1.1.88), recalling that f strict implies f^* commutes with sheafification (2.9.37). ☐

2.9.42 Lemma. *Let $f : \mathsf{C} \to \mathsf{D}$ be a topological functor, and let $\mathcal{P}$ and $\mathcal{Q}$ be properties of morphisms in C and D (respectively) preserved under pullback. Suppose that D is perfect and $\mathcal{Q}$ is local on the target* (2.1.5)*. If f sends pullbacks of $\mathcal{P}$-morphisms to pullbacks of $\mathcal{Q}$-morphisms, then so does the left Kan extension functor $f_! : \mathsf{Shv}(\mathsf{C}) \to \mathsf{Shv}(\mathsf{D})$.*

Proof. It was shown in (1.5.115) that the presheaf left Kan extension functor $f_! : \mathsf{P}(\mathsf{C}) \to \mathsf{P}(\mathsf{D})$ sends pullbacks of $\mathcal{P}$-morphisms to pullbacks of $\mathcal{Q}$-morphisms. Now the inclusion of sheaves into presheaves is continuous, and sheafification preserves all finite limits (**??**). It follows that the sheaf left Kan extension $f_! : \mathsf{Shv}(\mathsf{C}) \to \mathsf{Shv}(\mathsf{D})$ sends pullbacks of $\mathcal{P}$-morphisms to pullbacks. To show that sheaf left Kan extension $f_!$ sends $\mathcal{P}$-morphisms to $\mathcal{Q}$-morphisms, it suffices to show that sheafification $\mathsf{P}(\mathsf{D}) \to \mathsf{Shv}(\mathsf{D})$ preserves $\mathcal{Q}$-morphisms. Consider a $\mathcal{Q}$-morphism $F \to G$ in $\mathsf{P}(\mathsf{D})$, and let us show that $F^\# \to G^\#$ is also a $\mathcal{Q}$-morphism. Fix a map $d \to G^\#$ from some $d \in \mathsf{D}$, and let us show $F^\# \times_{G^\#} d \to d$ has $\mathcal{Q}$. Since D is perfect and $\mathcal{Q}$ is local on the target, we may wlog replace d with the elements of an open cover. In particular, we may assume wlog that the morphism $d \to G^\#$ lifts to G (**??**). Since sheafification preserves pullbacks, we have $F^\# \times_{G^\#} d = (F \times_G d)^\# \to d^\# = d$. The morphism $F \times_G d \to d$ in $\mathsf{P}(\mathsf{D})$ lies in the full subcategory $\mathsf{D} \subseteq \mathsf{P}(\mathsf{D})$ and has $\mathcal{Q}$, so sheafification does nothing since D is subcanonical. ☐

We saw just above that the ∞-category of sheaves $\mathsf{Shv}(\mathsf{C})$ on a topological ∞-site C satisfies a purely categorical universal property (2.9.32). It is natural to ask whether the topological functor $\mathsf{C} \hookrightarrow \mathsf{Shv}(\mathsf{C})$ satisfies a universal property which characterizes $\mathsf{Shv}(\mathsf{C})$ uniquely as a topological ∞-site (like we proved just above for presheaves (2.9.31)).

* *

2.9.43 Definition (Morita equivalence)**.** A topological functor $f : \mathsf{C} \to \mathsf{D}$ is called a *Morita equivalence* when $f_! : \mathsf{Shv}(\mathsf{C}) \to \mathsf{Shv}(\mathsf{D})$ (equivalently, its right adjoint $f^* : \mathsf{Shv}(\mathsf{D}) \to \mathsf{Shv}(\mathsf{C})$) is an equivalence of ∞-categories. A strict topological functor which is a Morita equivalence is called a *strict Morita equivalence*.

2.9.44 Example. Let $f : X \to Y$ be a continuous map of topological spaces whose inverse image map $f^{-1} = \mathsf{Open}(f) : \mathsf{Open}(Y) \to \mathsf{Open}(X)$ is an isomorphism. In this case $\mathsf{Open}(f)$ is a Morita equivalence, though it is only strict when f itself is an isomorphism (which is not implied by $\mathsf{Open}(f)$ being an isomorphism).

2.9.45 Exercise. Show that if f is a strict Morita equivalence, then $f_!$ and f^* are equivalences of topological ∞-sites (that is, they are both strict).

2.9.46 Definition (Topologically fully faithful)**.** Let $f : \mathsf{C} \to \mathsf{D}$ be a topological functor. For given $c \in \mathsf{C}$, we may consider the map of presheaves $よ(c) = \operatorname{Hom}(-, c) \to \operatorname{Hom}(f(-), f(c)) = f^*よ(f(c)) = f^*f_!よ(c)$ on C. When this map induces an isomorphism on sheafifications, we say that f is *topologically fully faithful*.

2.9.47 Lemma. *Let $f : \mathsf{C} \to \mathsf{D}$ be a strict topological functor. The sheaf pushforward $f_! : \mathsf{Shv}(\mathsf{C}) \to \mathsf{Shv}(\mathsf{D})$ is fully faithful iff f is topologically fully faithful.*

Proof. The left adjoint $f_!$ is fully faithful iff the unit map $1 \to f^*f_!$ is an isomorphism (**??**). Since f is strict, the sheaf pullback f^* is cocontinuous (2.9.38), so the unit map $1 \to f^*f_!$ is a natural transformation between cocontinuous functors. Every object of $\mathsf{Shv}(\mathsf{C})$ is a colimit of objects in the image of sheafified Yoneda $\#よ : \mathsf{C} \to \mathsf{Shv}(\mathsf{C})$, so the unit map is an isomorphism iff its pullback under $\#よ$ is an isomorphism. The pullback of the unit map under $\#よ$ being an isomorphism is exactly what it means for f to be topologically fully faithful (2.9.46). $\square$

* *

⋆ **2.9.48 Definition** (Perfect)**.** A subcanonical topological site C is called *perfect* when every C-stack which admits an open covering by objects of C is itself an object of C.

The subcanonical topological site Top is perfect. This was proven in (**??**) and amounts to the fact that topological spaces can be glued together along open sets. The same argument shows that Sm is perfect, as is the category $\mathsf{Vect} \rtimes \mathsf{Top}$ of topological spaces equipped with a vector bundle (2.9.3.5).

Given a perfect topological ∞-site C, the ∞-category of C-stacks $\mathsf{Shv}(\mathsf{C})$ provides a useful context in which to make constructions which may not *a priori* work in C itself. For example, while C need not be complete, the ∞-category $\mathsf{Shv}(\mathsf{C})$ is always complete, and the inclusion $\mathsf{C} \hookrightarrow \mathsf{Shv}(\mathsf{C})$ reflects and lifts limits (these assertions hold for presheaves, hence also for the reflective subcategory of sheaves (1.1.79)). Thus when studying limits in C, it is often useful to enlarge our focus to $\mathsf{Shv}(\mathsf{C})$. Note that we cannot use this strategy for colimits since the opposite of a topological ∞-site is not a topological ∞-site.

2.9.49 Example (Locality of limits)**.** Let C be a perfect topological site, and let us consider the question of whether a given limit $\lim_\alpha X_\alpha$ exists in C or not. This limit certainly exists in $\mathsf{Shv}(\mathsf{C})$, so it is a question of whether this limit in $\mathsf{Shv}(\mathsf{C})$ is representable. Since C is perfect, it is enough to show that $\lim_\alpha^{\mathsf{Shv}(\mathsf{C})} X_\alpha$ is locally representable. Given a map of diagrams $U_\alpha \to X_\alpha$ (over the same indexing shape) where all but finitely many of the constituent maps $U_\alpha \to X_\alpha$ are isomorphisms and all are open embeddings, the resulting map $\lim_\alpha U_\alpha \to \lim_\alpha X_\alpha$ of limits in $\mathsf{Shv}(\mathsf{C})$ is an open embedding (it is a finite iterated pullback of the open embeddings $U_\alpha \to X_\alpha$). In view of the canonical map $|\lim_\alpha^{\mathsf{Shv}(\mathsf{C})} X_\alpha| \to \lim_\alpha |X_\alpha|$, a collection of such 'open subdiagrams' will cover $\lim_\alpha X_\alpha$ provided the open embeddings $\lim_\alpha |U_\alpha| \to \lim_\alpha |X_\alpha|$ cover.

* *

2.9.50 Definition (Perfection)**.** Let C be a topological ∞-site. Its *perfection* $\mathsf{C}^\# \subseteq \mathsf{Shv}(\mathsf{C})$ is the full subcategory spanned by those objects which admit an open covering by objects which are in the essential image of the sheafified Yoneda functor

$$\mathsf{C} \overset{よ}{\hookrightarrow} \mathsf{P}(\mathsf{C}) \xrightarrow{\#} \mathsf{Shv}(\mathsf{C}). \tag{2.9.50.1}$$

Sheafified Yoneda $\#よ$ is a strict topological functor (2.9.25)(2.9.34), so $\mathsf{C}^\#$ is a topological ∞-site and $\mathsf{C} \to \mathsf{C}^\#$ is a strict topological functor (2.9.20). The topological ∞-site $\mathsf{C}^\#$ is perfect by (**??**).

Concretely, morphisms in $\mathsf{C}^\#$ are obtained from morphisms in C by sheafifying. Indeed, using the adjunction $(\#, i)$ and the Yoneda Lemma (**??**), we have

$$\begin{aligned} \operatorname{Hom}_{\mathsf{C}^\#}(X,Y) &= \operatorname{Hom}_{\mathsf{Shv}(\mathsf{C})}(\#よX, \#よY) && (2.9.50.2)\\ &= \operatorname{Hom}_{\mathsf{P}(\mathsf{C})}(よX, \#よY) = (\#よ(Y))(X). && (2.9.50.3) \end{aligned}$$

That is, we obtain $\operatorname{Hom}_{\mathsf{C}^\#}(X,Y)$ by sheafifying the presheaf $\operatorname{Hom}_{\mathsf{C}}(-,Y)$ on C (equivalently, on $(\mathsf{C} \downarrow^{\mathsf{opemb}} X) = \mathsf{Open}(|X|)$) and evaluating it on X.

* *

We now show how to freely adjoin *finite cosifted limits* (equivalently, freely adjoin *finite limits modulo preserving finite products*) to any topological ∞-site C admitting finite products (this is an adaptation of the purely categorical construction (1.5.112)). Note that while the constructions $\mathsf{C} \hookrightarrow \mathsf{P}(\mathsf{C})$ and $\mathsf{C} \hookrightarrow \mathsf{Shv}(\mathsf{C})$ for topological ∞-sites adjoin certain *colimits* (2.9.31)(**??**), the present discussion of adjoining *limits* need not be related, as the notion of a topological ∞-site is not invariant under passing to opposites.

2.9.51 Definition (Extension of $|\cdot|$ to formal limits)**.** Let C be a topological ∞-site. We equip the ∞-category $\mathsf{Lim}(\mathsf{C}) = \mathsf{Fun}(\mathsf{C}, \mathsf{Spc})^{\mathsf{op}}$ of 'formal limits in C' (**??**) with the unique continuous functor $|\cdot|_{\mathsf{Lim}(\mathsf{C})} : \mathsf{Lim}(\mathsf{C}) \to \mathsf{Top}$ (**??**) extending $|\cdot|_{\mathsf{C}}$.

$$\begin{tikzcd} \mathsf{C} \arrow[r] \arrow[dr, "|\cdot|_{\mathsf{C}}"'] & \mathsf{Lim}(\mathsf{C}) \arrow[d, dashed, "|\cdot|_{\mathsf{Lim}(\mathsf{C})}"] \\ & \mathsf{Top} \end{tikzcd} \tag{2.9.51.1}$$

Concretely, if $p : K \to \mathsf{C}$ is a diagram, then $|p|_{\mathsf{Lim}(\mathsf{C})} = \lim_K |p|_{\mathsf{C}}$. We may also drop the subscripts $_{\mathsf{C}}$ and $_{\mathsf{Lim}(\mathsf{C})}$, in which case we will write $\lim |p|$ for $|p|_{\mathsf{Lim}(\mathsf{C})}$, while $|p|$ denotes the pushforward of p under $(|\cdot|_{\mathsf{C}})_! : \mathsf{Lim}(\mathsf{C}) \to \mathsf{Lim}(\mathsf{Top})$.

A formal limit $p \in \mathsf{Lim}(\mathsf{C})$ in a topological ∞-site C may be nontrivial yet have $\lim |p| = \varnothing$. The next definition describes formal limits p which are, in a certain precise sense, 'germs' around the space $\lim |p|$.

⋆ **2.9.52 Definition** (Corporeal)**.** Let $p \in \mathsf{Lim}(\mathsf{C})$ be a formal limit in a topological ∞-site C. For any open embedding $U \hookrightarrow M$ in C, we may consider the following diagram.

$$\begin{array}{ccc} \operatorname{Hom}(p,U) & \longrightarrow & \operatorname{Hom}(p,M) \\ \downarrow & & \downarrow \\ \operatorname{Hom}(\lim|p|,|U|) & \longrightarrow & \operatorname{Hom}(\lim|p|,|M|) \end{array} \tag{2.9.52.1}$$

When this diagram is a pullback for every open embedding $U \hookrightarrow M$ in C, we say that the formal limit $p \in \mathsf{Lim}(\mathsf{C})$ is *corporeal*. We denote the full subcategory spanned by corporeal formal limits by $\mathsf{Lim}_{\mathsf{cp}}(\mathsf{C}) \subseteq \mathsf{Lim}(\mathsf{C})$.

2.9.53 Exercise. Show that all objects of $\mathsf{C} \subseteq \mathsf{Lim}(\mathsf{C})$ are corporeal (use the definition of open embeddings).

2.9.54 Exercise. Show that the formal fiber product $\mathbb{R} \times_{\mathbb{R}^2} \mathbb{R} \in \mathsf{Lim}(\mathsf{Top})$ of the two axes $\mathbb{R} \hookrightarrow \mathbb{R}^2 \hookleftarrow \mathbb{R}$ is not corporeal. Show that the formal inverse limit $\varprojlim_n (-\frac{1}{n}, \frac{1}{n}) \in \mathsf{Lim}(\mathsf{Top})$ is corporeal.

While corporeality of a formal limit (2.9.52) is not quite a special case of locality in the sense of (1.5.100), we will see that the ∞-category of corporeal formal limits $\mathsf{Lim}_{\mathsf{cp}}(\mathsf{C}) \subseteq \mathsf{Lim}(\mathsf{C})$ satisfies many of the same properties as local presheaves do.

2.9.55 Lemma. *The functor $\mathsf{C} \to \mathsf{Lim}_{\mathsf{cp}}(\mathsf{C})$ preserves open embeddings and their pullbacks.*

Proof. Let $U \hookrightarrow M$ be an open embedding in C. To say that $U \hookrightarrow M$ is an open embedding in $\mathsf{Lim}_{\mathsf{cp}}(\mathsf{C})$ is the assertion that for any corporeal $p \in \mathsf{Lim}(\mathsf{C})$, the map $\operatorname{Hom}(p,U) \to \operatorname{Hom}(p,M)$ is the pullback of $\operatorname{Hom}(\lim|p|,|U|) \to \operatorname{Hom}(\lim|p|,|M|)$, which is exactly what it means for p to be corporeal.

Now suppose $X' \to Y'$ is the pullback of an open embedding $X \to Y$ in C. For $q \in \mathsf{Lim}(\mathsf{C})$, applying $\operatorname{Hom}(q,-) \to \operatorname{Hom}(\lim|q|,|-|)$ to this pullback square produces a cube. The $\operatorname{Hom}(\lim|q|,|-|)$ face of the cube is a pullback since $|\cdot|_{\mathsf{C}}$ preserves pullbacks of open embeddings (2.9.5). If q is corporeal, two other faces are pullbacks by definition (2.9.52.1). Using cancellation (1.1.52), we deduce that the $\operatorname{Hom}(q,-)$ face is a pullback for q corporeal. □

⋆ **2.9.56 Lemma.** *The full subcategory of corporeal diagrams $\mathsf{Lim}_{\mathsf{cp}}(\mathsf{C}) \subseteq \mathsf{Lim}(\mathsf{C})$ is coreflective, and for any topological functor $f : \mathsf{C} \to \mathsf{D}$, the functor $f(-)_{\mathrm{cp}} : \mathsf{Lim}(\mathsf{C}) \to \mathsf{Lim}_{\mathsf{cp}}(\mathsf{D})$ sends corporealizations to isomorphisms.*

Proof. Let $p : K \to \mathsf{C}$ be a diagram, and let us construct its corporealization p_{cp}.

Let $|p| : K \to \mathsf{Top}$ denote the composition of p with the forgetful functor $|\cdot| : \mathsf{C} \to \mathsf{Top}$, and let $\lim |p|$ denote its limit. For any vertex $\alpha \in K$, let $(\lim |p| \downarrow \mathsf{Open}(p(\alpha)))$ denote the category of open subsets of $|p(\alpha)|$ which contain the image of the map $\lim |p| \to p(\alpha)$. We will show that the corporealization of p is the diagram

$$p_{\mathrm{cp}} : (\lim |p| \downarrow \mathsf{Open}(p)) \rtimes K \to \mathsf{C} \tag{2.9.56.1}$$

where a map $Z \to (\lim |p| \downarrow \mathsf{Open}(p)) \rtimes K$ is a map $f : Z \to K$ together with, for every vertex $z \in Z$, a choice of open set $\lim |p| \to U_z \subseteq p(f(z))$, such that for every edge $e : z \to z'$, we have $U_z \subseteq (p(f(e)) : p(f(z)) \to p(f(z')))^{-1}(U_{z'})$, and p_{cp} sends such a map out of Z to the evident diagram $Z \to \mathsf{C}$ given by $z \mapsto U_z$ (2.9.8). There is an evident inclusion $K \subseteq (\lim |p| \downarrow \mathsf{Open}(p)) \rtimes K$ given by taking $U_z = p(f(z))$ for all z, giving a map of formal limits $p_{\mathrm{cp}} \to p$. Note that the natural map $\lim |p_{\mathrm{cp}}| \to \lim |p|$ is an isomorphism.

Let us show that p_{cp} is corporeal. That is, we should show that for any open embedding $U \hookrightarrow M$ in C and any map $f : \lim |p| \to U$, the map

$$\operatorname*{colim}_{((\lim |p| \downarrow \mathsf{Open}(p)) \rtimes K)^{\mathsf{op}}} \mathrm{Hom}(p, U)_f \to \operatorname*{colim}_{((\lim |p| \downarrow \mathsf{Open}(p)) \rtimes K)^{\mathsf{op}}} \mathrm{Hom}(p, M)_f \tag{2.9.56.2}$$

is an isomorphism, where $\mathrm{Hom}_f \subseteq \mathrm{Hom}$ denotes the maps whose pullback to $\lim |p|$ is f. Since the map $\pi : (\lim |p| \downarrow \mathsf{Open}(p)) \rtimes K \to K$ is a cartesian fibration (**??**) (inspection), these colimits may be expressed as colimits over K^{op} of the fiberwise colimit pushforwards under π (**??**). We claim that the map between fiberwise colimits (diagrams over K^{op}) is already an isomorphism. That is, we claim that for every $X \in \mathsf{C}$, every subset $A \subseteq |X|$, and every function $f : A \to |U|$, the map

$$\operatorname*{colim}_{(A \downarrow \mathsf{Open}(|X|))^{\mathsf{op}}} \mathrm{Hom}(-, U)_f \to \operatorname*{colim}_{(A \downarrow \mathsf{Open}(|X|))^{\mathsf{op}}} \mathrm{Hom}(-, M)_f \tag{2.9.56.3}$$

is an isomorphism. This is evident since both sides are the set of germs of maps near A agreeing with f.

Now we claim that $p_{\mathrm{cp}} \to p$ is the corporealization of p. That is, we claim that for any corporeal diagram $q : L \to \mathsf{C}$, the composition map

$$\mathrm{Hom}_{\mathsf{Lim}(\mathsf{C})}(q, p_{\mathrm{cp}}) \to \mathrm{Hom}_{\mathsf{Lim}(\mathsf{C})}(q, p) \tag{2.9.56.4}$$

is an isomorphism. Both sides map to (the discrete set) $\mathrm{Hom}(\lim |q|, \lim |p|)$, so we may restrict to the fiber $\mathrm{Hom}_f \subseteq \mathrm{Hom}$ over a particular map $f : \lim |q| \to \lim |p|$. This restriction may be written as

$$\lim_{(\lim |p| \downarrow \mathsf{Open}(p)) \rtimes K} \mathrm{Hom}_{\mathsf{Lim}(\mathsf{C})}(q, p_{\mathrm{cp}}(-))_f \to \lim_K \mathrm{Hom}_{\mathsf{Lim}(\mathsf{C})}(q, p(-))_f. \tag{2.9.56.5}$$

We claim that after pushing forward the diagram on the left to K (fiberwise limit), we obtain an isomorphism of diagrams over K (and hence the map above is an isomorphism). It is enough to show that for any open embedding $U \hookrightarrow M$ and any map $f : \lim |q| \to U$, the map

$$\mathrm{Hom}_{\mathsf{Lim}(\mathsf{C})}(q, U)_f \to \mathrm{Hom}_{\mathsf{Lim}(\mathsf{C})}(q, M)_f \tag{2.9.56.6}$$

is an isomorphism. That this is an isomorphism now follows from the fact that q is corporeal (2.9.52.1). □

⋆ **2.9.57 Exercise** (A formal limit and its corporealization are 'topologically' equivalent)**.** Use the fact that $\mathsf{Lim}_{\mathsf{cp}}(\mathsf{C}) \subseteq \mathsf{Lim}(\mathsf{C})$ is coreflective (2.9.56) and contains $\mathsf{C} \subseteq \mathsf{Lim}(\mathsf{C})$ (2.9.53) to show that the map $\lim p_{\mathrm{cp}} \to \lim p$ is an isomorphism for every formal limit $p \in \mathsf{Lim}(\mathsf{C})$ (in the sense that if either exists, then so does the other, and in this case the map is an isomorphism). Use the fact that $\mathrm{cp} \circ f_!$ sends corporealizations to isomorphisms (2.9.56) to conclude that $|\cdot|_{\mathsf{Lim}_{\mathsf{cp}}(\mathsf{C})}$ is continuous and that a topological functor $f : \mathsf{C} \to \mathsf{D}$ preserves the limit of $p \in \mathsf{Lim}(\mathsf{C})$ iff if preserves the limit of p_{cp}.

2.9.58 Exercise. Note that open embeddings in $\mathsf{Lim}_{\mathsf{cp}}(\mathsf{C})$ are preserved under pullback by (2.9.4) since $|\cdot|_{\mathsf{Lim}_{\mathsf{cp}}(\mathsf{C})}$ preserves pullbacks (in fact, preserves all limits (2.9.57)).

2.9.59 Lemma. *A left fibration $p : K \to \mathsf{C}$ is corporeal iff it satisfies the right lifting property with respect to pairs $(\Delta^1, 1) \wedge (\Delta^k, \partial\Delta^k)$ mapping to C via the projection to Δ^1 followed by an open embedding $\Delta^1 \to \mathsf{C}$, say denoted $U \to M$, for which (and this only has content when $k = 0$) the induced map $\lim |p| \to |M|$ lands inside the open set $|U| \subseteq |M|$.*

Proof. This is a direct translation of the condition that (2.9.52.1) be a pullback (compare (1.5.101)). □

2.9.60 Lemma. *The corporealization functor $\mathsf{Lim}(\mathsf{C}) \to \mathsf{Lim}_{\mathsf{cp}}(\mathsf{C})$ sends a diagram in C to the result of applying the small object argument to the lifting problems in* (2.9.59) *(as in* (1.5.101)*).*

Proof. Following the proof of (1.5.101), it suffices to check that if $K \to \mathsf{C}$ is any diagram and $\hat{K} \to \mathsf{C}$ denotes the result of forming the pushout of a lifting problem as in (2.9.59), then for any left fibration $E \to \mathsf{C}$ satisfying the lifting property (2.9.59), the simplicial mapping space from $\hat{K}$ to E over C maps via a trivial Kan fibration to the simplicial mapping space from K to E over C. To see this, it is enough to argue that the smash product $(\Delta^1, 1) \wedge (\Delta^k, \partial\Delta^k) \wedge (\Delta^r, \partial\Delta^r)$ is filtered by pushouts of $(\Delta^1, 1) \wedge (\Delta^a, \partial\Delta^a)$ (mapping to C as in (2.9.59)). □

Recall the full subcategory $\mathsf{Cosif}(\mathsf{C}) \subseteq \mathsf{Lim}(\mathsf{C})$ of formal cosifted limits (1.5.112).

2.9.61 Lemma. *The corporealization functor $\mathsf{Lim}(\mathsf{C}) \to \mathsf{Lim}_{\mathsf{cp}}(\mathsf{C})$ restricts to an endofunctor of the full subcategory $\mathsf{Cosif}(\mathsf{C}) \subseteq \mathsf{Lim}(\mathsf{C})$ of formal cosifted limits.*

Proof. Let $p : K \to \mathsf{C}$ be a cosifted diagram, and let us show that its corporealization p_{cp} is cosifted. The domain of the corporealization p_{cp} (2.9.56.1) is $(\lim |p| \downarrow \mathsf{Open}(p)) \rtimes K$. The functor $(\lim |p| \downarrow \mathsf{Open}(|p|)) \rtimes K \to K$ is cartesian (2.9.8) and K is cosifted, so it suffices to show its fibers are cosifted (**??**). The fiber over $\alpha \in K$ is the poset category $(\lim |p| \downarrow \mathsf{Open}(p(\alpha)))$, which is a cofiltered poset (by intersection of open sets), hence cosifted (**??**). □

⋆ **2.9.62 Definition** ($\mathsf{Cosif}_{\mathrm{cp}}(\mathsf{C})$)**.** For a topological ∞-site C, we let $\mathsf{Cosif}_{\mathrm{cp}}(\mathsf{C}) = \mathsf{Lim}_{\mathrm{cp}}(\mathsf{C}) \cap \mathsf{Cosif}(\mathsf{C}) \subseteq \mathsf{Lim}(\mathsf{C})$, and we denote by $\mathsf{Cosif}_{\mathrm{cp,fin}}(\mathsf{C}) \subseteq \mathsf{Cosif}_{\mathrm{cp}}(\mathsf{C})$ the full subcategory spanned by finite limits of objects of C.

Let us summarize what we know about $\mathsf{Cosif}_{\mathrm{cp}}(\mathsf{C})$ so far. Since the corporealization functor $\mathsf{Lim}(\mathsf{C}) \to \mathsf{Lim}_{\mathrm{cp}}(\mathsf{C})$ preserves cosiftedness (2.9.61), it restricts to a coreflection of the inclusion $\mathsf{Cosif}_{\mathrm{cp}}(\mathsf{C}) \subseteq \mathsf{Cosif}(\mathsf{C})$. Hence $\mathsf{Cosif}_{\mathrm{cp}}(\mathsf{C}) \subseteq \mathsf{Lim}(\mathsf{C})$ is a coreflective subcategory, with coreflection given by the coreflection of $\mathsf{Cosif}(\mathsf{C}) \subseteq \mathsf{Lim}(\mathsf{C})$ (**??**) followed by (the restriction of) corporealization. We have $\mathsf{C} \subseteq \mathsf{Cosif}_{\mathrm{cp}}(\mathsf{C})$ (2.9.52).

The ∞-category $\mathsf{Cosif}_{\mathrm{cp,fin}}(\mathsf{C})$ has all finite limits of objects of C, essentially by definition. To show that it has all finite limits is more subtle and relies on the following key observation.

2.9.63 Corollary. *For any finite diagram $p : K \to \mathsf{C}$, every morphism to the associated object $p \in \mathsf{Cosif}_{\mathrm{cp,fin}}(\mathsf{C})$ from another object of $\mathsf{Cosif}_{\mathrm{cp,fin}}(\mathsf{C})$ is induced from a finite diagram $q : L \to \mathsf{C}$, an inclusion $K \hookrightarrow L$, and an isomorphism $q|_K = p$.*

Proof. Given the lifting property characterization of $\mathsf{Cosif}_{\mathrm{cp}}(\mathsf{C})$ (1.5.112)(1.5.101)(2.9.59), we may proceed as in (1.5.98). □

We equip $\mathsf{Cosif}_{\mathrm{cp}}(\mathsf{C})$ with the functor $|\cdot|_{\mathsf{Cosif}_{\mathrm{cp}}(\mathsf{C})}$ given by the restriction of $|\cdot|_{\mathsf{Lim}(\mathsf{C})}$. The restriction $|\cdot|_{\mathsf{Cosif}(\mathsf{C})}$ preserves cosifted limits since $\mathsf{Cosif}(\mathsf{C}) \subseteq \mathsf{Lim}(\mathsf{C})$ is closed under cosifted limits (**??**). Since $|\cdot|_{\mathsf{Lim}(\mathsf{C})}$ sends corporealizations to isomorphisms (**??**) and the coreflection $\mathsf{Cosif}(\mathsf{C}) \to \mathsf{Cosif}_{\mathrm{cp}}(\mathsf{C})$ is the restriction of corporealization $\mathsf{Lim}(\mathsf{C}) \to \mathsf{Lim}_{\mathrm{cp}}(\mathsf{C})$ (2.9.61), we conclude that the restriction $|\cdot|_{\mathsf{Cosif}_{\mathrm{cp}}(\mathsf{C})}$ also preserves cosifted limits. We note that it need not preserve all limits (e.g. finite products in C remain products in $\mathsf{Cosif}_{\mathrm{cp}}(\mathsf{C})$, and $|\cdot|_{\mathsf{C}}$ need not preserve finite products). However, if $|\cdot|_{\mathsf{C}}$ does preserve finite product, then $|\cdot|_{\mathsf{Cosif}(\mathsf{C})}$ preserves all limits (**??**), hence so does $|\cdot|_{\mathsf{Cosif}_{\mathrm{cp}}(\mathsf{C})}$.

⋆ **2.9.64 Lemma.** *$\mathsf{Cosif}_{\mathrm{cp,fin}}(\mathsf{C})$ is a topological ∞-site, and every open embedding in $\mathsf{Cosif}_{\mathrm{cp,fin}}(\mathsf{C})$ is a pullback in $\mathsf{Lim}_{\mathrm{cp}}(\mathsf{C})$ of an open embedding in C.*

Proof. Fix $X \in \mathsf{Cosif}_{\mathrm{cp,fin}}(\mathsf{C})$, and let us show how to realize every open subset of $\lim |X|$ by an open embedding in $\mathsf{Cosif}_{\mathrm{cp,fin}}(\mathsf{C})$ which is a pullback in $\mathsf{Lim}_{\mathrm{cp}}(\mathsf{C})$ of an open embedding in C. Represent X as the corporealization $X = p_{\mathrm{cp}}$ of an object $p \in \mathsf{Cosif}_{\mathrm{fin}}(\mathsf{C})$. Every $p \in \mathsf{Cosif}_{\mathrm{fin}}(\mathsf{C})$ is a truncated cosimplicial object $p : \mathbf{\Delta} \to \mathsf{C}$ (**??**). Now we have $\lim |X| = \lim |p_{\mathrm{cp}}| = \lim |p|$ which is a subspace of $|p^0|$. Hence every open subset of $\lim |X|$ is the pullback of an open subset of $|p^0|$. Lift this open subset to an open embedding $U^0 \hookrightarrow p^0$ in C, and consider the truncated cosimplicial object $U^\bullet = p^\bullet \times_{p^0} U^0 : \mathbf{\Delta} \to \mathsf{C}$ and its associated object $U \in \mathsf{Cosif}_{\mathrm{fin}}(\mathsf{C})$ (**??**) (to see that $U^\bullet$ is truncated, note that any limit of d-truncated objects is d-truncated, since the inclusion of d-truncated objects is right adjoint to truncation (1.3.13)). Note that $U^i = p^i \times_{p^0} U^0$ in $\mathsf{Lim}_{\mathrm{cp}}(\mathsf{C})$ since $\mathsf{C} \to \mathsf{Lim}_{\mathrm{cp}}(\mathsf{C})$ preserves pullbacks of open embeddings (2.9.55). Thus by taking the limit in $\mathsf{Lim}_{\mathrm{cp}}(\mathsf{C})$, we find that $U_{\mathrm{cp}} \to p_{\mathrm{cp}}$ is the pullback of $U^0 \to p^0$ in $\mathsf{Lim}_{\mathrm{cp}}(\mathsf{C})$. Now open embeddings in $\mathsf{Lim}_{\mathrm{cp}}(\mathsf{C})$ are preserved under pullback (2.9.58) and such pullbacks are preserved by $|\cdot|_{\mathsf{Lim}_{\mathrm{cp}}(\mathsf{C})}$. Thus $U_{\mathrm{cp}} \to p_{\mathrm{cp}}$ is an open embedding in $\mathsf{Lim}_{\mathrm{cp}}(\mathsf{C})$, hence also in $\mathsf{Cosif}_{\mathrm{cp,fin}}(\mathsf{C})$, corresponding to the correct open subset of $\lim |X| = \lim |p|_{\mathrm{cp}}$. □

2.9.65 Lemma. *Let $i : \mathsf{C} \to \overline{\mathsf{C}}$ be a strict topological functor, and suppose every open embedding in $\overline{\mathsf{C}}$ is a pullback of an open embedding in C. For all $f : \overline{\mathsf{C}} \to \mathsf{E}$ preserving pullbacks and all $\pi : |\cdot|_{\mathsf{E}} \circ f \to |\cdot|_{\overline{\mathsf{C}}}$, if f and π send open embeddings in C to open embeddings and pullbacks (respectively), then they do the same for open embeddings in $\overline{\mathsf{C}}$.*

For example, the inclusion $\mathsf{C} \to \mathsf{Cosif}_{\mathsf{cp,fin}}(\mathsf{C})$ is a strict topological functor, and every open embedding in $\mathsf{Cosif}_{\mathsf{cp,fin}}(\mathsf{C})$ is a pullback of one in C (2.9.64).

Proof. Fix $f : \overline{\mathsf{C}} \to \mathsf{E}$ satisfying $\overline{\alpha}$ and $\pi : |\cdot|_{\mathsf{E}} \circ f \to |\cdot|_{\overline{\mathsf{C}}}$. Suppose that f and π send open embeddings in C to open embeddings and pullbacks (respectively), and let us show they do the same for open embeddings in $\overline{\mathsf{C}}$. Fix an open embedding $X \to Y$ in $\overline{\mathsf{C}}$, and express it as the pullback of an open embedding $U \to M$ in C.

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ U \arrow[r] & M \end{tikzcd} \tag{2.9.65.1}$$

Since f preserves pullbacks, we see that $f(X) \to f(Y)$ is a pullback of $f(U) \to f(M)$. The latter is an open embedding by hypothesis on f, hence so is the former since open embeddings are preserved under pullback in any topological ∞-site E (2.9.5). Applying π to the pullback square (2.9.65.1) yields a cubical diagram.

$$\begin{tikzcd} {|f(X)|_{\mathsf{E}}} \arrow[rr] \arrow[dd] \arrow[dr] & & {|f(Y)|_{\mathsf{E}}} \arrow[dd] \arrow[dr] & \\ & {|X|_{\mathsf{C}}} \arrow[rr] \arrow[dd] & & {|Y|_{\mathsf{C}}} \arrow[dd] \\ {|f(U)|_{\mathsf{E}}} \arrow[rr] \arrow[dr] & & {|f(M)|_{\mathsf{E}}} \arrow[dr] & \\ & {|U|_{\mathsf{C}}} \arrow[rr] & & {|M|_{\mathsf{C}}} \end{tikzcd} \tag{2.9.65.2}$$

By hypothesis on π, the lower square $\pi(U \to M)$ is a pullback. The squares $|\cdot|_{\mathsf{C}}(2.9.65.1)$ and $|f(\cdot)|_{\mathsf{E}}(2.9.65.1)$ are pullbacks since $|\cdot|_{\mathsf{C}}$ and $|\cdot|_{\mathsf{E}}$ preserve pullbacks of open embeddings (2.9.5). It follows from cancellation (1.1.52) that the top square $\pi(X \to Y)$ is a pullback. □

⋆ **2.9.66 Definition** (Derived smooth manifold)**.** The topological ∞-site Der of derived smooth manifolds is the perfection (2.9.50) of the topological ∞-site $\mathsf{Cosif}_{\mathsf{cp,fin}}(\mathsf{Sm})$.

⋆ **2.9.67 Theorem.** *The ∞-category of derived smooth manifolds* (2.9.66) *satisfies the axioms* (2.10.2).

Proof. □

2.10 Derived smooth manifolds

The ∞-category of derived smooth manifolds Der is an enlargement of the category of smooth manifolds Sm, obtained by *formally adjoining finite limits, modulo transverse limits in* Sm (equivalently, formally adjoining cosifted limits as in (**??**)) within the realm of topological ∞-sites. For example, every diagram of smooth manifolds $X \to Y \leftarrow Z$ has a fiber product $X \times_Y Z$ in the category of derived smooth manifolds. When the diagram is transverse, this is simply the usual fiber product; otherwise it is a more exotic sort of object. The theory of derived smooth manifolds originates in work of Spivak [83, 84] with further developments by Joyce [39, 43] and many others. It is a homotopical analogue of the theory of locally finitely presented C^∞-schemes [16, 65, 42] and falls within the general framework of derived geometry of Lurie [56] and Toën–Vezzosi [87, 88]. We introduce a new axiomatic approach to the ∞-category of derived smooth manifolds.

* *

The ∞-category of derived smooth manifolds is best understood through certain properties it satisfies. It is not hard to see that these properties characterize it uniquely, so they may be regarded as the definition of derived smooth manifolds, modulo a proof of existence. Recall the notion of a 'perfect topological ∞-site' (2.9) which is an '∞-category of topological spaces equipped with additional local structure'.

⋆ **2.10.1 Definition** (Topological preservation of limits)**.** Let C be a topological ∞-site, and recall that to each sheaf $F : \mathsf{C}^{\mathsf{op}} \to \mathsf{Spc}$ we may associate a strict topological functor $F : \mathsf{C} \to \mathsf{Shv}(-)^{\mathsf{op}} \rtimes \mathsf{Top}$ lifting $|\cdot| : \mathsf{C} \to \mathsf{Top}$ (**??**). Given a limit in C which is preserved by $|\cdot|$, we say that F *topologically preserves* this limit when its associated functor $\mathsf{C} \to \mathsf{Shv}(-)^{\mathsf{op}} \rtimes \mathsf{Top}$ preserves said limit (equivalently, sends it to a relative limit (1.5.87) over Top).

Concretely, a diagram $K^{\triangleleft} \to \mathsf{Shv}(-)^{\mathsf{op}} \rtimes \mathsf{Top}$ encoding a diagram of sheaved topological spaces (X_α, F_α) all receiving a map from a sheaved topological space (X, F) is a relative limit diagram when the natural map

$$\operatorname*{colim}_{\alpha} \pi_\alpha^* F_\alpha \to F \tag{2.10.1.1}$$

is an isomorphism (that is, relative limits in $\mathsf{Shv}(-)^{\mathsf{op}} \rtimes \mathsf{Top} \to \mathsf{Top}$ are limits in fibers (**??**)(2.2.15)).

⋆ **2.10.2 Definition** (Derived smooth manifold)**.** The ∞-category Der of derived smooth manifolds together with the functor $\mathsf{Sm} \to \mathsf{Der}$ is defined by the following properties:

(2.10.2.1) $\mathsf{Sm} \to \mathsf{Der}$ is a strict topological functor between perfect topological ∞-sites (2.9).

(2.10.2.2) $\mathsf{Sm} \to \mathsf{Der}$ is fully faithful and preserves finite products.

(2.10.2.3) Der has finite limits, and every object of Der is locally isomorphic to a finite limit of smooth manifolds.

(2.10.2.4) $|\cdot| : \mathsf{Der} \to \mathsf{Top}$ preserves finite limits.

(2.10.2.5) For any $N \in \mathsf{Sm}$, the Yoneda sheaf $\mathrm{Hom}(-, N) \in \mathsf{Shv}(\mathsf{Der})$ topologically preserves finite cosifted limits (**??**).

(2.10.2.6) (Universal Property) For every complete perfect topological site E, the ∞-category of topological functors $\mathsf{Der} \to \mathsf{E}$ preserving finite cosifted limits is equivalent via restriction to the ∞-category of topological functors $\mathsf{Sm} \to \mathsf{E}$.

The universal property (**??**) evidently characterizes the functor $\mathsf{Sm} \to \mathsf{Der}$ uniquely up to contractible choice, provided it exists. The existence of this functor (satisfying all the above properties) shown in (2.9.67) below.

* *

We now discuss the notion of transversality for diagrams of smooth manifolds.

⋆ **2.10.3 Definition** (Transverse diagram of vector spaces)**.** A diagram of vector spaces $D : K \to \mathsf{Vect}$ is called *transverse* when the canonical map

$$\lim_K^{\mathsf{Vect}} D \to \lim_K^{\mathsf{K}(\mathsf{Vect})} D \tag{2.10.3.1}$$

(from the limit of D in the category of vector spaces to the limit of D in the ∞-category of complexes of vector spaces (**??**)) is an isomorphism. Transversality of D evidently depends only on its class in $\mathsf{Lim}(\mathsf{Vect})$.

Since $\mathsf{K}^{\geq 0}(\mathsf{Vect}) \subseteq \mathsf{K}(\mathsf{Vect})$ is closed under limits, we could just as well replace the limit in $\mathsf{K}(\mathsf{Vect})$ with the limit in $\mathsf{K}^{\geq 0}(\mathsf{Vect})$ in the definition of transversality. Since $\mathsf{Vect} \subseteq \mathsf{K}^{\geq 0}(\mathsf{Vect})$ is a coreflective subcategory, the comparison map from the limit in Vect to the limit in $\mathsf{K}^{\geq 0}(\mathsf{Vect})$ is an isomorphism iff the limit in $\mathsf{K}^{\geq 0}(\mathsf{Vect})$ lies in the full subcategory $\mathsf{Vect} \subseteq \mathsf{K}^{\geq 0}(\mathsf{Vect})$. In other words, a diagram $D : K \to \mathsf{Vect}$ is transverse iff its limit in $\mathsf{K}^{\geq 0}(\mathsf{Vect})$ lies in $\mathsf{Vect} \subseteq \mathsf{K}^{\geq 0}(\mathsf{Vect})$ (equivalently, has no higher cohomology).

It may help to recall that the limit of a diagram $D : K \to \mathsf{K}^{\geq 0}(\mathsf{Vect})$ is given explicitly by the total complex

$$\prod_{\sigma:[0]\to K} D(\sigma(0)) \to \prod_{\sigma:[1]\to K} D(\sigma(1)) \otimes \mathfrak{o}(1)^\vee \to \prod_{\sigma:[2]\to K} D(\sigma(2)) \otimes \mathfrak{o}(2)^\vee \to \cdots \tag{2.10.3.2}$$

which may be regarded as 'simplicial cochains on K with coefficients in D' (**??**). Also recall that the limit of a cosimplicial vector space $p : \mathbf{\Delta} \to \mathsf{Vect}$ is the object of $\mathsf{K}^{\geq 0}(\mathsf{Vect})$ associated to p by the Dold–Kan correspondence (**??**).

2.10.4 Exercise. Show that a diagram of vector spaces $V \to W \leftarrow U$ is transverse iff the sum map $V \oplus U \to W$ is surjective.

The notion of transversality for a diagram D evidently depends only on the 'formal ∞-limit' represented by D, namely its limit in the ∞-category $\mathsf{Lim}(\mathsf{C}) = \mathsf{P}(\mathsf{C}^{\mathsf{op}})^{\mathsf{op}} = \mathsf{Fun}(\mathsf{C}, \mathsf{Spc})^{\mathsf{op}}$ of formal ∞-limits (**??**). Indeed, recall that every object of $\mathsf{Lim}(\mathsf{C})$ is represented by a diagram $K \to \mathsf{C}$, and two diagrams represent the same object iff they are related by pullback under initial functors. Thus a property of objects of $\mathsf{Lim}(\mathsf{C})$ (i.e. a property of formal ∞-limits in C) is a property of diagrams in C which is invariant under pullback under initial functors.

2.10.5 Lemma. *A formal limit of vector spaces is transverse iff its cosiftedization is transverse.*

Proof. The functor $\mathsf{Vect} \to \mathsf{K}^{\geq 0}(\mathsf{Vect})$ preserves finite products, hence commutes with cosiftedization (**??**). □

⋆ **2.10.6 Definition** (Transverse diagram of smooth manifolds)**.** Let $D : K \to \mathsf{Sm}$ be a diagram. A point

$$p \in \lim_K^{\mathsf{Top}} D \tag{2.10.6.1}$$

of the limit of D in the category of topological spaces determines a lift of D to Sm_* (pointed smooth manifolds and basepoint preserving maps).

$$\begin{tikzcd} & \mathsf{Sm}_* \arrow[d] \\ K \arrow[ur, dashed, "D_p"] \arrow[r, "D"] & \mathsf{Sm} \end{tikzcd} \tag{2.10.6.2}$$

We can now compose this lift D_p with the 'tangent space at the basepoint' functor $T_* : \mathsf{Sm}_* \to \mathsf{Vect}_{\mathbb{R}}$ to obtain a diagram

$$T_pD : K \to \mathsf{Vect}_{\mathbb{R}}. \tag{2.10.6.3}$$

We say that D is *transverse at p* when T_pD is transverse (2.10.3), and we say that D is *transverse* when it is transverse at every point of its topological limit $\lim_K^{\mathsf{Top}} D$. Transversality of D evidently depends only on its class in $\mathsf{Lim}(\mathsf{Sm})$.

2.10.7 Lemma. *Show, using the corresponding statement for vector spaces* (2.10.5)*, that a formal limit of smooth manifolds is transverse iff its cosiftedization is transverse.*

2.10.8 Exercise. Show that a diagram of smooth manifolds $D : J \to \mathsf{Sm}$ with only 0-cells and 1-cells is transverse in the sense of (2.10.6) iff it is transverse in the sense of (2.6.8).

* *

We now study presentations of derived smooth manifolds by cosimplicial smooth manifolds. The relation between cosimplicial smooth manifolds csSm and derived smooth manifolds Der is analogous to the relation between the category of complexes $\mathsf{Kom}^{\geq 0}(\mathsf{Vect})$ (1.1.99)(**??**) and the ∞-category of complexes $\mathsf{K}^{\geq 0}(\mathsf{Vect})$ (**??**). Recall that a cosimplicial object is called *n-truncated* when its matching maps in degrees $> n$ are isomorphisms (1.3.13)(1.3.16) and is called *truncated* when it is n-truncated for some $n < \infty$.

⋆ **2.10.9 Lemma** (Existence of cosimplicial presentations)**.** *Any derived smooth manifold may be expressed locally as the limit of a truncated cosimplicial smooth manifold. Any map of derived smooth manifolds may be expressed locally as a levelwise submersive map of truncated cosimplicial smooth manifolds.*

Proof. Every derived smooth manifold is locally the limit of a finite diagram of smooth manifolds. Since $\mathsf{Sm} \to \mathsf{Der}$ preserves finite products, this limit unchanged by applying cosiftedization, which turns a finite diagram into a truncated cosimplicial diagram (**??**).

Every map of derived smooth manifolds is locally the map from the limit of a finite diagram of smooth manifolds to the limit of a subdiagram thereof (**??**), and upon applying cosimplicialization this becomes a levelwise submersion of truncated cosimplicial smooth manifolds. $\square$

Recall that a cosimplicial object is called *Reedy* $\mathcal{P}$ when its matching maps have $\mathcal{P}$ (1.3.17) (any property of morphisms $\mathcal{P}$). We saw earlier that a map of cosimplicial vector spaces $V^\bullet \to W^\bullet$ is Reedy surjective iff the corresponding map of chain complexes is surjective (1.3.26). Recall that a 'point' x of a cosimplicial smooth manifold $X^\bullet$ means a point of its topological limit $\lim_{\mathbf{\Delta}}^{\mathsf{Top}} X^\bullet$ or, equivalently, a map $x : * \to X^\bullet$ from the constant cosimplicial smooth manifold $*$. Thus for a point x of a cosimplicial smooth manifold $X^\bullet$, a map $X^\bullet \to Y^\bullet$ is levelwise submersive at x iff it is Reedy submersive at x.

Let us now see how to upgrade levelwise (equivalently, Reedy) submersivity over the topological limit to (true) levelwise submersivity and Reedy submersivity over an open cosimplicial submanifold containing the topological limit.

2.10.10 Exercise. Let $X^\bullet$ be a cosimplicial smooth manifold. Given an open subset $V^k \subseteq X^k$, consider the cosimplicial smooth manifold $U^\bullet$ with a levelwise open embedding $U^\bullet \to X^\bullet$ defined by

$$U^j = \bigcap_{f:[j]\to[k]} (X^j \xrightarrow{f_*} X^k)^{-1}(V^k). \tag{2.10.10.1}$$

Show that if $M^iX^\bullet$ exists, then so does $M^iU^\bullet$ and the map $M^iU^\bullet \to M^iX^\bullet$ is an open embedding. Show that if $i > k$, then the induced square of matching maps

$$\begin{tikzcd} U^i \arrow[r] \arrow[d] & X^i \arrow[d] \\ M^iU^\bullet \arrow[r] & M^iX^\bullet \end{tikzcd} \tag{2.10.10.2}$$

is a pullback (so, in particular, if the ith matching map of $X^\bullet$ is an isomorphism, then so is that of $U^\bullet$).

2.10.11 Corollary. *Every map of derived smooth manifolds $X \to Y$ is, locally near any point $x \in X$, a finite composition $X = Z_N \to \cdots \to Z_0 \to Z_{-1} = Y$ in which $Z_i \to Z_{i-1}$ is locally a pullback of the ith diagonal of $\mathbb{R}^{a_i}$ for some integers $a_i \geq 0$.*

Proof. Realize our given map $X \to Y$ (locally) as a levelwise submersion of truncated cosimplicial smooth manifolds $X^\bullet \to Y^\bullet$ (2.10.9). By replacing $X^\bullet$ with an open cosimplicial submanifold thereof, we may assume $X^\bullet \to X^\bullet$ is also Reedy submersive (**??**). Since $X^\bullet \to Y^\bullet$ is Reedy submersive, its relative matching maps all exist (1.3.18). Now for any map of cosimplicial objects $X^\bullet \to Y^\bullet$, the induced map on totalizations $X = \lim_{\mathbf{\Delta}} X^\bullet \to$

$\lim_{\mathbf{\Delta}} Y^\bullet = Y$ factors canonically as a (co-transfinite) composition $X = \varprojlim_i Z_i \to \cdots \to Z_2 \to Z_1 \to Z_0 \to Z_{-1} = Y$ where each map $Z_i \to Z_{i-1}$ is a pullback of the ith diagonal of the ith matching map $X^i \to M^i X^\bullet \times_{M^i Y^\bullet} Y^i$ (??). In our case, the inverse limit is achieved at some finite i (indeed, $X^\bullet$ and $Y^\bullet$ are both k-truncated for some $k < \infty$, so their ith matching maps are isomorphisms for $i > k$ (1.3.16), so the inverse limit is achieved at all $i \geq k$). The ith matching map is submersive at x since $X^\bullet \to Y^\bullet$ is Reedy submersive, and the diagonal of a pullback is a pullback of the diagonal (1.1.60). □

A given cosimplicial presentation of a derived smooth manifold (or of a morphism of derived smooth manifolds) may be much larger than necessary. Our next goal is to show (2.10.16) how to transform a given cosimplicial presentation into one which is 'minimal' in the following sense.

⋆ **2.10.12 Definition** (Minimal)**.** A cosimplicial vector space will be called minimal when the associated complex of vector spaces (1.3.20) has vanishing differential. A cosimplicial smooth manifold $X^\bullet$ will be called minimal at a point $x \in X^\bullet$ when the cosimplicial vector space $T_x X^\bullet$ is minimal. More generally, a levelwise submersion of cosimplicial smooth manifolds $X^\bullet \to Y^\bullet$ will be called minimal at $x \in X^\bullet$ when the cosimplicial vector space $T_x(X^\bullet/Y^\bullet)$ is minimal. The term 'minimal' without qualification means minimal at all points.

Recall that the chain complex $[\mathbb{Z}[k+1] \to \mathbb{Z}[k]] \in \mathsf{Kom}_{\geq 0}(\mathsf{Ab})$ corresponds under Dold–Kan (1.3.20) to the simplicial abelian group $C^k_{\mathrm{cell}}(\Delta^\bullet)$ (1.3.23.1). In what follows, we will default to real coefficients, so $C^k_{\mathrm{cell}}(\Delta^\bullet) = C^k_{\mathrm{cell}}(\Delta^\bullet; \mathbb{R})$ corresponds to $[\mathbb{R}[k+1] \to \mathbb{R}[k]]$. The dual cosimplicial vector spaces $C_k^{\mathrm{cell}}(\Delta^\bullet)$ will be of interest to us as cosimplicial smooth manifolds. The augmented cosimplicial diagram $* \to C_k^{\mathrm{cell}}(\Delta^\bullet)$ is a transverse limit diagram in Sm since the complex of vector spaces corresponding to the cosimplicial vector space $C_k^{\mathrm{cell}}(\Delta^\bullet)$ is acyclic.

2.10.13 Lemma. *For every $k \geq 0$, the augmented cosimplicial diagram $* \to C_k^{\mathrm{cell}}(\Delta^\bullet)$ is a limit diagram in* Der*.*

Proof. It suffices to show that they have the same space of maps to $\mathbb{R}$. The space of maps from $\lim_{\mathbf{\Delta}} C_k^{\mathrm{cell}}(\Delta^\bullet)$ to $\mathbb{R}$ is the colimit $\operatorname{colim}_{\mathbf{\Delta}} C^\infty(C_k^{\mathrm{cell}}(\Delta^\bullet))_0$ by (2.10.2.5) (where the subscript $_0$ indicates taking germs near zero). Thus we should show that the augmented simplicial diagram $C^\infty(C_k^{\mathrm{cell}}(\Delta^\bullet))_0 \to \mathbb{R}$ is a colimit diagram. Note that this augmented simplicial diagram can really just be denoted $C^\infty(C_k^{\mathrm{cell}}(\Delta^\bullet))_0$ if we follow the convention that $\Delta^{-1} = \varnothing$. It suffices to prove the extension property for maps from $(\Delta^r, \partial\Delta^r)$ to $C^\infty(C_k^{\mathrm{cell}}(\Delta^\bullet))_0$. That is, given a collection of smooth functions $f_I : C_k^{\mathrm{cell}}(\Delta^I) \to \mathbb{R}$ (or rather germs near zero of such) for every $I \subsetneqq \{0, \ldots, r\}$, we should produce a function $C_k^{\mathrm{cell}}(\Delta^r) \to \mathbb{R}$ whose restriction to $\Delta^I \subseteq \Delta^r$ is f_I for every $I \subsetneqq \{0, \ldots, r\}$ (the case $I = \varnothing$ corresponds to Δ^{-1}). We can take the function

$$\sum_{J \subsetneqq \{0,\ldots,r\}} (-1)^{r-1-|J|} f_J \circ (\Delta^J \to \Delta^r)^! \tag{2.10.13.1}$$

where $(\Delta^J \to \Delta^r)^! : C_k^{\mathrm{cell}}(\Delta^r) \to C_k^{\mathrm{cell}}(\Delta^J)$ is the brutal restriction of chains (simply throw away any k-simplices not contained in $\Delta^J \subseteq \Delta^r$). We should check that evaluating this function on $C_k^{\mathrm{cell}}(\Delta^I) \subseteq C_k^{\mathrm{cell}}(\Delta^r)$ yields f_I for every $I \subsetneqq \{0,\ldots,r\}$. It suffices to consider the case $I = \{0,\ldots,r\} \setminus a$ for some $0 \leq a \leq r$. The term $J = I$ gives the desired result, and the remaining terms cancel in pairs with the same intersection $J \cap I$. □

2.10.14 Definition (Elementary derived open embedding)**.** A map of truncated cosimplicial smooth manifolds $X^\bullet \to Y^\bullet$ will be called an *elementary derived open embedding* when it is a finite composition of the following sorts of maps:

(2.10.14.1) A levelwise open embedding $X^\bullet \to Y^\bullet$.

(2.10.14.2) A levelwise submersive pullback

$$\begin{array}{ccc} X^\bullet & \longrightarrow & Y^\bullet \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & C_k^{\mathrm{cell}}(\Delta^\bullet) \end{array}$$

for some $k \geq 0$.

If $X^\bullet \to Y^\bullet$ is an elementary derived open embedding, then the induced map on derived limits $\lim_{\mathbf{\Delta}} X^\bullet \to \lim_{\mathbf{\Delta}} Y^\bullet$ is an open embedding since $\mathsf{Sm} \to \mathsf{Der}$ preserves submersive pullbacks (**??**) and $\lim_{\mathbf{\Delta}} C_k^{\mathrm{cell}}(\Delta^\bullet) = *$ (2.10.13).

2.10.15 Exercise. Show that for any elementary derived open embedding $f : X^\bullet \to Y^\bullet$, the induced map on tangent spaces $T_x X^\bullet \to T_{f(x)} Y^\bullet$ corresponds to a quasi-isomorphism of cochain complexes under Dold–Kan (note the use of (1.3.24)).

⋆ **2.10.16 Proposition** (Existence of minimal cosimplicial presentations)**.** *Let $X^\bullet \to Y^\bullet$ be a levelwise submersion of cosimplicial smooth manifolds, and suppose $X^\bullet$ is truncated. For every point $x \in X^\bullet$, there exists an elementary derived open embedding $U^\bullet \to X^\bullet$ and a lift of x to $u \in U^\bullet$ such that the composition $U^\bullet \to Y^\bullet$ is submersive and minimal* (2.10.12) *at u.*

Proof. Suppose that f is non-minimal at x. That is, the simplicial vector space $T_x^*(X^\bullet/Y^\bullet)$ is non-minimal, meaning that the corresponding chain complex $N_\bullet T_x^*(X^\bullet/Y^\bullet)$ has non-vanishing differential. Intuitively, this means that there are some 'transverse directions' of $X^\bullet$ at x which we can 'cancel' (take transverse limit in Sm) while preserving submersivity of f.

Fix an injective map $[\mathbb{R}[k+1] \to \mathbb{R}[k]] \to N_\bullet T_x^*(X^\bullet/Y^\bullet)$ for some $k \geq 0$. Denote by $C^\infty(X^\bullet, x) \subseteq C^\infty(X^\bullet)$ the smooth functions vanishing at x, and let us try to find a lift $[\mathbb{R}[k+1] \to \mathbb{R}[k]] \to N_\bullet C^\infty(X^\bullet, x)$.

$$\begin{tikzcd} & N_\bullet C^\infty(X^\bullet, x) \arrow[d] \\ {[\mathbb{R}[k+1] \to \mathbb{R}[k]]} \arrow[r] \arrow[ur, dashed] & N_\bullet T_x^*(X^\bullet/Y^\bullet) \end{tikzcd} \tag{2.10.16.1}$$

The maps $C^\infty(X^\bullet, x) \to T_x^* X^\bullet \to T_x^*(X^\bullet/Y^\bullet)$ are levelwise surjective, so the corresponding maps of complexes $N_\bullet C^\infty(X^\bullet, x) \to N_\bullet T_x^* X^\bullet \to N_\bullet T_x^*(X^\bullet/Y^\bullet)$ are degreewise surjective

(1.3.24), hence the desired lift exists. This lift corresponds to a linear map $C^k_{\mathrm{cell}}(\Delta^\bullet) \to C^\infty(X^\bullet, x)$ (1.3.23.1), which is equivalently a smooth map

$$(X^\bullet, x) \to (C_k^{\mathrm{cell}}(\Delta^\bullet), 0). \tag{2.10.16.2}$$

By construction, the derivative of this map at x is the map $C^k_{\mathrm{cell}}(\Delta^\bullet) \to T^*_x(X^\bullet/Y^\bullet)$ corresponding to our chosen injection $[\mathbb{R}[k+1] \to \mathbb{R}[k]] \to N_\bullet T^*_x(X^\bullet/Y^\bullet)$. The derivative $C^k_{\mathrm{cell}}(\Delta^\bullet) \to T^*_x(X^\bullet/Y^\bullet)$ is thus also injective (1.3.24), so our map (2.10.16.2) is levelwise submersive at x.

We would now like to form the pullback of $0 \to C_k^{\mathrm{cell}}(\Delta^\bullet)$ under our map (2.10.16.2).

$$\begin{tikzcd} U^\bullet \arrow[r] \arrow[d] & X^\bullet \arrow[d] \\ 0 \arrow[r] & C_k^{\mathrm{cell}}(\Delta^\bullet) \end{tikzcd} \tag{2.10.16.3}$$

Since $X^\bullet \to C_k^{\mathrm{cell}}(\Delta^\bullet)$ is submersive at x, we may use (**??**) to replace $X^\bullet$ by an open cosimplicial submanifold thereof over which the map $X^\bullet \to C_k^{\mathrm{cell}}(\Delta^\bullet)$ is levelwise submersive, thus ensuring that the pullback $U^\bullet$ exists.

The map $U^\bullet \to Y^\bullet$ is levelwise submersive at $u = x \times_0 0$ by construction. Applying (**??**) again, we may find inside $U^\bullet$ an open cosimplicial submanifold over which this map is levelwise submersive. Finally, the rank of the differential of $N_\bullet T_x(U^\bullet/Y^\bullet)$ is one less than that of $N_\bullet T_x(X^\bullet/Y^\bullet)$, so by iterating this construction we eventually reach a $U^\bullet$ which is minimal over $Y^\bullet$ at u. □

⋆ **2.10.17 Corollary** (Finite products generate finite transverse limits)**.** *The category of smooth manifolds* Sm *has all finite transverse limits* (2.10.6)*, and a topological functor* Sm $\to$ C *preserves finite transverse limits iff it preserves finite products of copies of* $\mathbb{R}$*. In particular,* Sm $\to$ Der *preserves finite transverse limits.*

Proof. Due to the local nature of limits in topological ∞-sites (2.9.49), a topological functor Sm $\to$ C preserves finite products iff it preserves finite products of copies of $\mathbb{R}$. By the universal property of Sm $\to$ Der (**??**), a topological functor Sm $\to$ C preserving finite products extends uniquely to a topological functor Der $\to$ C preserving finite limits. It therefore suffices to show that Sm has finite transverse limits and that they are preserved by Sm $\to$ Der.

Fix a finite transverse diagram of smooth manifolds $D : J \to$ Sm, and let us show that $\lim D$ exists in Sm and is preserved by Sm $\to$ Der. Since Sm has finite products and Sm $\to$ Der preserves finite products, we may wlog replace D with its cosiftedization, which is represented by a truncated cosimplicial object $X^\bullet : \mathbf{\Delta} \to$ Sm (**??**). We may moreover assume $X^\bullet$ is minimal by (2.10.16) (noting that an elementary derived open embedding also induces an open embedding on limits in Sm provided these exist, and that it preserves transversality (2.10.15)). Now if $X^\bullet$ is minimal at $x \in X^\bullet$ and transverse, we can construct an open embedding covering x which is constant. Indeed, just work by induction applying (2.10.10) to replace each level X^n with an open subset over which the nth matching map is an open embedding; this makes all matching maps isomorphisms, hence we win.

The functor $\mathsf{Sm} \to \mathsf{Der}$ preserves finite products (2.10.2.2), hence preserves finite transverse limits. $\square$

* *

We saw earlier that every morphism of derived smooth manifolds is, locally on the source, a finite composition of maps which are, locally on the source, a pullback of $\mathbb{R} \to *$ or one of its iterated diagonals (2.10.11). The 'amplitude' of a morphism of derived smooth manifolds records which iterated diagonals are relevant.

⋆ **2.10.18 Definition** (Amplitude)**.** Let $I \subseteq \mathbb{Z}_{\geq 0}$. A morphism of derived smooth manifolds is said to have *amplitude* $\subseteq I$ when it is, locally on the source, a finite composition of maps which are, locally on the source, a pullback of the ith diagonal (1.1.57) of $\mathbb{R}$ for some non-negative integer $i \in I$.

Let us get a handle on the iterated diagonals of $\mathbb{R} \to *$. The first diagonal is $\mathbb{R} \to \mathbb{R} \times \mathbb{R}$, which is a pullback of $* \to \mathbb{R}$, and conversely $* \to \mathbb{R}$ is a pullback of $\mathbb{R} \to \mathbb{R} \times \mathbb{R}$; both are transverse pullbacks in Sm, hence are pullbacks in Der as well. Now the diagonal of a pullback is a pullback of the diagonal (1.1.60), so being a pullback of the kth diagonal of $\mathbb{R} \to *$ is, for $k \geq 1$, equivalent to being a pullback of the $(k-1)$st diagonal of $* \to \mathbb{R}$. The ath iterated diagonal of $* \to \mathbb{R}$ is $* \to \Omega^a \mathbb{R}$, where Ω^a is the ath based loop space in the ∞-categorical sense, namely the limit over the D^a-shaped diagram in Der taking the value $*$ on the boundary and the value $\mathbb{R}$ in the interior.

2.10.19 Exercise. Show that having amplitude $\subseteq I$ is preserved under pullback and closed under composition. Show that if $X \to Y$ has amplitude $\subseteq I$, then its relative diagonal $X \to X \times_Y X$ has amplitude $\subseteq I+1$. Formulate the resulting cancellation (1.1.62) statement for amplitude. Conclude, in particular, that every morphism of smooth manifolds has amplitude ≤ 1. Unwind the reasoning to explicitly express any morphism of smooth manifolds as the composition of an immersion followed by a submersion.

2.10.20 Definition (Submersion)**.** A morphism of derived smooth manifolds is called a submersion when it has amplitude 0 (equivalently, when it is, locally on the source, a pullback of $\mathbb{R}^a \to *$).

We now explain the notion of a vector bundle on a derived smooth manifold. Since this is an ∞-categorical context, a certain amount of abstraction is required to describe the relevant systems of higher homotopies in a manageable way. Explicitly, a vector bundle on a derived smooth manifold X is an open cover $X = \bigcup_i U_i$, integers $n_i \geq 0$, transition functions $\varphi_{ij} : U_i \cap U_j \to \operatorname{Hom}(\mathbb{R}^{n_j}, \mathbb{R}^{n_i})$ ($\varphi_{ii} = \mathbf{1}$), homotopies $\varphi_{ijk} : \varphi_{ij}\varphi_{jk} \to \varphi_{ik}$ over $U_i \cap U_j \cap U_k$, and higher homotopies $\varphi_{i_0 \cdots i_p}$ over $U_{i_0} \cap \cdots \cap U_{i_p}$ for all $p \geq 3$; morphisms of vector bundles may be defined similarly. It becomes prohibitively complex to manipulate explicitly such systems of homotopies, so a more categorical perspective is required.

⋆ **2.10.21 Definition** (Tangent complex)**.** The tangent complex functor on derived smooth manifolds is a section $T : \mathsf{Der} \to \mathsf{Perf}^{\geq 0} \rtimes \mathsf{Der}$ of the cartesian functor $\mathsf{Perf}^{\geq 0} \rtimes \mathsf{Der} \to \mathsf{Der}$ encoding the functor $\mathsf{Perf}^{\geq 0} : \mathsf{Der} \to \mathsf{Cat}_\infty$. In other words, it assigns to each derived smooth manifold X a perfect complex $TX \in \mathsf{Perf}^{\geq 0}(X)$, to each morphism of derived smooth manifolds $f : X \to Y$ a morphism $TX \to f^*TY$, and coherent homotopies for every chain of morphisms $X_0 \to \cdots \to X_p$ with $p \geq 2$.

The tangent complex functor is defined uniquely up to contractible choice by the requirement that it preserve finite limits and that the following diagram commute.

$$\begin{tikzcd}
\mathsf{Vect} \rtimes \mathsf{Sm} \arrow[r] \arrow[d] & \mathsf{Perf}^{\geq 0} \rtimes \mathsf{Der} \arrow[d] \\
\mathsf{Sm} \arrow[r] \arrow[u, bend left, "T"] & \mathsf{Der} \arrow[u, bend left, "T"]
\end{tikzcd} \tag{2.10.21.1}$$

In other words, the tangent functor on derived smooth manifolds preserves finite limits and solves the following lifting problem.

$$\begin{tikzcd}
\mathsf{Sm} \arrow[r] \arrow[d] & \mathsf{Perf}^{\geq 0} \rtimes \mathsf{Der} \arrow[d] \\
\mathsf{Der} \arrow[r, equal] \arrow[ur, dashed] & \mathsf{Der}
\end{tikzcd} \tag{2.10.21.2}$$

The tangent bundle functor on smooth manifolds $T : \mathsf{Sm} \to \mathsf{Vect} \rtimes \mathsf{Sm}$ (2.6.10) preserves finite products, as do the inclusions $\mathsf{Vect} \rtimes \mathsf{Sm} \hookrightarrow \mathsf{Vect} \rtimes \mathsf{Der} \hookrightarrow \mathsf{Perf}^{\geq 0} \rtimes \mathsf{Der}$, hence so does their composition $\mathsf{Sm} \to \mathsf{Perf}^{\geq 0} \rtimes \mathsf{Der}$. Now the universal property of $\mathsf{Sm} \hookrightarrow \mathsf{Der}$, namely that it freely adjoins finite limits modulo preserving finite products within the realm of perfect topological sites, implies the space of lifts (**??**) is contractible.

Concretely, the tangent complex of a derived smooth manifold X may be described as follows. Suppose X is the limit of a finite diagram $p : K \to \mathsf{Sm}$ of smooth manifolds. The pair $(TX, X) \in \mathsf{Perf}^{\geq 0} \rtimes \mathsf{Der}$ is then the limit of $Tp : K \to \mathsf{Vect} \rtimes \mathsf{Sm} \subseteq \mathsf{Perf}^{\geq 0} \rtimes \mathsf{Der}$. This limit may be computed by first taking the limit in Der and then taking the relative limit (1.5.87) in $\mathsf{Perf}^{\geq 0} \rtimes \mathsf{Der} \to \mathsf{Der}$, which in this case is the limit in a fiber (**??**). Thus $TX \in \mathsf{Perf}^{\geq 0}(X)$ is the limit of the diagram $K \to \mathsf{Vect}(X) \subseteq \mathsf{Perf}^{\geq 0}(X)$ obtained by pulling back the diagram Tp to X.

2.10.22 Example. Consider a finite diagram of smooth manifolds $p : K \to \mathsf{Sm}$, and consider its derived limit $\lim_K^{\mathsf{Der}} p$. The fiber of the tangent complex of $\lim_K^{\mathsf{Der}} p$ at a point x is the limit $\lim_K^{\mathsf{K}^{\geq 0}(\mathsf{Vect})} T_x p$ (since the 'fiber at x' functor $\mathsf{Perf}^{\geq 0}(X) \to \mathsf{Perf}^{\geq 0}(*) = \mathsf{K}^{\geq 0}(\mathsf{Vect})$ preserves finite limits (**??**)). Recall that the diagram p is called transverse at x precisely when this limit lies in $\mathsf{Vect} \subseteq \mathsf{K}^{\geq 0}(\mathsf{Vect})$ (2.10.6). Thus if p is not transverse, then the tangent complex of $\lim_K^{\mathsf{Der}} p$ is not concentrated in degree zero, and so the derived limit $\lim_K^{\mathsf{Der}} p$ is not a smooth manifold. Thus the result that $\mathsf{Sm} \to \mathsf{Der}$ preserves transverse limits (2.10.17) is sharp: if p is non-transverse, then its derived limit does not lie in the full subcategory $\mathsf{Sm} \subseteq \mathsf{Der}$.

2.10.23 Exercise. Let $X = s^{-1}(0)$ be the derived zero set of a section $s : M \to E$ of a vector bundle E over a smooth manifold M. The map of vector bundles $ds : TM \to E$ on M depends on a choice of connection on E. Fixing any choice of connection, show that the cone of this map, restricted to X, is the tangent complex TX.

2.10.24 Exercise. Show that for any point x of a derived smooth manifold X, there exists a function $(X, x) \to (\mathbb{R}, 0)$ with any prescribed derivative $T^0_x X \to \mathbb{R}$ at x.

2.10.25 Definition (Relative tangent complex)**.** For any map of derived smooth manifolds $f : X \to Y$, the relative tangent complex $T_{X/Y}$ is the fiber product

$$\begin{tikzcd} T_{X/Y} \arrow[r] \arrow[d] & TX \arrow[d, "Tf"] \\ 0 \arrow[r] & f^*TY \end{tikzcd} \tag{2.10.25.1}$$

in $\mathsf{Perf}^{\geq 0}(X)$ (in other words, it is the cone $T_{X/Y} = [TX \to f^*TY[-1]]$).

$\star$ **2.10.26 Proposition** (Minimal amplitude factorization)**.** *Every map of derived smooth manifolds $X \to Y$ is, locally near any point $x \in X$, a finite composition $X = Z_N \to \cdots \to Z_0 \to Z_{-1} = Y$ in which $Z_i \to Z_{i-1}$ is locally a pullback of the ith diagonal of $T^i_x(X/Y)$.*

Proof. Recall the argument of (2.10.11), which showed that presenting our input map $X \to Y$ by a submersive and Reedy submersive map of cosimplicial smooth manifolds $X^\bullet \to Y^\bullet$ (such a presentation always exists) gives rise to a factorization in which $Z_i \to Z_{i-1}$ is a pullback of the ith diagonal of (the vertical tangent space of) the ith relative matching map $X^i \to M^i X^\bullet \times_{M^i Y^\bullet} Y^i$. The key to the present result is the fact that every Reedy submersive presentation $X^\bullet \to Y^\bullet$ can be refined to one which is *minimal* at x (2.10.16), which we recall means that the cosimplicial vector space $T_x(X^\bullet/Y^\bullet)$ maps under the Dold–Kan correspondence to a cochain complex with vanishing differential.

It suffices therefore to match the vertical tangent space of the ith relative matching map of $X^\bullet \to Y^\bullet$ at x with $T^i_x(X/Y)$ when $X^\bullet \to Y^\bullet$ is minimal at x; this is now simply a matter of unwinding definitions. The vertical tangent space of the ith matching map of $X^\bullet \to Y^\bullet$ is the kernel of the ith matching map of $T_x X^\bullet \to T_x Y^\bullet$ (the pullbacks involved in the construction (1.3.18) of the matching map of $X^\bullet \to Y^\bullet$ are all submersive, so they are preserved by passing to the tangent space at x). The kernel of the ith matching map of $T_x X^\bullet \to T_x Y^\bullet$ is in turn identified (1.3.26) with the kernel of the map on normalized cochain complexes $N^\bullet T_x X^\bullet \to N^\bullet T_x Y^\bullet$ in degree i. Now $X^\bullet \to Y^\bullet$ is levelwise submersive, so $N^\bullet T_x X^\bullet \to N^\bullet T_x Y^\bullet$ is degreewise surjective (1.3.24), so its kernel is its fiber in $\mathsf{K}^{\geq 0}(\mathsf{Vect})$. Since $X^\bullet \to Y^\bullet$ is minimal, the kernel of $N^\bullet T_x X^\bullet \to N^\bullet T_x Y^\bullet$ has vanishing differential, so the space in question is thus $H^i(N^\bullet T_x X^\bullet \to N^\bullet T_x Y^\bullet[-1])$. The normalized cochain complex of a cosimplicial vector space is the same as its limit in $\mathsf{K}^{\geq 0}(\mathsf{Vect})$ (**??**), so this is the same as the ith cohomology of $[\lim_{\boldsymbol{\Delta}} T_x X^\bullet \to \lim_{\boldsymbol{\Delta}} T_x Y^\bullet[-1]] = T_x(X/Y)$ as desired. $\square$

$\star$ **2.10.27 Definition** (Derived Lie group)**.** A *derived Lie group* is a group object (**??**) in the ∞-category of derived smooth manifolds Der.

A derived Lie group is Hausdorff and paracompact for the same reason as a Lie group (2.6.26).

⋆ **2.10.28 Definition** (Universal tangent vector τ)**.** We denote by τ the derived zero set of the function x^2, namely the fiber product

$$\begin{tikzcd} \tau \arrow[r] \arrow[d] & * \arrow[d, "0"] \\ \mathbb{R} \arrow[r, "x\mapsto x^2"] & \mathbb{R} \end{tikzcd} \tag{2.10.28.1}$$

in the category Der.

2.10.29 Proposition. *Let M be a smooth manifold, and consider the derived zero set $s^{-1}(0) \in \mathsf{Der}$ of a function $s : M \to \mathbb{R}$ whose (topological) zero set has empty interior. The sheaf $C^\infty_{s^{-1}(0)}$ of real valued functions on $s^{-1}(0)$ is the quotient (as a sheaf of sets) C^∞_M/s of the sheaf C^∞_M of real valued functions on M by the equivalence relation of equality modulo s.*

Proof. We present the derived zero set $s^{-1}(0)$ as a cosimplicial smooth manifold as follows. The point $*$ is the limit of the cosimplicial smooth manifold $C_0^{\mathrm{cell}}(\Delta^\bullet)$ (2.10.13), and the augmentation map $C_0^{\mathrm{cell}}(\Delta^\bullet) \to \mathbb{R}$ is surjective, hence submersive, by inspection. Hence

$$M \times_{\mathbb{R}} 0 = M \times_{\mathbb{R}} \lim_{\boldsymbol{\Delta}} C_0^{\mathrm{cell}}(\Delta^\bullet) = \lim_{\boldsymbol{\Delta}} (M \times_{\mathbb{R}} C_0^{\mathrm{cell}}(\Delta^\bullet)) \tag{2.10.29.1}$$

is a cosimplicial smooth manifold presenting the derived zero set $s^{-1}(0)$. It is truncated since $C_0^{\mathrm{cell}}(\Delta^\bullet)$ is truncated. Formation of the sheaf of smooth functions commutes with totalizations of truncated cosimplicial objects (by axiom (2.10.2.5) of derived smooth manifolds), so we obtain

$$C^\infty_{s^{-1}(0)} = \operatorname*{colim}_{\boldsymbol{\Delta}} C^\infty_{M\times_{\mathbb{R}} C_0^{\mathrm{cell}}(\Delta^\bullet)}|_{s^{-1}(0)}. \tag{2.10.29.2}$$

Recall that the colimit functor $\operatorname{colim}_{\boldsymbol{\Delta}} : \mathsf{sSet} \to \mathsf{Spc}$ simply amounts to regarding a simplicial set as a space in the obvious way (**??**).

Explicitly, the cosimplicial smooth manifold $M \times_{\mathbb{R}} C_0^{\mathrm{cell}}(\Delta^\bullet)$ is as follows.

$$M \overset{\substack{(p,s(p))\\(p,0)}}{\rightrightarrows} M \times \mathbb{R} \overset{\substack{(p,s(p),x)\\(p,x,x)\\(p,x,0)}}{\rightrightarrows} M \times \mathbb{R}^2 \overset{\substack{(p,s(p),x,y)\\(p,x,x,y)\\(p,x,y,y)\\(p,x,y,0)}}{\rightrightarrows} M \times \mathbb{R}^3 \overset{\substack{(p,s(p),x,y,z)\\(p,x,x,y,z)\\(p,x,y,y,z)\\(p,x,y,z,z)\\(p,x,y,z,0)}}{\rightrightarrows} \cdots \tag{2.10.29.3}$$

Every function on $M \times \mathbb{R}$ is uniquely of the form $f(p) + x(g(p) + (x - s(p))h(p, x)))$ by Hadamard's Lemma (2.6.33) (applied twice). Such a function pulls back under the maps $M \rightrightarrows M \times \mathbb{R}$ to a pair of functions on M of the form $(f(p), f(p) + s(p)g(p))$. Thus two elements of $C^\infty(M)$ are joined by an edge in $C^\infty(M \times_{\mathbb{R}} C_0^{\mathrm{cell}}(\Delta^\bullet))$ iff they are congruent modulo s. We conclude that $\pi_0 C^\infty(M \times_{\mathbb{R}} C_0^{\mathrm{cell}}(\Delta^\bullet)) = C^\infty(M)/s$.

Now it remains to show that $\pi_k C^\infty(M \times_{\mathbb{R}} C_0^{\mathrm{cell}}(\Delta^\bullet)) = 0$ for $k > 0$. This is a simplicial abelian group, hence a Kan complex (1.4.13), so it suffices to show that every map $(\Delta^k, \partial\Delta^k) \to (C^\infty(M \times_{\mathbb{R}} C_0^{\mathrm{cell}}(\Delta^\bullet)), 0)$ for $k > 0$ is null-homotopic. Such a map consists of a function $F : M \times \mathbb{R}^k \to \mathbb{R}$ whose the restrictions

$$F(p, s(p), y_1, \ldots, y_{k-1}) \tag{2.10.29.4}$$
$$F(p, y_1, y_1, \ldots, y_{k-1}) \tag{2.10.29.5}$$
$$\vdots$$
$$F(p, y_1, \ldots, y_i, y_i, \ldots, y_{k-1}) \tag{2.10.29.6}$$
$$\vdots$$
$$F(p, y_1, \ldots, y_{k-1}, y_{k-1}) \tag{2.10.29.7}$$
$$F(p, y_1, \ldots, y_{k-1}, 0) \tag{2.10.29.8}$$

all vanish. By Hadamard's Lemma (2.6.33), the last of these vanishing conditions implies that $F(p, x_1, \ldots, x_k) = x_k G(p, x_1, \ldots, x_k)$ for some smooth G. Now consider the $(k+1)$-simplex given by $(z_k - z_{k+1})G(p, z_1, \ldots, z_k)$. Its face $z_{k+1} = 0$ is our given simplex Δ^k, so it suffices to show that all other faces are zero. That is, we should show that

$$G(p, s(p), y_1, \ldots, y_{k-1}) \tag{2.10.29.9}$$
$$G(p, y_1, y_1, \ldots, y_{k-1}) \tag{2.10.29.10}$$
$$\vdots$$
$$G(p, y_1, \ldots, y_i, y_i, \ldots, y_{k-1}) \tag{2.10.29.11}$$
$$\vdots$$
$$G(p, y_1, \ldots, y_{k-1}, y_{k-1}) \tag{2.10.29.12}$$

all vanish. These are the same pullbacks which are known to annihilate F. The difference is that now we have divided by x_k, so it suffices to show that the inverse image of the locus $\{x_k = 0\}$ along all such pullbacks is nowhere dense. When $k > 1$, every such inverse image is simply $\{y_{k-1} = 0\}$, which is evidently nowhere dense. For $k = 1$, the inverse image is $s^{-1}(0) \subseteq M$, which is nowhere dense by hypothesis. □

2.10.30 Exercise. Deduce from (2.10.29) (formally, without repeating the proof) a similar characterization of the sheaf of maps to any smooth manifold N (use Hadamard's Lemma (2.6.33) to show that there is a well defined notion of 'equality modulo s' for maps to N, independent of the choice of coordinate charts of N used to define it).

⋆ **2.10.31 Proposition.** *The functor* $\underline{\mathrm{Hom}}(\tau, -) : \mathsf{Der} \to \mathsf{Der}$ *exists and is canonically isomorphic to the tangent functor* $T : \mathsf{Der} \to \mathsf{Der}$.

Proof. For smooth manifolds M and N, the sheaf of functions $M \times \tau \to N$ is computed in (2.10.29) to equal the sheaf of functions $M \times \mathbb{R} \to N$ modulo x^2 (indeed, $M \times \tau \to M \times \mathbb{R}$

is the derived zero set of the function $(p,x)\mapsto x^2$ and has empty interior). The sheaf of functions $M\times\mathbb{R}\to N$ modulo x^2 is in turn identified with the sheaf of functions $M\to TN$ by taking derivative in the x direction by Hadamard's Lemma (2.6.33). These identifications are functorial in M, hence exhibit TN as representing the functor $\underline{\mathrm{Hom}}(\tau,N)$ on Sm. They are also functorial in N, hence define in fact an isomorphism $\underline{\mathrm{Hom}}(\tau,-)=T$ of functors $\mathsf{Sm}\to\mathsf{Sm}$.

Now the $\underline{\mathrm{Hom}}(\tau,-)$ and $よ_{\mathsf{Der}}\circ T$ are both topological functors $\mathsf{Der}\to\mathsf{Shv}(\mathsf{Der})$ which preserve finite limits. The isomorphism between their restrictions to Sm extends uniquely to Der by the universal property of $\mathsf{Sm}\to\mathsf{Der}$ (**??**). □

⋆ **2.10.32 Definition** (Tangent functor $T:\mathsf{Shv}(\mathsf{Der})\to\mathsf{Shv}(\mathsf{Der})$)**.** The tangent functor on derived smooth stacks $T:\mathsf{Shv}(\mathsf{Der})\to\mathsf{Shv}(\mathsf{Der})$ is the unique cocontinuous extension (2.9.32) of the tangent functor on derived smooth manifolds $T:\mathsf{Der}\to\mathsf{Der}$ (2.10.21).

$$\begin{tikzcd} \mathsf{Der} \arrow[r, "T_{\mathsf{Der}}"] \arrow[d] & \mathsf{Der} \arrow[d] \\ \mathsf{Shv}(\mathsf{Der}) \arrow[r, dashed, "T_{\mathsf{Shv}(\mathsf{Der})}"] & \mathsf{Shv}(\mathsf{Der}) \end{tikzcd} \tag{2.10.32.1}$$

In other words, $T_{\mathsf{Shv}(\mathsf{Der})}$ is the left Kan extension $(T_{\mathsf{Der}})_!$ of $T_{\mathsf{Der}}:\mathsf{Der}\to\mathsf{Der}$ (2.9.35). Alternatively, $T_{\mathsf{Shv}(\mathsf{Der})}$ is the sheaf pullback functor $(-\times\tau)^*:\mathsf{Shv}(\mathsf{Der})\to\mathsf{Shv}(\mathsf{Der})$ under multiplication by the universal tangent vector $\times\tau:\mathsf{Der}\to\mathsf{Der}$ (2.10.28). Indeed, the sheaf pullback functor $(-\times\tau)^*$ is cocontinuous since $\times\tau$ preserves underlying topological spaces (2.9.38), and its restriction to $\mathsf{Der}\subseteq\mathsf{Shv}(\mathsf{Der})$ is canonically identified with $T:\mathsf{Der}\to\mathsf{Der}$ (2.10.31). This description shows that $T_{\mathsf{Shv}(\mathsf{Der})}$ is continuous (every sheaf pullback functor is continuous (2.9.35)).

2.10.33 Exercise (Tangent space of $\underline{\mathrm{Sec}}$)**.** Recall that $T_{Q/M}$ is the pullback of $TQ\to TM$ under the zero section $M\to TM$. Conclude that a map $Z\to T_{Q/M}$ from a derived smooth manifold Z is a diagram of the following shape.

$$\begin{tikzcd} Z\times\tau \arrow[r] \arrow[d, "p_Z"'] & Q \arrow[d] \\ Z \arrow[r] & M \end{tikzcd} \tag{2.10.33.1}$$

Conclude that a map $Z\to\underline{\mathrm{Sec}}(M,T_{Q/M})$ is the same as a diagram

$$\begin{tikzcd} Z\times M\times\tau \arrow[r] \arrow[d, "p_{Z\times M}"'] & Q \arrow[d] \\ Z\times M \arrow[r, "p_M"] & M \end{tikzcd} \tag{2.10.33.2}$$

which in turn is the same as a map $Z\times\tau\to\underline{\mathrm{Sec}}(M,Q)$, thus identifying $T\underline{\mathrm{Sec}}(M,Q)=\underline{\mathrm{Sec}}(M,T_{Q/M})$. Generalize this argument to show that $T(\underline{\mathrm{Sec}}_B(M,Q)/B)=\underline{\mathrm{Sec}}_B(M,T_{Q/M})$.

* *

It is sometimes possible to deduce statements about derived smooth manifolds as a formal consequence of the special case of smooth manifolds. The underlying engine behind such results is the left Kan extesion functor $\mathsf{Shv}(\mathsf{Sm}) \hookrightarrow \mathsf{Shv}(\mathsf{Der})$ (fully faithful since $\mathsf{Sm} \hookrightarrow \mathsf{Der}$ is fully faithful (2.9.41)) and a characterization of its essential image based on (2.10.2.5). Stated informally, a sheaf $F : \mathsf{Der}^{\mathsf{op}} \to \mathsf{Spc}$ is left Kan extended from Sm when every morphism $Q \to F$ from a derived smooth manifold Q factors, uniquely up to contractible choice, through a smooth manifold $Q \to M \to F$. Formulated in this way, it is not so surprising that certain $F \in \mathsf{Shv}(\mathsf{Der})$ being left Kan extended from Sm allows us to deduce results about derived smooth manifolds from the special case of smooth manifolds.

⋆ **2.10.34 Proposition.** *The essential image of the left Kan extension functor*

$$(\mathsf{Sm} \to \mathsf{Der})_! : \mathsf{Shv}(\mathsf{Sm}) \to \mathsf{Shv}(\mathsf{Der}) \tag{2.10.34.1}$$

consists precisely of those sheaves on Der *which topologically preserve* (2.10.1) *finite cosifted limits* (**??**) *(equivalently, finite cosifted limits of smooth manifolds).*

Proof. Yoneda functors $\operatorname{Hom}(-, N)$ for $N \in \mathsf{Sm}$ topologically preserve finite cosifted limits by definition (2.10.2.5). The essential image of left Kan extension is the closure of such Yoneda functors under colimits (**??**). We claim that the collection of sheaves on Der which topologically preserve finite cosifted limits is closed under colimits. It suffices to show that for any fixed diagram $K^{\triangleleft} \to \mathsf{Top}$, the collection of lifts to $\mathsf{Shv}(-)^{\mathsf{op}} \rtimes \mathsf{Top}$ which are relative limit diagrams is closed under limits inside the ∞-category of all lifts. Since relative limits are limits in fibers (functorially) (**??**) and the pullback functors between sheaf categories preserve colimits (being left adjoints), we are reduced to the fact (**??**) that limit diagrams inside $\mathsf{Fun}(K^{\triangleleft}, \mathsf{E})$ are closed under limits for any ∞-category E (in this case $\mathsf{E} = \mathsf{Shv}(X)$ for $X \in \mathsf{Top}$ the cone point of our fixed diagram $K^{\triangleleft} \to \mathsf{Top}$).

We have thus shown that if $F \in \mathsf{Shv}(\mathsf{Der})$ is left Kan extended from Sm then F topologically preserves finite cosifted limits. To show the converse, it suffices (1.1.81) to check that if $F, G \in \mathsf{Shv}(\mathsf{Der})$ both topologically preserve finite cosifted limits, then a morphism $F \to G$ is an isomorphism iff its restriction to Sm is an isomorphism. This fact follows immediately from the fact that every derived smooth manifold is locally a finite cosifted limit of smooth manifolds (2.10.2.3) (a finite limit of smooth manifolds is also a finite cosifted limit by applying cosiftedization, since $\mathsf{Sm} \to \mathsf{Der}$ preserves finite products (2.10.2.2)(**??**))—that is just axiom (2.10.2.5). □

We now argue that certain stacks of interest on Der are left Kan extended from Sm using the criterion (2.10.34). This will allow us to deduce facts about derived smooth manifolds as formal consequences of corresponding facts about smooth manifolds.

2.10.35 Proposition. *Let M be a compact Hausdorff derived smooth manifold, and let N be a smooth manifold. The derived smooth stack $\underline{\operatorname{Hom}}(M, N)$ is left Kan extended from* Sm.

2.10.36 Corollary. *The stack of proper submersions on* Der *is left Kan extended from* Sm.

Proof. Proper submersions in Sm and Der are locally trivial (2.6.30)(**??**). It follows that the stack of proper submersions (on both Sm and Der) is the disjoint union over all diffeomorphism classes of compact Hausdorff smooth manifolds F of the stack quotient

$$*/\underline{\mathrm{Diff}}(F) = \operatorname{colim}\Big(\cdots \substack{\rightarrow\\\rightarrow\\\rightarrow} \underline{\mathrm{Diff}}(F)^2 \rightrightarrows \underline{\mathrm{Diff}}(F) \to *\Big) \tag{2.10.36.1}$$

where $\underline{\mathrm{Diff}}(F) \subseteq \underline{\mathrm{Hom}}(F, F)$ denotes the open substack of diffeomorphisms. Left Kan extension preserves $\underline{\mathrm{Hom}}(F, F)$ (2.10.35), open substacks (2.9.35), finite products (2.9.40), and colimits (since it is a left adjoint). $\square$

2.11 Derived smooth stacks

We now study derived smooth stacks, seeking a theory parallel to that of smooth stacks (2.7).

2.11.1 Definition (Submersive atlas)**.** A submersive atlas on a derived smooth stack X is a representable submersion $U \to X$ from a derived smooth manifold U.

Here is a generalization of (2.7.10).

2.11.2 Lemma. *Let X be a derived smooth stack, and let $U \to X$ be a submersion from a derived smooth manifold U. For $x \in X$, consider the map $U \times_X x \to U$ (from a smooth manifold to a derived smooth manifold). This map factors locally as the composition of a surjective submersion with vertical tangent space $\ker((T_{U/X})_u \to T^0U)$ and a map from the resulting quotient manifold with tangent space $\operatorname{im}((T_{U/X})_u \to T^0U)$ to U acting on tangent spaces via the tautological inclusion.*

Proof. The relative tangent bundle of the map $U \times_X x \to U$ is the cone of $T_{U/X} \to TU$ and is identified with (the pullback of) the relative tangent bundle of $* \to X$, namely the constant bundle with fiber $T_xX[-1]$. It follows that the rank of the tangent cohomology T^iU is constant over $U \times_X x$ as is the rank of the map $(T_{U/X})_u = T(U \times_X u) \to T^0U$. Now we can express U (locally) as the limit of a cosimplicial smooth manifold $U^\bullet$, and the functor of maps from smooth manifolds to U is (by the universal property of limits) the functor of maps to U^0 which land inside $|U| \subseteq |U^0|$. The structure theorem for maps of smooth manifolds with constant rank derivative thus applies to the map $U \times_X x \to U$. ☐

2.11.3 Definition (Minimal atlas)**.** Given a derived smooth stack X with a submersive atlas, a submersion $U \to X$ from $U \in \mathsf{Der}$ is called *minimal* at $u \in U$ when the map $T_{U/X} \to TU$ vanishes at u.

2.11.4 Lemma (Proper atlas from proper diagonal)**.** *Let X be a derived smooth stack with proper diagonal, and let $U \to X$ be a submersion which is minimal at $p \in U$. For every sufficiently small open neighborhood $p \in V \subseteq U$, we have $p \times_X p = p \times_X V$ and the map $V \to X$ is proper over an open substack of X containing the image of p.*

Proof. Given the purely topological result (2.3.17), it suffices to show that $p \times_X p \subseteq p \times_X U$ is open, which follows from minimality (2.11.2). ☐

2.11.5 Lemma (Existence of a minimal atlas)**.** *Let X be a derived smooth stack, and let $x \in X$ be a point. If X admits an submersive atlas, then it admits a submersive atlas which is minimal at some lift of x.*

Proof. Begin with an arbitrary submersive atlas $U \to X$ and a lift $u \in U$ of x. If $V \subseteq U$ is the zero set of a map $U \to \mathbb{R}^k$, then $V \to X$ is a submersion iff the relative tangent complex $T_{V/X}$ is supported in degree zero. This relative tangent complex is the cone $[T_{U/X} \to \mathbb{R}^k[-1]]$ of the composition $T_{U/X} \to TU \to \mathbb{R}^k$, so $V \to X$ is a submersion iff this composition is surjective over V. Now the image of the map $T_{U/X} \to TU$ at u is some subspace of T^0_uU. Choose a map $U \to \mathbb{R}^k$ vanishing at u whose derivative at u restricted to this subspace T^0_uU is an isomorphism (2.10.24). Now the resulting submersion $V \to X$ is minimal. ☐

2.12 Hybrid categories

In this section, we introduce 'hybrid categories'. The simplest of these is the category we denote by TopSm, whose objects we will call topological-smooth spaces. Topological-smooth spaces are locally modelled on products $Z \times \mathbb{R}^n$ for topological spaces Z. Morphisms of topological-smooth spaces, called continuous-smooth maps, are maps $Z \times \mathbb{R}^n \to Z' \times \mathbb{R}^{n'}$ which locally preserve the decomposition into 'leaves' $z \times \mathbb{R}^n$ and whose derivatives to all orders along the leaves (i.e. in the $\mathbb{R}^n$ coordinate) exist and are continuous. This category allows one to make sense of notions such as 'a family of smooth manifolds parameterized by a topological space'. It also provides a context in which to define topological spaces $\underline{\mathrm{Hom}}(X, Y)$ of smooth maps between smooth manfolds X and Y via a universal property analogous to that used to define the topological spaces $\underline{\mathrm{Hom}}(X, Y)$ of continuous maps between topological spaces X and Y (2.4.1).

* *

Let us now introduce the category of topological-smooth spaces TopSm.

2.12.1 Definition (Continuous-smooth map)**.** Let Z be a topological space and let $n \geq 0$. We consider maps defined on the product $Z \times \mathbb{R}^n$ or any open subset thereof.

A map $f : Z \times \mathbb{R}^n \to Z'$ (any topological space Z') will be called *continuous-smooth* when it is, locally on the source, the composition of the projection $Z \times \mathbb{R}^n \to Z$ and a continuous map $Z \to Z'$. A map $f : Z \times \mathbb{R}^n \to \mathbb{R}$ will be called *continuous-smooth* iff its derivative $D^\alpha f : Z \times \mathbb{R}^n \to \mathbb{R}$ exists and is continuous for every multi-index α on $\mathbb{R}^n$. A map $f : Z \times \mathbb{R}^n \to Z' \times \mathbb{R}^{n'}$ will be called *continuous-smooth* when its coordinate factors $Z \times \mathbb{R}^n \to Z'$ and $Z \times \mathbb{R}^n \to \mathbb{R}$ are all continuous-smooth.

The notion of a continuous-smooth map manifestly depends on the expression of its source and target as the product of a topological space and a Euclidean space.

2.12.2 Exercise. Determine what are the continuous-smooth maps

$$|\mathbb{R}^n| \times * \to |\mathbb{R}^n| \times * \qquad\qquad * \times \mathbb{R}^n \to |\mathbb{R}^n| \times * \tag{2.12.2.1}$$

$$|\mathbb{R}^n| \times * \to * \times \mathbb{R}^n \qquad\qquad * \times \mathbb{R}^n \to * \times \mathbb{R}^n \tag{2.12.2.2}$$

where we write $|\mathbb{R}^n|$ to denote the topological space underlying the smooth manifold $\mathbb{R}^n$ (so as to distinguish the topological and smooth factors). Do the same with the domains replaced with arbitrary open subsets thereof.

2.12.3 Lemma. *A composition of continuous-smooth maps is continuous-smooth.*

Proof. We consider a composition $Z \times \mathbb{R}^n \to Z' \times \mathbb{R}^{n'} \to Z'' \times \mathbb{R}^{n''}$. It evidently suffices to consider the case that the target is simply Z'' or $\mathbb{R}$.

The case of the target Z'' is evident: the map $Z' \times \mathbb{R}^{n'} \to Z''$ locally factors through the projection to Z', and the map $Z \times \mathbb{R}^n \to Z'$ locally factors through the projection to Z, so we are done.

Now consider a composition $h = f \circ (t; g_1 \ldots, g_{n'})$ with target $\mathbb{R}$.

$$Z \times \mathbb{R}^n \xrightarrow{(t;g_1,\ldots,g_{n'})} Z' \times \mathbb{R}^{n'} \xrightarrow{f} \mathbb{R} \tag{2.12.3.1}$$

By the chain rule, the derivative $D^\alpha h$ is a continuous function of the derivatives of $g_1, \ldots, g_{n'}$ and f. That is, $D^\alpha h$ is the composition of a continuous function with the product of the continuous functions

$$Z \times \mathbb{R}^n \xrightarrow{(t;g_1,\ldots,g_{n'})} Z' \times \mathbb{R}^{n'} \xrightarrow{D^\gamma f} \mathbb{R} \tag{2.12.3.2}$$

$$Z \times \mathbb{R}^n \xrightarrow{D^\beta g_i} \mathbb{R} \tag{2.12.3.3}$$

for multi-indices β and γ. It is thus continuous, as desired. □

⋆ **2.12.4 Definition** (Category of topological-smooth spaces TopSm)**.** A *topological-smooth space* is a topological space X equipped with an atlas of charts (as in (2.6.1)) from open subsets of products $Z \times \mathbb{R}^n$ for topological spaces Z and integers $n \geq 0$, whose transition maps are continuous-smooth. A morphism of topological-smooth spaces is a continuous map of topological spaces which, when viewed via the charts, is continuous-smooth. The category of topological-smooth spaces is denoted TopSm.

There are tautological fully faithful embeddings of the categories of topological spaces Top and smooth manifolds Sm into the category of topological-smooth spaces TopSm.

$$\mathsf{Top} \hookrightarrow \mathsf{TopSm} \hookleftarrow \mathsf{Sm} \tag{2.12.4.1}$$

2.12.5 Exercise. Show that $Z \times \mathbb{R}^n \in \mathsf{TopSm}$ is the categorical product of $Z \in \mathsf{Top} \subseteq \mathsf{SmTop}$ and $\mathbb{R}^n \in \mathsf{Sm} \subseteq \mathsf{SmTop}$.

2.12.6 Exercise. Show that the embedding $\mathsf{Top} \subseteq \mathsf{TopSm}$ has right adjoint given by the underlying topological space functor $|\cdot| : \mathsf{TopSm} \to \mathsf{Top}$ and left adjoint given by the 'collapse leaves' functor $\mathsf{TopSm} \to \mathsf{Top}$.

2.12.7 Exercise (Locally connected)**.** Show that for a topological space X, the following are equivalent (in which case X is called *locally connected*):

(2.12.7.1) Every open subset of X is a disjoint union of connected open subsets.

(2.12.7.2) Every point $x \in X$ has arbitrarily small connected open neighborhoods.

2.12.8 Exercise (Leaf structure)**.** Consider the presheaf on $\mathsf{Top}^{\mathsf{opemb}}$ (topological spaces and open embeddings) defined as follows. To a topological space X we associate the set of equivalence relations on X all of whose equivalence classes are connected and locally connected subspaces of X. Such an equivalence relation on X restricts to an equivalence relation on any open subset $U \subseteq X$ all of whose equivalence classes are locally connected, but not necessarily connected. Splitting each such naively restricted equivalence class into its connected components (2.12.7.1) defines the restriction operation for our presheaf. Show that this presheaf is separated. A section of its sheafification is called a *leaf structure*.

2.12.9 Definition (Submersion)**.** A map of topological-smooth spaces is called a *submersion* when it is locally on the source a pullback of $\mathbb{R}^n \to *$.

Since the category TopSm is a topological site (**??**), we can form the category of *topological-smooth stacks* Shv(TopSm).

* *

Now let us define a category LogTopSm of log topological-smooth spaces. The local models of such spaces are fiber products $Q \times_B Z$ where $Q \to B$ is a submersion of log smooth manifolds (2.8.78) and Z is an arbitrary log topological space mapping to B. Recall that a submersion of log smooth manifolds is locally monomial (2.8.81), so this class of local models is equivalent to $X_Q \times_{X_P} Z$ for injective maps of polyhedral cones $P \to Q$ and maps $Z \to X_P$. In applications, we will only ever need the case that $Q \to B$ is an exact submersion (2.8.91), but this assumption is unnecessary for setting up the general theory (the reader is warned that submersions of log smooth manifolds are not preserved under pullback (**??**), in contrast to exact submersions (2.8.97)).

2.12.10 Definition (Log continuous-smooth map)**.** Given $\pi : Q \to B$ a submersion of log smooth manifolds and $w : Z \to B$ a map from an arbitrary log topological space Z, we introduce a formal symbol $Q \times_B Z$. Associated to such a formal symbol is an 'underlying log topological space' $|Q \times_B Z|$ given by the indicated fiber product in the category of log topological spaces. We wish to define a notion of 'log continuous-smooth map' $Q \times_B Z \to Q' \times_{B'} Z'$ between such formal symbols; such a map will, in particular, specialize to a log map on underlying log topological spaces. A log continuous-smooth map $Q \times_B Z \to Q' \times_{B'} Z'$ is, by definition, given locally by a diagram

$$\begin{tikzcd} {|Q \times_B Z|} \arrow[r] \arrow[d] & {|Q' \times_{B'} Z'|} \arrow[d] \\ Z \arrow[r] & Z' \end{tikzcd} \tag{2.12.10.1}$$

satisfying a certain differentiability property in the vertical direction. More formally, a map $Q \times_B Z \to Q' \times_{B'} Z'$ defined over an open subset $U \subseteq |Q \times_B Z|$ consists of a section of the sheaf $\operatorname{Hom}(-, |Q' \times_{B'} Z'|) \times_{\operatorname{Hom}(-,Z')} p_Z^* \operatorname{Hom}(-, Z')$ (where Hom denotes morphisms in LogTop) over U whose associated map $|Q \times_B Z| \to Q'$ is 'vertically smooth' in the sense we are about to define.

Chapter 3

Analysis

3.1 Function spaces

We recall various standard function spaces and their basic properties. References include [75, 2, 33, 91]. By '(smooth) manifold' we mean 'paracompact Hausdorff smooth manifold', and by 'vector bundle' we mean 'finite-dimensional smooth real (or complex) vector bundle'.

* *

We begin with some generalities about topological vector spaces.

⋆ **3.1.1 Definition** (Topological vector space)**.** A (real or complex) *topological vector space* is a (real or complex) vector space V whose addition $V \times V \to V$ and scalar multiplication $\mathbb{R} \times V \to V$ (or $\mathbb{C} \times V \to V$) are continuous.

3.1.2 Exercise. Show that the category of topological vector spaces and continuous linear maps has all limits and that these limits are preseved by the forgetful functor to topological spaces.

3.1.3 Exercise (Vector space topology generated by neighborhoods of the origin)**.** Let V be a real vector space, and let $\{U_\alpha \subseteq V\}_\alpha$ be a collection of subsets containing zero satisfying the following axioms:

(3.1.3.1) There is at least one α.
(3.1.3.2) For every pair α, β, there exists γ such that $U_\gamma \subseteq U_\alpha \cap U_\beta$.
(3.1.3.3) For every α, there exists β such that $U_\beta + U_\beta \subseteq U_\alpha$.
(3.1.3.4) For every α, there exists β such that $(-1, 1) \cdot U_\beta \subseteq U_\alpha$.
(3.1.3.5) For every α and every v, there exists $\varepsilon > 0$ such that $(-\varepsilon, \varepsilon) \cdot v \subseteq U_\alpha$.

Declare a set $U \subseteq V$ to be open iff for every $u \in U$ we have $u + U_\alpha \subseteq U$ for some α. Show that this defines the coarsest vector space topology on V in which every U_α is a neighborhood of the origin. Show that, conversely, given any vector space topology on V, the collection of all neighborhoods of the origin satisfies the above axioms, and generates the input topology in the above sense.

3.1.4 Definition (Semi-norm)**.** A *semi-norm* on a real (resp. complex) vector space V is a map $\|\cdot\| : V \to \mathbb{R}_{\geq 0}$ satisfying linearity $\|av\| = |a|\|v\|$ for $a \in \mathbb{R}$ (resp. $a \in \mathbb{C}$) and the triangle inequality $\|x + y\| \leq \|x\| + \|y\|$.

3.1.5 Definition (Norm)**.** A *norm* is a semi-norm for which $\|v\| = 0$ implies $v = 0$.

3.1.6 Definition (Complete topological vector space)**.** Let V be a topological vector space. A swarm (2.1.20) $v : S^* \to V$ is called *Cauchy* when for every neighborhood of zero $0 \in U \subseteq V$, there exists a neighborhood of the basepoint $A \subseteq S$ such that $v(a) - v(a') \in U$ for all $a, a' \in A^*$. A topological vector space is called *complete* when every Cauchy swarm has a unique limit.

3.1.7 Exercise. Show that a complete topological vector space is Hausdorff.

3.1.8 Definition (Smooth map to a complete topological vector space)**.** Let V be a complete topological vector space, and let us define a notion of when a map from a smooth manifold to V is smooth. A continuous map $f : Z \to V$ from a smooth manifold Z is said to be of class C^1 (continuously differentiable) when the map $Z \times Z \times (\mathbb{R} \setminus 0) \to V$ given by $(x, y, t) \mapsto t^{-1}(f(x) - f(y))$ extends continuously to the 'deformation to the tangent bundle' $\mathbb{P}(Z)$ (2.6.36), in which case the restriction to $TZ \subseteq \mathbb{P}(Z)$ of this (necessarily unique) continuous extension is called the derivative $Tf : TZ \to V$ of f. For $k \geq 2$, we say f is of class C^k when it is of class C^1 and its derivative Tf is of class C^{k-1}. Note that the derivative $Tf : TZ \to V$ is automatically linear (every relation $x + y = z$ in TZ is a limit of triples (p, q, t), (q, r, t), and (p, r, t) in $\mathbb{P}(Z)$). This notion of smoothness is respected by pre-composition with smooth maps of smooth manifolds and by post-composition with continuous linear maps of complete topological vector spaces.

* *

3.1.9 Definition (Multi-index notation)**.** Let V be a finite-dimensional real vector space. The symmetric algebra $\operatorname{Sym} V = \bigoplus_{r\geq 0} \operatorname{Sym}^r V$ (where $\operatorname{Sym}^r V = (V^{\otimes r})_{S_r}$) is the space of translation invariant differential operators on V. Given a basis $v_1, \ldots, v_n \in V$, there is an induced basis $\operatorname{Sym} V$ consisting of all possible monomials in $v_1, \ldots, v_n$. These monomials are in natural bijection with *multi-indices* $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n_{\geq 0}$; the *degree* of a multi-index α is denoted $|\alpha| = \sum_{i=1}^n \alpha_i$. The differential operator associated to a multi-index α is denoted D^α. Usually $V = \mathbb{R}^n$ with the standard basis.

⋆ **3.1.10 Definition** (Smooth functions C^∞_{loc} and C^∞_c)**.** Let M be a manifold, and let E/M be a vector bundle. The space of smooth sections $f : M \to E$ is denoted

$$C^\infty_{\mathrm{loc}}(M, E). \tag{3.1.10.1}$$

We denote by $C^\infty_c \subseteq C^\infty_{\mathrm{loc}}$ the subspace of functions which are compactly supported, and we denote by $C^\infty_K \subseteq C^\infty_{\mathrm{loc}}$ the subspace of functions supported inside a given compact set $K \subseteq M$. The corresponding spaces of k times continuously differentiable are denoted by C^k with the same subscripts.

⋆ **3.1.11 Definition** (C^∞_{loc}-topology and C^k_{loc}-topology)**.** For any integer $k \geq 0$, the C^k-norm of a smooth function f on $\mathbb{R}^n$ is the supremum of the sum of the absolute values of its derivatives (in the principal coordinate directions) of order $\leq k$.

$$\|f\|_{C^k(\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n} \sum_{|\alpha|\leq k} |D^\alpha f(x)| \tag{3.1.11.1}$$

Now let M be a manifold and E/M a vector bundle. Given a coordinate chart $\alpha : \mathbb{R}^n \supseteq U \hookrightarrow M$, a smooth section $\varphi : \alpha(U) \to E^*$ of compact support, and an integer $k \geq 0$, we may consider the semi-norm $f \mapsto \|\alpha^*(\varphi f)\|_{C^k(\mathbb{R}^n)}$ on $C^k_{\mathrm{loc}}(M, E)$. A C^k_{loc}-semi-norm is a semi-norm of this form (or one which is bounded by a finite sum of semi-norms of this form). The topology generated by the family of all C^k_{loc}-semi-norms is called the C^k_{loc}-*topology* (concretely, it is the topology of uniform convergence of all derivatives of order $\leq k$ over compact subsets). A C^∞_{loc}-semi-norm is a C^k_{loc}-semi-norm for some $k < \infty$. The topology generated by the family of all C^∞_{loc}-semi-norms is called the C^∞_{loc}-*topology* (concretely, it is the topology of uniform convergence of all derivatives over compact subsets).

3.1.12 Exercise. Show that $C^k_{\mathrm{loc}}(M, E)$ is complete with respect to the C^k_{loc}-topology for all $k \geq 0$ as well as for $k = \infty$. Show that $C^k_K \subseteq C^k_{\mathrm{loc}}$ is a closed subspace with respect to the C^k_{loc}-topology (hence complete in the subspace topology). Show that $C^k_c \subseteq C^k_{\mathrm{loc}}$ is a dense subspace with respect to the C^k_{loc}-topology.

3.1.13 Exercise. Let $k \geq 0$ be an integer. Fix a collection of coordinate charts $\alpha_i : \mathbb{R}^n \supseteq U_i \hookrightarrow M$ and smooth sections $\varphi_i : \alpha_i(U_i) \to E^*$ of compact support. Show that if the values $\varphi_i(x) \in E^*_x$ span for every $x \in M$, then every C^k_{loc}-semi-norm is bounded by a finite sum of the C^k_{loc}-semi-norms $f \mapsto \|\alpha_i^*(\varphi_i f)\|_{C^k(\mathbb{R}^n)}$.

⋆ **3.1.14 Lemma** (C^∞_c-topology)**.** *The directed colimit*

$$C^\infty_c = \varinjlim_{K \text{ compact}} C^\infty_K \tag{3.1.14.1}$$

*exists in the category of locally convex topological vector spaces and commutes with the forgetful functor to vector spaces; we call this the C^∞_c-*topology*. The C^∞_c-topology is complete and is generated by the family of semi-norms $\sum_i \|\varphi_i f\|_{C^{k_i}}$ (which we call C^∞_c-semi-norms) associated to locally finite collections of smooth functions φ_i and integers $k_i < \infty$.*

Proof. □

3.1.15 Definition (Bundle of densities)**.** We denote by Ω_M the bundle of densities on M. It is a smooth real line bundle defined by the existence of a canonical integration map $\int_M : C^\infty_c(M, \Omega_M) \to \mathbb{R}$. In fact, it is the line bundle associated to a principal $\mathbb{R}_{>0}$-bundle, so it has powers Ω^t_M for any $t \in \mathbb{R}$.

3.1.16 Example (Delta function)**.** The delta function $\delta_p \in C^{-\infty}_c(\mathbb{R}^n)$ is the distribution given by the linear functional 'evaluate at $p \in \mathbb{R}^n$'. On a manifold, the delta function is naturally a distribution valued in densities $\delta_p \in C^{-\infty}_c(M, \Omega_M)$.

3.1.17 Definition (Schwartz functions $\mathcal{S}$)**.** The space of Schwartz functions $\mathcal{S}(\mathbb{R}^n)$ consists of those infinitely differentiable functions all of whose norms

$$\|f\|_{\mathcal{S},A,B} = \sup_{x \in \mathbb{R}^n} (1 + |x|^A) \sum_{|\alpha| \leq B} |D^\alpha_x f(x)| \tag{3.1.17.1}$$

are finite. The space $\mathcal{S}$ is complete with respect to this family of norms.

3.1.18 Definition (Fourier transform)**.** The Fourier transform is a linear map $\mathcal{S}(\mathbb{R}^n,\mathbb{C}) \to \mathcal{S}(\mathbb{R}^n,\mathbb{C})$ denoted $f \mapsto \hat{f}$ and given by the formula

$$\hat{f}(\xi) = \int e^{-2\pi i\langle \xi,x\rangle} f(x)\,dx. \tag{3.1.18.1}$$

The Fourier transform is continuous: the required decay properties of $\hat{f}$ and its derivatives follow from integration by parts. Intrinsically speaking, the Fourier transform of a function on a finite-dimensional real vector space V is a function on its dual V^* valued in $\det(V)$.

The Fourier transform also makes sense for Schwartz functions valued in a complex vector space E. When E is a real vector space, the Fourier transform maps $\mathcal{S}(\mathbb{R}^n,E)$ to the subspace of $\mathcal{S}(\mathbb{R}^n,E\otimes_{\mathbb{R}}\mathbb{C})$ consisting of those functions g satisfying $g(-\xi)=\overline{g(\xi)}$ (and conversely, the Fourier transform of such a function g lies in the subspace $\mathcal{S}(\mathbb{R}^n,E)\subseteq\mathcal{S}(\mathbb{R}^n,E\otimes_{\mathbb{R}}\mathbb{C})$).

3.1.19 Exercise. Show that $\int_{\mathbb{R}^n} e^{-\pi|x|^2}\,dx = 1$ by reducing to the case $n=2$ and using polar coordinates.

3.1.20 Lemma (Fourier inversion)**.** *For $f\in\mathcal{S}(\mathbb{R}^n)$, we have $\hat{\hat{f}}(x)=f(-x)$.*

Proof. Note that $\hat{f}(\xi)e^{-\pi(\xi/N)^2}\to\hat{f}(\xi)$ in $\mathcal{S}(\mathbb{R}^n)$ as $N\to\infty$. It thus suffices to show that

$$\int e^{2\pi i\langle\xi,x\rangle}\hat{f}(\xi)e^{-\pi(\xi/N)^2}\,d\xi\to f(x). \tag{3.1.20.1}$$

The left hand side may be written as

$$\iint e^{2\pi i\langle\xi,x-y\rangle}f(y)e^{-\pi(\xi/N)^2}\,dy\,d\xi=\int\left(\int e^{-\pi(\xi/N)^2+2\pi i\langle\xi,z\rangle}\,d\xi\right)f(x+z)\,dz. \tag{3.1.20.2}$$

Now we may compute the inner integral of ξ by completing the square, moving the contour, and appealing to the identity $\int e^{-\pi x^2}\,dx=1$ (3.1.19). The result is $N^ne^{-\pi(Nz)^2}$, making the desired convergence to $f(x)$ clear upon appealing to (3.1.19) for a second time. $\square$

3.1.21 Exercise (Fourier transform and convolution)**.** Show that for $f,g\in\mathcal{S}(\mathbb{R}^n)$, we have $\widehat{f*g}=\hat{f}\hat{g}$, where $(f*g)(x)=\int f(y)g(x-y)\,dy$ denotes convolution. Using Fourier inversion, conclude that $\widehat{fg}=\hat{f}*\hat{g}$, and specialize this to conclude that $\int fg=\int\hat{f}\hat{g}$, so in particular $\|f\|_{L^2}=\|\hat{f}\|_{L^2}$ (Plancherel's formula).

* *

⋆ **3.1.22 Definition** (Sobolev spaces H^s)**.** The Sobolev H^s-norm of a function $f\in\mathcal{S}(\mathbb{R}^n)$ is

$$\|f(x)\|_{H^s(\mathbb{R}^n)}=\left\|(1+|\xi|^2)^{s/2}\hat{f}(\xi)\right\|_{L^2(\mathbb{R}^n)}. \tag{3.1.22.1}$$

When s is a non-negative integer, differentiation under the integral sign and Plancherel (3.1.21) imply that the H^s-norm is equivalent to $\sum_{|\alpha|\le s}\|D^\alpha f\|_{L^2(\mathbb{R}^n)}$.

Now let M be a manifold and let E/M be a vetor bundle. Given a coordinate chart $\alpha : \mathbb{R}^n \supseteq U \hookrightarrow M$ and a smooth section $\varphi : \alpha(U) \to E^*$ of compact support, we may consider the semi-norm $f \mapsto \|\alpha^*(\varphi f)\|_{H^s(\mathbb{R}^n)}$, which we call an H^s_{loc}*-semi-norm.* The topology generated by the set of all H^s_{loc}-semi-norms is called the H^s_{loc}-topology. The description of generating families as in (3.1.13) for C^∞_{loc}-semi-norms continues to hold, though the proof is more complicated, see (3.1.29) below.

3.1.23 Exercise (Dual norm)**.** Recall the *dual norm* (**??**): for any norm on $C^\infty_c(M,E)$, we can take its dual $\|\cdot\|'$ on $C^\infty_c(M, E^* \otimes \Omega_M)$ given by

$$\|f\|' = \sup_{\|g\|\leq 1} \int fg. \tag{3.1.23.1}$$

Show that if $\|f\|' < \infty$ for all f, then $\|\cdot\|'$ is a norm and $\|\cdot\|'' \leq \|\cdot\|$.

3.1.24 Exercise. Show using the Cauchy–Schwarz inequality that the L^2-norm on $\mathbb{R}^n$ is self-dual (where we implicitly trivialize $\Omega_{\mathbb{R}^n}$ by $dx_1 \cdots dx_n$). More generally, show that for any positive function $w : \mathbb{R}^n \to \mathbb{R}_{>0}$, the dual of the $L^2_w(\mathbb{R}^n)$-norm $f \mapsto (\int w|f|^2)^{1/2}$ is the $L^2_{w^{-1}}$-norm. Conclude that the dual of the $H^s(\mathbb{R}^n)$-norm is the $H^{-s}(\mathbb{R}^n)$-norm.

⋆ **3.1.25 Definition** (Interpolation norm)**.** Let $a < b < c \in \mathbb{R}$. Given norms $\|\cdot\|_a$ and $\|\cdot\|_c$ on a vector space X, we define a third norm $\|\cdot\|_b$ by the formula

$$\|v\|_b = \left(N^{-1}_{a,b,c} \int_{-\infty}^{\infty} \inf_{x+y=v} \left(e^{2(b-a)t}\|x\|_a^2 + e^{2(b-c)t}\|y\|_c^2\right) dt\right)^{1/2}, \tag{3.1.25.1}$$

where $N_{a,b,c} = \int \inf_{x+y=1}\left(e^{2(b-a)t}x^2 + e^{2(b-c)t}y^2\right) dt$ is a normalization factor. This is known, more precisely, as the *K-interpolation norm at $q = 2$.*

3.1.26 Exercise. Show that

$$\int_{-\infty}^{\infty} \inf_{x+y=1}\left(e^{2(b-a)t}r^{2a}x^2 + e^{2(b-c)t}r^{2c}y^2\right) dt = N_{a,b,c}r^{2b} \tag{3.1.26.1}$$

by reparameterizing t (any $r > 0$). Use this to show that

$$\left.\begin{matrix}\|v\|_a \leq Mr^a \\ \|v\|_c \leq Mr^c\end{matrix}\right\} \implies \|v\|_b \leq Mr^b \tag{3.1.26.2}$$

or equivalently that

$$\|v\|_b \leq \|v\|_a^{\frac{c-b}{c-a}}\|v\|_c^{\frac{b-a}{c-a}} \tag{3.1.26.3}$$

with equality with $\dim X = 1$ (to prove this, consider the infimum over x and y both multiples of v).

3.1.27 Exercise. Consider interpolation triples $(\|\cdot\|_a, \|\cdot\|_b, \|\cdot\|_c)$ on vector spaces X and Y, and consider a linear map $A : X \to Y$ which is (a,a)-bounded and (c,c)-bounded. Show that A is (b,b)-bounded with (b,b)-norm bounded by

$$\|A\|_{(b,b)} \leq \|A\|_{(a,a)}^{\frac{c-b}{c-a}} \|A\|_{(c,c)}^{\frac{b-a}{c-a}}. \tag{3.1.27.1}$$

To show this, bound the infimum over $A(v) = z + w$ (appearing in $\|A(v)\|_b$) by the infinum over $v = x + y$ (taking $z = A(x)$ and $w = A(y)$), and then reparameterize the integral over t to obtain the desired constant factor times $\|v\|_b$.

⋆ **3.1.28 Lemma** (Interpolation for Sobolev norms)**.** *For $a < b < c \in \mathbb{R}$, the H^b-norm is the interpolation of the H^a-norm and H^c-norm on $\mathcal{S}(\mathbb{R}^n)$.*

Proof. The square of the H^b-norm of u may be written using the identity (3.1.26.1) as

$$N_{a,b,c}^{-1} \iint \inf_{x+y=\hat{u}(\xi)} \bigl(e^{2(b-a)t}(1+|\xi|^2)^a|x|^2 + e^{2(b-c)t}(1+|\xi|^2)^c|y|^2\bigr)\,dt\,d\xi. \tag{3.1.28.1}$$

The square of the interpolation of the H^a-norm and H^c-norm may be written as

$$N_{a,b,c}^{-1} \int \inf_{f+g=u} \left[\int \bigl(e^{2(b-a)t}(1+|\xi|^2)^a|\hat{f}(\xi)|^2 + e^{2(b-c)t}(1+|\xi|^2)^c|\hat{g}(\xi)|^2\bigr)\,d\xi\right] dt. \tag{3.1.28.2}$$

The inequality (3.1.28.1) $\leq$ (3.1.28.2) is immediate. To show equality, it suffices to note that the pointwise minimizing pair $(x(\xi), y(\xi))$ with $x + y = \hat{u}$ from (3.1.28.1) can be well approximated by pairs of the form $(\hat{f}(\xi), \hat{g}(\xi))$ with $f + g = u$. □

3.1.29 Example. Consider the operator $M_f : C_c^\infty(\mathbb{R}^n) \to C_c^\infty(\mathbb{R}^n)$ given by multiplication by a smooth function $f : \mathbb{R}^n \to \mathbb{R}$ all of whose derivatives are bounded. Direct calculation shows that M_f is bounded $H^s \to H^s$ for every integer $s \geq 0$. It follows from interpolation (3.1.27)(3.1.28) that M_f is bounded $H^s \to H^s$ for real $s \geq 0$. The adjoint of M_f is itself, so it follows from duality (3.1.24)(**??**) that M_f is bounded $H^s \to H^s$ for all s.

Now let $\phi : \mathbb{R}^n \to \mathbb{R}^n$ be a compactly supported diffeomorphism, and consider the pushforward map $\phi_* : C_c^\infty(\mathbb{R}^n) \to C_c^\infty(\mathbb{R}^n)$. This operator ϕ_* is bounded $H^s \to H^s$ for integer $s \geq 0$ by direct calculation, hence for all real $s \geq 0$ by interpolation. The same applies to pullback ϕ^* since $\phi^* = (\phi^{-1})_*$. The adjoint of ϕ_* is $M_{\det D\phi} \circ \phi^*$, so by duality we conclude that ϕ_* (hence also ϕ^*) is bounded $H^s \to H^s$ for all s.

⋆ **3.1.30 Exercise** (Sobolev norm on a manifold)**.** Let M be a manifold, and let E/M be a vector bundle. Given a coordinate chart $\mathbb{R}^n \supseteq U \subseteq M$ and a smooth function of compact support $\varphi : U \to E^*$, we may consider the semi-norm $u \mapsto \|\varphi u\|_{H^s(\mathbb{R}^n)}$ on $C_{\mathrm{loc}}^\infty(M, E)$. These are called Sobolev H_{loc}^s-semi-norms on $C_{\mathrm{loc}}^\infty(M, E)$, and the topology they generate is called the H_{loc}^s-topology (also called the H^s-topology when M is compact). Show that the H_{loc}^s-semi-norms associated to a particular collection of pairs $(\mathbb{R}^n \supseteq U_i \subseteq M, \varphi_i : U_i \to E^*)$ generate the H_{loc}^s-topology provided the $\varphi_i(x)$ span E_x^* at every point $x \in M$ (note that it suffices to show that any single semi-norm $\|\psi u\|_{H^s(\mathbb{R}^n)}$ is bounded by a sum of semi-norms $\|\varphi_i u_i\|_{H^s(\mathbb{R}^n)}$, and then prove this using (3.1.29)).

⋆ **3.1.31 Lemma** (Sobolev embedding)**.** *For integer $k \geq 0$ and real $s > k + \frac{n}{2}$, we have $\|u\|_{C^k} \leq \mathrm{const}_{K,k,s}\|u\|_{H^s}$ for $\operatorname{supp} u \subseteq K$ (K compact).*

Proof. Differentiation D^α on $\mathbb{R}^n$ is bounded $H^s \to H^{s-|\alpha|}$ by direct calculation for integer $s \geq |\alpha|$, hence for all real $s \geq |\alpha|$ by interpolation (3.1.27)(3.1.28). It therefore suffices to treat the case $k = 0$, where we are supposed to show that

$$|u(0)|^2 = \left|\int e^{2\pi i\langle\xi,x\rangle}\hat{u}(\xi)\,d\xi\right|^2 \leq \mathrm{const}_s \int |\hat{u}(\xi)|^2(1+|\xi|)^{2s}\,d\xi. \tag{3.1.31.1}$$

This follows from Cauchy–Schwarz provided $\int(1+|\xi|)^{-2s}\,d\xi < \infty$, which is the case for $s > \frac{n}{2}$ (using polar coordinates, it is equivalent to $\int_1^\infty r^{-2s}r^{n-1}\,dr < \infty$). □

3.1.32 Lemma (Sobolev restriction)**.** *For a codimension d submanifold $N \subseteq M$, we have $\|u|_N\|_s \leq \mathrm{const}_{K,s}\|u\|_{s+d/2}$ for $\operatorname{supp} u \subseteq K$ (K compact) and $s > 0$.*

Proof. It suffices to show that restriction of smooth functions $\mathbb{R}^n$ to $\mathbb{R}^{n-1} \times 0$ is bounded $H^{s+1/2}(\mathbb{R}^n) \to H^s(\mathbb{R}^{n-1})$ provided $s > 0$.

Fix coordinates $(x, y) \in \mathbb{R}^{n-1} \times \mathbb{R}$ and dual coordinates $(\xi, \eta) \in \mathbb{R}^{n-1} \times \mathbb{R}$. The Fourier transform of the restriction $f(x, 0)$ is the integral $\int \hat{f}(\xi, \eta)\,d\eta$. Our desired estimate is thus

$$\int(1+|\xi|^2)^s\left|\int \hat{f}(\xi,\eta)\,d\eta\right|^2 d\xi \leq \mathrm{const}_s \int(1+|\xi|^2+|\eta|^2)^{s+\frac{1}{2}}|\hat{f}(\xi,\eta)|^2\,d\eta\,d\xi. \tag{3.1.32.1}$$

Cauchy–Schwarz gives

$$\left|\int \hat{f}(\xi,\eta)\,d\eta\right|^2 \leq \int(1+|\xi|^2+|\eta|^2)^{-s-\frac{1}{2}}\,d\eta \int(1+|\xi|^2+|\eta|^2)^{s+\frac{1}{2}}|\hat{f}(\xi,\eta)|^2\,d\eta, \tag{3.1.32.2}$$

so it suffices to show that

$$\int(1+|\xi|^2+|\eta|^2)^{-s-\frac{1}{2}}\,d\eta \leq \mathrm{const}_s(1+|\xi|^2)^{-s}. \tag{3.1.32.3}$$

Writing $1+|\xi|+|\eta| = (1+|\xi|)(1+\frac{|\eta|}{1+|\xi|})$ and performing the change of variables $\eta = (1+|\xi|)t$, we are reduced to the inequality $\int(1+|t|)^{-2s-1}\,dt \leq \mathrm{const}_s$, which holds since $s > 0$. □

3.1.33 Rellich Lemma ([77, 48])**.** *For every $s < t \in \mathbb{R}$ and every $\varepsilon > 0$, there exists a finite list of functions $\rho_1, \ldots, \rho_N \in C_c^\infty(\mathbb{R}^n)$ such that we have*

$$\|u\|_s \leq \varepsilon\|u\|_t + \sum_i \left|\int \rho_i u\right| \tag{3.1.33.1}$$

for all $u \in C^\infty(\mathbb{R}^n)$ supported inside the unit ball. The same holds for any manifold M and functions supported in any given compact set $K \subseteq M$.

Proof. For functions on the torus $M = \mathbb{R}^n/\mathbb{Z}^n$, the Sobolev norm is expressible in Fourier series $\|u\|_s^2 = \sum_m |\hat{u}(m)|^2(1+|m|^2)^s$, so we may take $\rho_1, \ldots, \rho_N$ to be large multiples of the Fourier modes $e^{2\pi imx}$ for $|m|^2 \le M$ for suitable $M < \infty$ depending on $\varepsilon > 0$. By embedding the unit ball into the torus, we conclude the case of the unit ball as well. The Sobolev norm on a general manifold is defined in terms of the Sobolev norm on the unit ball using a partition of unity (3.1.30), so the case of the unit ball implies the general case. □

* *

The next result allows us to define Sobolev spaces of maps to non-linear targets (i.e. manifolds) whenever $H^s \subseteq C^0$ and s is an integer.

3.1.34 Proposition (Moser [67, §2])**.** *Let $s \ge 0$ be an integer for which $H^s \subseteq C^0$, and suppose $F : \mathbb{R}^n \to \mathbb{R}$ is smooth and vanishes at the origin. For compact $K \subseteq \mathbb{R}^m$, we have*

$$\|F(g)\|_{H^s} \le \mathrm{const}_{K,s}\|F\|_{C^s(g(K))}\|g\|_{H^s} \tag{3.1.34.1}$$

for $\operatorname{supp} g \subseteq K$.

Proof. Derivatives of $F(g)$ of order $\le s$ are sums of terms of the form

$$(D^\alpha F)(g)\prod_{i=1}^{|\alpha|} D^{\beta_i} g \tag{3.1.34.2}$$

with $|\beta_i| \ge 1$ and $\sum_i |\beta_i| \le s$ (in particular, $|\alpha| \le s$). In the case $\alpha = 0$, we note that $\|F(g)\|_{L^2}$ is bounded by a constant times $\|g\|_{L^2}$ since $F(0) = 0$. For $|\alpha| \ge 1$, the factor $(D^\alpha F)(g)$ is bounded uniformly by $\|F\|_{C^s(K)}$, so it suffices to show that

$$\left\|\prod_{i=1}^{|\alpha|} D^{\beta_i} g\right\|_{L^2} \le \mathrm{const}_{K,s}\|g\|_{H^s}. \tag{3.1.34.3}$$

Using Hölder's inequality (**??**) and compactness of K, it suffices to show $\|D^\beta g\|_{L^{2s/|\beta|}} \le \mathrm{const}_{K,s}\|g\|_{H^s}$. That is, we should show that $H^{s-r} \to L^{2s/r}$ is bounded for integers $r = 1, \ldots, s$. For $r = 0$, this is the assumption that $H^s \to C^0$ is bounded, and for $r = s$ this is the definition $H^0 = L^2$. It thus follows for general $r \in [0, s]$ by interpolation (3.1.27)(3.1.28)(**??**). □

3.1.35 Corollary. *In the setup of* (3.1.34)*, if F vanishes to order $m \ge 1$ at the origin and $\|g\|_{C^0} \le 1$, then $\|F(g)\|_{H^s} \le \mathrm{const}_{K,s,m}\|F\|_{C^{\max(s,m)}(B(\|g\|_{C^0}))}\|g\|_{C^0}^{m-1}\|g\|_{H^s}$.*

Proof. Let $F_\varepsilon(x) = \varepsilon^{-m}F(\varepsilon x)$, and note that

$$\|F_\varepsilon\|_{C^s(B(1))} \le \mathrm{const}_{s,m}\|F\|_{C^{\max(s,m)}(B(\varepsilon))} \tag{3.1.35.1}$$

for all $0 < \varepsilon \le 1$ since F vanishes to order m at the origin. Now take $\varepsilon = \|g\|_{C^0}$ and write

$$\|F(g)\|_{H^s} = \varepsilon^m\|F_\varepsilon(\varepsilon^{-1}g)\|_{H^s} \overset{(3.1.34)}{\le} \mathrm{const}_s \cdot \varepsilon^m\|F_\varepsilon\|_{C^s(B(1))}\|\varepsilon^{-1}g\|_{H^s} \tag{3.1.35.2}$$

which combines with (3.1.35.1) to give the desired bound. □

3.1.36 Exercise ($H^s \subseteq C^0$ is an algebra)**.** Suppose $H^s \subseteq C^0$. Use (3.1.34) and rescaling as in (3.1.35) to show that $\|fg\|_{H^s} \leq \operatorname{const}_{K,s}(\|f\|_{C^0}\|g\|_{H^s} + \|f\|_{H^s}\|g\|_{C^0})$ for $\operatorname{supp} f, \operatorname{supp} g \subseteq K$.

3.1.37 Exercise. Suppose $H^s \subseteq C^0$. Show that if A vanishes along $\mathbb{R}^n \times \mathbb{R}^m \times 0$ and vanishes to order two along $\mathbb{R}^n \times 0 \times 0$, then

$$\|A(x, f(x), g(x))\|_{H^s} \leq \operatorname{const}_{K,s}\|A\|_{C^{\max(s,2)}(B(1))} \cdot (\|f\|_{H^s}\|g\|_{C^0} + (\|f\|_{C^0} + \|g\|_{C^0})\|g\|_{H^s}) \quad (3.1.37.1)$$

for $\|f\|_{C^0}, \|g\|_{C^0} \leq 1$ and $\operatorname{supp} f, \operatorname{supp} g \subseteq K$ (split into the two cases $\|f\|_{C^0} \geq \|g\|_{C^0}$ and $\|f\|_{C^0} \leq \|g\|_{C^0}$, and use (3.1.34) and rescaling as in (3.1.35)). Make a change of variables to conclude that if B vanishes along $\mathbb{R}^n \times \Delta_{\mathbb{R}^m}$ and to order two along $\mathbb{R}^n \times 0 \times 0$, then

$$\|B(x, f(x), g(x))\|_{H^s} \leq \operatorname{const}_{K,s}\|B\|_{C^{\max(s,2)}(B(1))} \cdot ((\|f\|_{H^s} + \|g\|_{H^s})\|f-g\|_{C^0} + (\|f\|_{C^0} + \|g\|_{C^0})\|f-g\|_{H^s}) \quad (3.1.37.2)$$

for $\|f\|_{C^0}, \|g\|_{C^0} \leq 1$ and $\operatorname{supp} f, \operatorname{supp} g \subseteq K$.

3.2 Differential operators

⋆ **3.2.1 Definition** (Differential operator)**.** On a manifold M carrying vector bundles E and F, a differential operator $L : C^\infty_{\mathrm{loc}}(M, E) \to C^\infty_{\mathrm{loc}}(M, F)$ of order $\leq m$ is a map which is given in local coordinates $M \supseteq U \subseteq \mathbb{R}^n$ by an expression of the form

$$Lf = \sum_{|\alpha| \leq m} c_\alpha D^\alpha f \tag{3.2.1.1}$$

where c_α are smooth functions taking values in $\operatorname{Hom}(E, F)$ (an operator being of this form is evidently preserved by diffeomorphisms).

The order m terms transform under diffeomorphisms independently of the others, so a differential operator L of order $\leq m$ has a well-defined order m term. Intrinsically, this order m term is an element of $\operatorname{Hom}(E, F) \otimes (TM^{\otimes m})_{S_m}$ or, equivalently, a homogeneous degree m polynomial map $T^*M \to \operatorname{Hom}(E, F)$ called the *order m symbol* of L.

It is evident that for a differential operator L of order $\leq m$, we have for all compact $K \subseteq M$ that

$$\|Lu\|_{C^k} \leq \mathrm{const}_{L,K,k} \|u\|_{C^{k+m}} \tag{3.2.1.2}$$

for $\operatorname{supp} u \subseteq K$. To interpret this estimate, we remind the reader of our convention for norms of functions on manifolds. Such a norm depends on a choice of covering family of charts and subordinate partition of unity, and is well-defined only up to constant factor. The constant appearing in estimates such as (3.2.1.2) thus depends on the choice of data used to define the C^k_K-norm on M, although we systematically omit this dependence from the notation (it can be regarded as part of the explicitly indicated dependence on k).

3.2.2 Definition (Formal adjoint)**.** For a differential operator $L : C^\infty_{\mathrm{loc}}(M, E) \to C^\infty_{\mathrm{loc}}(M, F)$, its formal adjoint is the differential operator

$$L^* : C^\infty_{\mathrm{loc}}(M, F^* \otimes \Omega_M) \to C^\infty_{\mathrm{loc}}(M, E^* \otimes \Omega_M) \tag{3.2.2.1}$$

defined by the property $\int_M \langle u, Lv \rangle = \int_M \langle L^* u, v \rangle$ (say for u and v of compact support), where Ω_M denotes the bundle of densities on M. In other words, L^* is obtained from L by formally integrating by parts.

3.2.3 Exercise. Show that a differential operator $L : C^\infty_{\mathrm{loc}}(M, E) \to C^\infty_{\mathrm{loc}}(M, F)$ admits a unique continuous extension $L : C^{-\infty}_{\mathrm{loc}}(M, E) \to C^{-\infty}_{\mathrm{loc}}(M, F)$.

3.2.4 Exercise. Let $w \in C^{-\infty}_{\mathrm{loc}}(\mathbb{R}^n)$ be a distribution supported (in the sense of (**??**)) at the origin. Show that w is a linear combination of the delta function (3.1.16) and its derivatives.

3.2.5 Exercise. Let $\Delta = \partial_x^2 + \partial_y^2$ on $\mathbb{R}^2$, and show that $\Delta(\log r) = 2\pi\delta_0$ (after first making precise how the function $\log r$ defines a distribution on $\mathbb{R}^2$).

3.2.6 Exercise. Let $\partial_{\bar z} = \frac{1}{2}(\partial_x + i\partial_y)$ on $\mathbb{C}$, and show that $\partial_{\bar z}(z^{-1}) = \pi\delta_0$ (after first making precise how the function z^{-1} defines a distribution on $\mathbb{C}$).

⋆ **3.2.7 Proposition.** *Let $L : E \to F$ be a differential operator on M of order $\leq m$. For any compact $K \subseteq M$, we have $\|Lu\|_s \leq \mathrm{const}_{L,K,s}\|u\|_{s+m}$ for $\operatorname{supp} u \subseteq K$.*

Proof. The statement is local, so it suffices to consider the case of differential operators on $M = \mathbb{R}^n$ and functions u supported inside the unit ball.

We begin by considering the case that L has *constant coefficients*, namely $L : C^\infty_{\mathrm{loc}}(\mathbb{R}^n, E) \to C^\infty_{\mathrm{loc}}(\mathbb{R}^n, F)$ for vector spaces E and F takes the form $Lf = \sum_{|\alpha|\leq m} c_\alpha D^\alpha f$ for constants $c_\alpha \in \mathrm{Hom}(E, F)$. In this case, we have

$$\widehat{Lu}(\xi) = P(2\pi i\xi)\hat{u}(\xi) \tag{3.2.7.1}$$

for $P(\xi) = \sum_{|\alpha|\leq m} c_\alpha \xi^\alpha$. Since P is a polynomial of degree $\leq m$, we conclude that $\|Lu\|_s \leq \mathrm{const}_{L,s}\|u\|_{s+m}$ for all u on $\mathbb{R}^n$.

We now consider the general case of variable coefficient operators L on $\mathbb{R}^n$. Since we are considering functions u supported inside a fixed compact set, we may assume the same for L. Write $Lf = \sum_{|\alpha|\leq m} c_\alpha D^\alpha f$ for functions c_α supported in the unit ball, and let $P(x, \xi) = \sum_{|\alpha|\leq m} c_\alpha(x)\xi^\alpha$, so we have

$$Lu(x) = \int e^{2\pi i\langle \xi, x\rangle} P(x, 2\pi i\xi)\hat{u}(\xi)\, d\xi \tag{3.2.7.2}$$

by differentiating under the integral sign. We may thus calculate

$$\widehat{Lu}(\zeta) = \int e^{-2\pi i\langle \zeta, x\rangle} \int e^{2\pi i\langle \xi, x\rangle} P(x, 2\pi i\xi)\hat{u}(\xi)\, d\xi\, dx \tag{3.2.7.3}$$

$$= \int \hat{u}(\xi) \int e^{2\pi i\langle \xi-\zeta, x\rangle} P(x, 2\pi i\xi)\, dx\, d\xi \tag{3.2.7.4}$$

$$= \int K_P(\zeta, \xi)\hat{u}(\xi)\, d\xi \quad \text{for } K_P(\zeta, \xi) = \int e^{2\pi i\langle \xi-\zeta, x\rangle} P(x, 2\pi i\xi)\, dx \tag{3.2.7.5}$$

where the interchange of integrals is justified by the fact that the x-support of P is compact and $\hat{u}(\xi)$ is rapidly decaying. Note that the kernel K_P is bounded by

$$|K_P(\zeta, \xi)| = \mathrm{const}_{L,N} \cdot (1 + |\xi - \zeta|)^{-N} \cdot (1 + |\xi|)^m \tag{3.2.7.6}$$

for any $N < \infty$ (integrate by parts N times in the direction of $\xi - \zeta$ if $|\xi - \zeta| \geq 1$).

We now claim that the desired bound

$$\int |\widehat{Lu}(\zeta)|^2(1 + |\zeta|^2)^s\, d\zeta \leq \mathrm{const}_{L,s} \int |\hat{u}(\xi)|^2(1 + |\xi|^2)^{s+m}\, d\xi \tag{3.2.7.7}$$

follows from the estimate (3.2.7.6) on the kernel $K_P(\zeta, \xi)$. First of all, we have

$$|\widehat{Lu}(\zeta)|^2 = \left(\int K_P(\zeta, \xi)\hat{u}(\xi)\, d\xi\right)^2 \tag{3.2.7.8}$$

$$\leq \mathrm{const}_{L,N}\left(\int |\hat{u}(\xi)|(1 + |\xi|)^m(1 + |\zeta - \xi|)^{-N}\, d\xi\right)^2 \tag{3.2.7.9}$$

$$\leq \mathrm{const}_{L,N} \int |\hat{u}(\xi)|^2(1 + |\xi|)^{2m}(1 + |\xi - \zeta|)^{-N}\, d\xi \tag{3.2.7.10}$$

by Cauchy–Schwarz. Now multiply this by $(1+|\zeta|^2)^s$ and integrate $\int d\zeta$. Then apply the bound $\int (1+|\xi-\zeta|)^{-N}(1+|\zeta|^2)^s\, d\zeta \leq \mathrm{const}_{s,N}(1+|\xi|^2)^s$ on the right to obtain (3.2.7.7). $\square$

3.2.8 Exercise. Prove (3.2.7) using interpolation and duality as in (3.1.29).

3.3 Elliptic operators

We review the theory of elliptic operators. References include Lawson–Michelsohn [52, III], Atiyah–Bott [5, §§3,6], Wells [91, IV 3–4], and Egorov–Schulze [17, §§1–4].

⋆ **3.3.1 Definition** (Elliptic operator)**.** A differential operator of order $\leq m$ is called *elliptic of order* m when its order m symbol $T^*M \to \operatorname{Hom}(E, F)$ sends nonzero elements $\xi \in T^*M$ to invertible elements of $\operatorname{Hom}(E, F)$.

3.3.2 Example. The operator $\sum_i \partial_{x_i}^2$ on functions $\mathbb{R}^n \to \mathbb{R}$ is elliptic (its symbol is $\sum_i \xi_i^2$). The operators $\partial_x \pm i\partial_y$ on functions $\mathbb{R}^2 \to \mathbb{C}$ are elliptic (their symbols are $\xi_1 \pm i\xi_2$).

Roughly speaking, an operator is elliptic when it is 'invertible on high frequencies' (indeed, the symbol of an operator describes, to leading order, its action on a given frequency). We shall see below that this implies elliptic operators are 'almost invertible', in the sense that every order m elliptic operator L has a *parametrix* Q, which is an operator of 'order $-m$' such that both operators $\mathbf{1} - LQ$ and $\mathbf{1} - QL$ have 'order $-\infty$'. Having 'order r' in the relevant sense, to be made precise below, means, in particular, being bounded $H^s \to H^{s-r}$. The existence of parametrices leads to many strong results about elliptic operators.

* *

Parametrices are not differential operators, rather they belong to the broader class of 'semi-local' operators which we now define.

3.3.3 Definition (Support of an operator)**.** Let $Q : C_c^\infty(M) \to C_{\mathrm{loc}}^\infty(M')$ be a linear operator. The *support* of Q is the closed subset $\operatorname{supp} Q \subseteq M' \times M$ defined by the property that $(p', p) \notin \operatorname{supp} Q$ iff there exist neighborhoods U, U' of p, p' such that $\operatorname{supp} u \subseteq U$ implies $Qu|_{U'} \equiv 0$.

3.3.4 Exercise. Show that the support of a differential operator on M is contained in the diagonal of $M \times M$.

3.3.5 Exercise. Let $Q : C_c^\infty(M) \to C^\infty(M')$ be a linear operator. Show that $\operatorname{supp} Qu$ is contained in the image of $Q \times_M \operatorname{supp} u \to M'$.

3.3.6 Exercise. Let $Q : C_c^\infty(M) \to C_{\mathrm{loc}}^\infty(M')$ be a linear operator. Show that:
(3.3.6.1) If $\operatorname{supp} Q$ is proper over M, then $Q : C_c^\infty \to C_c^\infty$.
(3.3.6.2) If $\operatorname{supp} Q$ is proper over M', then Q has a canonical extension $C_{\mathrm{loc}}^\infty \to C_{\mathrm{loc}}^\infty$.
(3.3.6.3) If $\operatorname{supp} Q$ is compact, then $Q : C_{\mathrm{loc}}^\infty \to C_c^\infty$.
We say that Q is *semi-local* when $\operatorname{supp} Q$ is proper over both M and M'.

3.3.7 Exercise. Let $P : C_c^\infty(M) \to C_{\mathrm{loc}}^\infty(M')$ and $Q : C_c^\infty(M') \to C_{\mathrm{loc}}^\infty(M'')$ be operators for which $\operatorname{supp} Q \times_{M'} \operatorname{supp} P \to M'' \times M$ is proper. Show that there is a canonical 'composition' $QP : C_c^\infty(M) \to C_{\mathrm{loc}}^\infty(M'')$ whose support is contained in the image of this map. Conclude that any composition of semi-local operators is defined and semi-local.

⋆ **3.3.8 Definition** (Operator of order $\leq m$)**.** A linear operator $Q : C_c^\infty(M) \to C_{\mathrm{loc}}^\infty(M')$ is said to have order $\leq m$ iff for every compact set $K \subseteq M$, every compactly supported smooth $\varphi : M' \to \mathbb{R}$, and every $s \in \mathbb{R}$, we have

$$\|\varphi \cdot Qu\|_s \leq \mathrm{const}_{K,\varphi,s} \|u\|_{s+m} \tag{3.3.8.1}$$

for $\operatorname{supp} u \subseteq K$. A *smoothing operator* is an operator of order $-\infty$, meaning order $\leq -N$ for every $N < \infty$.

3.3.9 Exercise. Let L be a differential operator. We saw in (3.2.7) that if L has order $\leq m$ as a differential operator, then it has order $\leq m$ in the sense of (3.3.8). Show the converse, namely that if L has order $\leq m$ in the sense of (3.3.8), then it has order $\leq m$ as a differential operator.

3.3.10 Exercise. Show that order is subadditive under composition, namely that if Q and Q' have order $\leq m$ and $\leq m'$ and satisfy the criterion for existence of the composition QQ' (3.3.7), then QQ' has order $\leq m + m'$.

⋆ **3.3.11 Definition** (Parametrix)**.** Let L be an elliptic operator of order m. A *parametrix* for L is a semi-local operator Q of order $\leq -m$ for which $\mathbf{1} - QL$ and $\mathbf{1} - LQ$ are both smoothing operators (thus a parametrix is an 'inverse modulo smoothing operators'). The analogous notion of a left (resp. right) parametrix requires that only $\mathbf{1} - QL$ (resp. $1 - LQ$) be a smoothing operator.

3.3.12 Exercise. Fix an elliptic operator L of order m. Show that if Q is a left parametrix for L and $Q' - Q$ is a smoothing operator, then Q' is a left parametrix for L. Show that if Q is a left parametrix and Q' is a right parametrix, then $Q - Q'$ is a smoothing operator (consider $Q(\mathbf{1} - LQ') - (\mathbf{1} - QL)Q'$), and hence both Q and Q' are parametrices for L.

* *

To construct parametrices for elliptic operators with variable coefficients, we will study the following general class of operators.

⋆ **3.3.13 Definition** (Pseudo-differential operator)**.** A pseudo-differential operator $T_A : C_c^\infty(\mathbb{R}^n) \to C_c^\infty(\mathbb{R}^n)$ is an operator of the form

$$(T_A u)(x) = \int e^{2\pi i \langle \xi, x \rangle} A(x,\xi) \hat{u}(\xi)\, d\xi. \tag{3.3.13.1}$$

where A has compact x-support and is a *symbol of order* $\leq m$, meaning that $|D_x^\alpha D_\xi^\beta A(x,\xi)| \leq \mathrm{const}_{\alpha,\beta} \cdot (1 + |\xi|)^{m-|\beta|}$.

3.3.14 Example. Compactly supported differential operators on $\mathbb{R}^n$ of order $\leq m$ are precisely the pseudo-differential operators (3.3.13.1) in which $A = \sum_{|\alpha| \leq k} c_\alpha(x)(2\pi i \xi)^\alpha$ is a polynomial of degree $\leq k$ in ξ with coefficients which are smooth compactly supported functions of x.

3.3.15 Lemma. *If A is a symbol of order $\leq m$, then T_A is an operator of order $\leq m$.*

Proof. The reasoning used to treat the special case of differential operators (3.2.7) applies with little change. We have

$$\widehat{T_A u}(\zeta) = \int K_A(\zeta,\xi)\hat{u}(\xi)\,d\xi \quad \text{where } K_A(\zeta,\xi) = \int e^{2\pi i\langle \xi-\zeta, x\rangle} A(x,\xi)\,dx. \tag{3.3.15.1}$$

The fact that A is a symbol of order $\leq m$ with compact x-support implies that

$$|K_A(\zeta,\xi)| \leq \mathrm{const}_{A,N} \cdot (1+|\xi-\zeta|)^{-N} \cdot (1+|\xi|)^m \tag{3.3.15.2}$$

for all $N < \infty$ by integrating by parts. This bound on K_A implies the desired estimate as in (3.2.7). □

⋆ **3.3.16 Proposition** (Composition of pseudo-differential operators)**.** *Fix operators T_A and T_B of the form* (3.3.13.1) *where A and B are symbols of order $\leq m_A$ and $\leq m_B$, respectively, of compact spatial support. We have $T_A \circ T_B = T_C$ where C is a symbol of order $\leq m_A + m_B$ and has asymptotic expansion*

$$C(x,\xi) \sim \sum_\alpha \frac{D_\xi^\alpha A(x,\xi) D_x^\alpha B(x,\xi)}{\alpha!(2\pi i)^{|\alpha|}} \tag{3.3.16.1}$$

where $\alpha! = \prod_i \alpha_i!$. The meaning of this asymptotic expansion (3.3.16.1) *is that the difference between C and the sum of terms on the right with $|\alpha| < N$ is a symbol of order $m_A + m_B - N$.*

Proof. As we have seen, the action of an operator of the form T_A on Fourier transforms is given by integration against a corresponding kernel K_A (3.3.15.1). The decay properties of these kernels (3.3.15.2) justify the exchange of integrals needed to show that a composition of such operators is given by the composition of their kernels:

$$(T_A T_B u)^\wedge(\eta) = \int K_C(\eta,\xi)\hat{u}(\xi)\,d\xi \quad \text{for } K_C(\eta,\xi) = \int K_A(\eta,\zeta)K_B(\zeta,\xi)\,d\zeta. \tag{3.3.16.2}$$

Now let us write the composed kernel K_C as follows.

$$K_C(\eta,\xi) = \iiint e^{2\pi i\langle \zeta-\eta, y\rangle} A(y,\zeta) e^{2\pi i\langle \xi-\zeta, x\rangle} B(x,\xi)\,dy\,dx\,d\zeta \tag{3.3.16.3}$$

$$= \int e^{2\pi i\langle \xi-\eta, y\rangle} \left[\iint e^{2\pi i\langle \zeta-\xi, y-x\rangle} A(y,\zeta)B(x,\xi)\,dx\,d\zeta\right] dy \tag{3.3.16.4}$$

$$= \int e^{2\pi i\langle \xi-\eta, y\rangle} \left[\iint e^{-2\pi i\langle \beta, t\rangle} A(y,\xi+\beta)B(y+t,\xi)\,dt\,d\beta\right] dy \tag{3.3.16.5}$$

At least formally, the bracketed expression in the middle will be our new symbol C, however we still need to justify the interchange of order of integration. We begin with the first triple integral (3.3.16.3). It is not absolutely convergent: it is defined (so that it equals the integral of K_A against K_B) by first integrating with respect to x and y (in which the integrand has compact support) to get something with rapid decay in ζ, and then integrating $d\zeta$. However, merely doing first the integral dx already gives us rapid decay in ζ, so we can interchange the y integral and ζ integral. This justifies the integral manipulation above. Now we define

$$C(y,\xi) = \iint e^{-2\pi i\langle\beta,t\rangle} A(y,\xi+\beta)B(y+t,\xi)\,dt\,d\beta \tag{3.3.16.6}$$

(note the order of integration: after doing the integral dt, we have rapid decay in β), so we have $T_A \circ T_B = T_C$.

It remains to show that C admits the asymptotic expansion (3.3.16.1) (and thus is a symbol of order $\leq m_A + m_B$). The key to proving the asymptotic expansion of C is to consider the Taylor expansion

$$A(y,\xi+\beta) \sim \sum_\alpha \frac{D_\xi^\alpha A(y,\xi)}{\alpha!}\beta^\alpha. \tag{3.3.16.7}$$

If in the definition of C we replace $A(y,\xi+\beta)$ by this Taylor expansion, we obtain precisely the asymptotic expansion (3.3.16.1) (we have $\iint e^{-2\pi i\langle\beta,t\rangle}\beta^\alpha B(y+t,\xi)\,dt\,d\beta = (2\pi i)^{-\alpha} D_y^\alpha B(y,\xi)$ by Fourier inversion and integration by parts). It thus suffices to show that the error in the Taylor expansion above contributes a symbol of order $\leq m_A + m_B - N$ to C.

We wish to show that the expression

$$R(y,\xi) = \int \left[\int e^{-2\pi i\langle\beta,t\rangle} B(y+t,\xi)\,dt\right]\left(A(y,\xi+\beta) - \sum_{|\alpha|<N} \frac{D_\xi^\alpha A(y,\xi)}{\alpha!}\beta^\alpha\right)d\beta, \tag{3.3.16.8}$$

is a symbol of order $\leq m_A + m_B - N$. That is, we should show that $D_y^\gamma D_\xi^\delta R(y,\xi)$ is bounded by $\mathrm{const}_{\gamma,\delta} \cdot (1+|\xi|)^{m_A+m_B-N-|\delta|}$. Now note that the derivatives D_y^γ and D_ξ^δ fall on A and B, producing symbols whose orders sum to $m_A + m_B - |\delta|$. The estimate for general (γ,δ) thus follows from the special case of $\gamma = \delta = 0$ (for different A and B). It therefore suffices to show that $|R(y,\xi)| \leq \mathrm{const}(1+|\xi|)^{m_A+m_B-N}$.

The function $R(y,\xi)$ is an integral $d\beta$ of a product of two factors, which we bound separately. The first factor (bracketed integral dt) is bounded by $\mathrm{const}_M(1+|\xi|)^{m_B}(1+|\beta|)^{-M}$ for any $M < \infty$ since B is a symbol of order m_B (3.3.15.2). The second factor (Taylor remainder in parentheses) is bounded by $\mathrm{const}_N|\beta|^N(1+|\xi|+|\beta|)^{m_A-N}$ by the Taylor remainder theorem since A is a symbol of order m_A. We are therefore left with showing that

$$\int (1+|\beta|)^{-M}(1+|\xi|+|\beta|)^{m_A-N}\,d\beta \leq \mathrm{const}_N(1+|\xi|)^{m_A-N} \tag{3.3.16.9}$$

for some $M < \infty$. Over the locus $|\beta| \geq |\xi|$, the integrand is bounded by $(1+|\beta|)^{m_A-N-M}$ (up to constant factor), hence the integral decays faster than any power of $|\xi|$ by taking M large. Over the locus $|\beta| \leq |\xi|$, the integrand is bounded by $(1+|\beta|)^{-M}(1+|\xi|)^{m_A-N}$ (up to constant factor), hence has integral bounded by $(1+|\xi|)^{m_A-N}$. $\square$

* *

We now explain how the existence of parametrices for elliptic operators is straightforward given the asymptotic composition formula (3.3.16). We consider the generality of symbols defined not on all of $\mathbb{R}^n$, rather on open subsets thereof.

3.3.17 Definition (Spaces of symbols S^m, S, $S^{-\infty}$)**.** For any domain $\Omega \subseteq \mathbb{R}^n$, denote by $S^m(\Omega)$ the space of symbols of order $\leq m$, namely smooth functions A on $\Omega \times \mathbb{R}^n$ satisfying $|D_x^\alpha D_\xi^\beta A(x,\xi)| \leq \mathrm{const}_{\alpha,\beta} \cdot (1+|\xi|)^{m-|\beta|}$. Denote by $S_c^m(\Omega) \subseteq S^m(\Omega)$ the symbols supported inside $K \times \mathbb{R}^n$ for some compact $K \subseteq \Omega$. Let $S = \bigcup_m S^m$ be the ascending union of the spaces S^m, and let $S^{-\infty} := \bigcap_m S^m$ be their intersection.

Associated to a symbol in $S(\Omega)$ (resp. $S_c(\Omega)$) is a pseudo-differential operator $C_c^\infty(\Omega) \to C_{\mathrm{loc}}^\infty(\Omega)$ (resp. $C_c^\infty(\Omega) \to C_c^\infty(\Omega)$. The operator associated to a symbol of order $\leq m$ has order $\leq m$ by (3.3.15). Composition of compactly supported symbols is defined by composition of operators (3.3.16).

3.3.18 Definition (φ-parametrix)**.** Let L be an elliptic operator of order m and let φ be a smooth function of compact support. A *left (resp. right) φ-parametrix* for L is a compactly supported (3.3.6.3) operator Q of order $\leq -m$ for which $\varphi - QL$ (resp. $\varphi - LQ$) is a smoothing operator.

3.3.19 Corollary. *Every elliptic operator L on an open set $\Omega \subseteq \mathbb{R}^n$ has left and right φ-parametrices for every $\varphi \in C_c^\infty(\Omega)$.*

Proof. By (**??**), there exists a symbol $Q \in S^{-m}(\Omega)$ which is inverse to L modulo smoothing operators. Also denote by Q the associated pseudo-differential operator $C_c^\infty(\Omega) \to C_{\mathrm{loc}}^\infty(\Omega)$. Now for $\psi \in C_c^\infty(\Omega)$ satisfying $\psi \equiv 1$ over a neighborhood of $\operatorname{supp}\varphi$, we claim that the operators $\varphi Q \psi, \psi Q \varphi : C_{\mathrm{loc}}^\infty(\Omega) \to C_c^\infty(\Omega)$ are our desired left and right parametrices. Indeed, the identities $\varphi Q \psi L \sim \varphi \sim L \psi Q \varphi$ follow by inspecting composition of symbols. □

3.3.20 Corollary. *Every elliptic operator L of order m has a parametrix Q.*

Proof. Let $M = \bigcup_i U_i$ be a locally finite open cover by Euclidean charts, and let $\varphi_i : M \to \mathbb{R}$ be a subordinate partition unity. By (3.3.19), there exist left and right φ_i-parametrices $Q_i, Q_i' : C_{\mathrm{loc}}^\infty(U_i) \to C_c^\infty(U_i)$ of order $\leq -m$. Their sums $Q = \sum_i Q_i$ and $Q' = \sum_i Q_i'$ are thus left and right parametrices for L. It follows formally (see (3.3.12)) that their difference $Q - Q'$ is a smoothing operator and hence that both Q and Q' are parametrices for L. □

* *

We now explore the consequences of the existence of parametrices for elliptic operators (3.3.20).

3.3.21 Corollary (Elliptic estimate)**.** *Let L be an elliptic operator of order m. We have*

$$\|u\|_s \leq \mathrm{const}_{L,K,s}\|Lu\|_{s-m} + \mathrm{const}_{L,K,N,s}\|u\|_{s-N} \tag{3.3.21.1}$$

for u supported inside compact $K \subseteq M$ and any $N < \infty$.

Proof. Let Q be a parametrix for L (3.3.20). Write $u = QLu + (\mathbf{1} - QL)u$, and note that $\|Q\|_{(s-m,s)} \leq \mathrm{const}_{L,K,s}$ and $\|\mathbf{1} - QL\|_{(s-N,s)} \leq \mathrm{const}_{L,K,N,s}$. □

3.3.22 Corollary (Kernel and cokernel of an elliptic operator)**.** *For an elliptic operator L of order m, the natural inclusions between the two-term complexes*

$$C^\infty_{\mathrm{loc}}(M,E) \xrightarrow{L} C^\infty_{\mathrm{loc}}(M,F) \tag{3.3.22.1}$$

$$H^s_{\mathrm{loc}}(M,E) \xrightarrow{L} H^{s-m}_{\mathrm{loc}}(M,F) \tag{3.3.22.2}$$

$$C^{-\infty}_{\mathrm{loc}}(M,E) \xrightarrow{L} C^{-\infty}_{\mathrm{loc}}(M,F) \tag{3.3.22.3}$$

are all quasi-isomorphisms. We denote by $\ker L$ *and* $\operatorname{coker} L$ *the kernel and cokernel of these operators; we have* $\ker L \subseteq C^\infty_{\mathrm{loc}}(M,E)$ *and* $C^\infty_{\mathrm{loc}}(M,F) \twoheadrightarrow \operatorname{coker} L$*. The same holds for the action of L on $C^\infty_c \subseteq H^s_c \subseteq C^{-\infty}_c$, giving spaces* $\ker_c L$ *and* $\operatorname{coker}_c L$*.*

Proof. Consider the case $H^s \hookrightarrow H^t$ for $s \geq t$ (the others are identical). It suffices to show that the total complex of the double complex

$$\begin{tikzcd} H^s_{\mathrm{loc}}(M,E) \arrow[r, "L"] \arrow[d, "\mathbf{1}"'] & H^{s-m}_{\mathrm{loc}}(M,F) \arrow[d, "\mathbf{1}"] \\ H^t_{\mathrm{loc}}(M,E) \arrow[r, "L"] & H^{t-m}_{\mathrm{loc}}(M,F) \end{tikzcd} \tag{3.3.22.4}$$

is acyclic. The endomorphism of this double complex given by

$$\begin{tikzcd} H^s_{\mathrm{loc}}(M,E) & H^{s-m}_{\mathrm{loc}}(M,F) \arrow[l, "Q"'] \\ H^t_{\mathrm{loc}}(M,E) \arrow[u, "\mathbf{1}-QL"] & H^{t-m}_{\mathrm{loc}}(M,F) \arrow[l, "Q"'] \arrow[u, "\mathbf{1}-LQ"'] \end{tikzcd} \tag{3.3.22.5}$$

is a chain homotopy between the identity map and the zero map (which implies acyclicity). Note that in writing the vertical arrows above, we are appealing to the fact that $\mathbf{1} - LQ$ and $\mathbf{1} - QL$ are smoothing operators. In the case of compactly supported functions, note that Q is semi-local. □

3.3.23 Corollary. *Let L be an elliptic operator of order m on a compact manifold M. If L is an isomorphism, then*

$$\|L^{-1}\|_{(s,s+m)} \leq \mathrm{const}_{M,L,s,a,b}\|L^{-1}\|_{(a,b)} \tag{3.3.23.1}$$

for any $s, a, b \in \mathbb{R}$. The same holds for a right inverse P provided $a \leq s$ and for a left inverse P' provided $b \geq s + m$.

Proof. Let Q be a parametrix for L, and write $L^{-1} = (\mathbf{1} - QL)L^{-1}(\mathbf{1} - LQ) + 2Q - QLQ$ or $P = (\mathbf{1} - QL)P + Q$ or $P' = P'(\mathbf{1} - LQ) + Q$. □

3.3.24 Corollary. *Let L be an elliptic operator of order m on M. For every $s \in \mathbb{R}$ and every compact $K \subseteq M$, there exist finitely many smooth functions $\rho_1, \ldots, \rho_N$ such that*

$$\|u\|_s \leq \mathrm{const}_{L,K,s} \|Lu\|_{s-m} + \sum_{i=1}^{N} \left| \int \rho_i u \right| \tag{3.3.24.1}$$

for u supported inside K.

Proof. Begin with the elliptic estimate $\|u\|_s \leq \mathrm{const}_{L,K,s}\|Lu\|_{s-m} + \mathrm{const}_{L,K,s}\|u\|_{s-1}$ (3.3.21), and apply the Rellich Lemma (3.1.33) to bound $\mathrm{const}_{L,K,s}\|u\|_{s-1}$ by $\frac{1}{2}\|u\|_s + \sum_{i=1}^{N}\left|\int \rho_i u\right|$. ☐

3.3.25 Corollary (Kernel finiteness)**.** *The kernel of an elliptic operator on a compact manifold is finite-dimensional.*

Proof #1. The estimate (3.3.24) implies that $\|u\|_s \leq \sum_{i=1}^{N}\left|\int \rho_i u\right|$ for all $u \in \ker L$, so the map $\ker L \to \mathbb{R}^N$ given by $u \mapsto (\int \rho_i u)_i$ is injective. ☐

Proof #2. It suffices to show that there exists a finite collection of points $P \subseteq M$ such that an element of $\ker L$ which vanishes on P must be zero. Take P to be any set of points such that the ε-balls centered at P cover M (we will choose $\varepsilon > 0$ later). Thus $\varphi|_P = 0$ implies that $\|\varphi\|_{C^0} \leq \varepsilon\|\varphi\|_{C^1}$. On the other hand, $\varphi \in \ker L$ implies that $\|\varphi\|_{C^1} \leq \mathrm{const}_L\|\varphi\|_{C^0}$ since L is elliptic (3.3.21). Thus if $\varphi|_P = 0$ and $\varphi \in \ker L$, then these combine to give $\|\varphi\|_{C^0} \leq \varepsilon \cdot \mathrm{const}_L\|\varphi\|_{C^0}$, which implies $\varphi = 0$ provided we choose $\varepsilon > 0$ sufficient small. ☐

3.3.26 Exercise. Explain the relation between the two proofs of kernel finiteness (3.3.25).

3.4 Rough coefficients

In the study of *smooth non-linear* elliptic equations, it is often necessary to have estimates for *linear* elliptic equations with *non-smooth* coefficients. We now generalize some of the results from (3.2)–(3.3) about linear differential operators with smooth coefficients to the setting of coefficients in some Sobolev space.

3.5 Ellipticity in cylindrical ends

In this section, we study cylindrical and asymptotically cylindrical elliptic operators. This means operators on a cylinder $\mathbb{R} \times N$ which are $\mathbb{R}$-equivariant; more generally, on a manifold M with ends modelled asymptotically on $\mathbb{R} \times N$, an asymptotically cylindrical operator is one which is asymptotically $\mathbb{R}$-equivariant in the ends. The reference for this section is Lockhart–McOwen [55].

3.5.1 Definition (Cylinder)**.** A 'cylinder' is a product $\mathbb{R} \times N$. The adjective 'cylindrical' when applied to objects living on a cylinder means $\mathbb{R}$-equivariant. For example, a cylindrical vector bundle on $\mathbb{R}\times N$ is one identified with the pullback of a vector bundle on N, and on such vector bundles we can consider cylindrical (i.e. $\mathbb{R}$-equivariant) operators $C^\infty_{\mathrm{loc}}(\mathbb{R} \times N, E) \to C^\infty_{\mathrm{loc}}(\mathbb{R} \times N, F)$.

⋆ **3.5.2 Definition** (Manifold with asymptotically cylindrical ends)**.** A manifold with asymptotically cylindrical ends M is a paracompact Hausdorff space with an atlas of charts from open subsets of $(0, \infty] \times \mathbb{R}^n$ whose transition functions take the form

$$(t, x) \mapsto (t + a(x) + o(1)_{C^\infty}, \phi(x) + o(1)_{C^\infty}) \quad \text{as } t \to \infty \tag{3.5.2.1}$$

for smooth $\phi : N \to N$ and $a : N \to \mathbb{R}$. Points of M at infinity (in the t coordinate) are called 'ideal points' and form a closed subset $M^{\mathrm{id}} \subseteq M$, which is a manifold. The complement $M \setminus M^{\mathrm{id}}$ is called the 'interior' $M^\circ \subseteq M$. A cylinder $\mathbb{R} \times N$ is the interior of a manifold with asymptotically cylindrical ends $[-\infty, \infty] \times N$.

The adjective 'asymptotically cylindrical' means built using functions of the form $f(x) + o(1)_{C^\infty}$ on charts $(0, \infty] \times \mathbb{R}^n$. Asymptotically cylindrical objects on M 'restrict' to cylindrical objects on $\mathbb{R} \times M^{\mathrm{id}}$ (their 'asymptotic limit'). Objects (vector bundles, almost complex structures, etc.) on asymptotically cylindrical manifolds are by default asymptotically cylindrical unless specified otherwise.

Asymptotic cylindricity is a special case of 'log smoothness' (2.8.69), so a manifold with asymptotically cylindrical ends is the same thing as a log smooth manifold of depth one (2.8.53), and asymptotically cylindrical objects (functions, vector bundles, etc.) are the same as log smooth objects.

Beware that one must be careful with the term 'compact' in the context of manifolds with asymptotically cylindrical ends. For example, if M is a manifold with asymptotically cylindrical ends, then compactness of M is distinct from compactness of M° (which implies $M^{\mathrm{id}} = \varnothing$, hence is a rather vacuous setting for our present discussion). Also contrast compactly supported functions on M with compactly supported functions on M°.

3.5.3 Example. Let C be a Riemann surface, and let $p \in C$ be a point. Given any local holomorphic chart $(D^2, 0) \to (C, p)$, we may glue $C \setminus p$ together with $(0, \infty] \times S^1$ via the identification of $z = e^{-t-i\theta} \in D^2$ with $(t, \theta) \in (0, \infty] \times S^1$. The coordinate change between any two such local holomorphic charts has the form $(t, \theta) \mapsto (t + a + O(e^{-t})_{C^\infty}, \theta + b + O(e^{-t})_{C^\infty})$ as $t \to \infty$ (by analyticity of holomorphic functions). These charts thus define a manifold with asymptotically cylindrical ends $\mathrm{Bl}_p C$ with interior $(\mathrm{Bl}_p C)^\circ = C \setminus p$.

* *

Let us now recall Sobolev norms on manifolds with asymptotically cylindrical ends.

⋆ **3.5.4 Definition** (Sobolev spaces H^s)**.** Let M be a manifold with asymptotically cylindrical ends carrying a vector bundle E. Given a coordinate chart $\alpha : (0,\infty] \times \mathbb{R}^n \supseteq U \hookrightarrow M$ and a smooth function of compact support $\varphi : \alpha(U) \to E^*$, we may consider the semi-norm $u \mapsto \|\alpha^*(\varphi u)\|_{H^s(\mathbb{R}\times\mathbb{R}^n)}$ on $C_c^\infty(M, M^{\mathrm{id}})$. We call these H^s_{loc}-semi-norms and the induced topology the H^s_{loc}-topology with completion $H^s_{\mathrm{loc}}(M, M^{\mathrm{id}}; E)$. The description of generating families of semi-norms from (3.1.30) continues to hold, for the same reason.

Here is an alternative presentation of the Sobolev norm on $\mathbb{R}^n$. Essentially it is a repeat of the argument given earlier for well definedness of Sobolev norms on manifolds (3.1.30) keeping track of constants.

3.5.5 Lemma. *A* bound on the geometry *of a collection of smooth functions $\varphi_i : \mathbb{R}^n \to \mathbb{R}$ is a sequence of constants $N, M_0, M_1, \ldots < \infty$ with the following properties:*

(3.5.5.1) $\|\varphi_i\|_{C^k} \le M_k$ *for all i and all $k < \infty$.*

(3.5.5.2) *Every ball of unit radius intersects* $\operatorname{supp}\varphi_i$ *for at most N indices i.*

For such a collection of functions, the maps

$$C_c^\infty(\mathbb{R}^n) \to \bigoplus_i C_c^\infty(\mathbb{R}^n) \qquad\qquad \bigoplus_i C_c^\infty(\mathbb{R}^n) \to C_c^\infty(\mathbb{R}^n) \tag{3.5.5.3}$$

$$u \mapsto \big(\varphi_i u\big)_i \qquad\qquad (u_i)_i \mapsto \sum_i \varphi_i u_i \tag{3.5.5.4}$$

are bounded in terms a bound on the geometry of the collection $(\varphi_i)_i$ and $s \in \mathbb{R}$, where we equip $C_c^\infty(\mathbb{R}^n)$ with the norm $u \mapsto \|u\|_s$ and we equip the direct sum $\bigoplus_i C_c^\infty(\mathbb{R}^n)$ with the norm $(u_i)_i \mapsto \big(\sum_i \|u_i\|_s^2\big)^{1/2}$.

Proof. For integer $s \ge 0$, express the H^s-norm squared as an integral of squares of derivatives of orders $\le s$, and note that there is a pointwise bound of integrands on $\mathbb{R}^n$ (for both maps). This implies boundedness for real $s \ge 0$ by interpolation (3.1.27)(3.1.28). The result follows for $s \le 0$ by duality (3.1.24)(**??**) since the two maps in question are adjoint. □

3.5.6 Corollary. *Let $\varphi_i : \mathbb{R}^n \to \mathbb{R}$ be a collection of smooth functions with geometry bounded by $N, M_0, M_1, \ldots < \infty$ in the sense of* (3.5.5)*. If in addition* $\sup_i |\varphi|_i \ge N^{-1} > 0$ *pointwise, then the norms $\|u\|_s$ and $\big(\sum_i \|\varphi_i u\|_s^2\big)^{1/2}$ are commensurate on $C_c^\infty(\mathbb{R}^n)$, with constant depending on $N, M_0, M_1, \ldots < \infty$.*

Proof. The inequality $\big(\sum_i \|\varphi_i u\|_s^2\big)^{1/2} \le \mathrm{const}_{N,M}\|u\|_s$ is boundedness of the first map in (3.5.5). For the reverse inequality, we appeal to boundedness of the second map in (3.5.5) for a different collection of functions $\psi_i : \mathbb{R}^n \to \mathbb{R}$. The hypothesis $\sup_i |\varphi|_i \ge N^{-1} > 0$ implies that there exists a collection of functions $\psi_i : \mathbb{R}^n \to \mathbb{R}$ with a bound on their geometry depending on $N, M_0, M_1, \ldots$ and with the property that $\sum_i \psi_i \varphi_i \equiv 1$. Boundedness of the second map in (3.5.5) for the collection ψ_i gives the desired inequality $\|u\|_s \le \mathrm{const}_{N,M}\big(\sum_i \|\varphi_i u\|_s^2\big)^{1/2}$. □

3.5.7 Exercise. Conclude from (3.5.6) that the $H^s_{\mathrm{loc}}(M, M^{\mathrm{id}})$-topology is generated by the semi-norms $u \mapsto \big(\sum_i \|\varphi_i u\|_s^2\big)^{1/2}$ for any collection of functions $\varphi_i : M^\circ \to \mathbb{R}$ of compact support which have combined support contained in a compact subset of M (hence the norms $\|\cdot\|_s$ can be taken with respect to any given finite collection of cylindrical charts covering this compact subset of M, and the result is well defined to overall constant) and which in local cylindrical coordinates on M have bounded geometry in the sense of (3.5.5).

⋆ **3.5.8 Definition** (Spaces C^∞_2)**.** Let M be a manifold with asymptotically cylindrical ends. We define

$$C^\infty_{2,\mathrm{loc}}(M, M^{\mathrm{id}}) = \bigcap_s H^s_{\mathrm{loc}}(M, M^{\mathrm{id}}) \subseteq C^\infty_{\mathrm{loc}}(M, M^{\mathrm{id}}) \tag{3.5.8.1}$$

to be the space of smooth functions on M which vanish on M^{id} and such that in local cylindrical coordinates on M, all their derivatives are square integrable. The $C^\infty_{2,\mathrm{loc}}$-topology is that generated by all $C^\infty_{2,\mathrm{loc}}$-semi-norms, which are simply all the H^s_{loc}-semi-norms for all s.

⋆ **3.5.9 Proposition.** *Let L be an asymptotically cylindrical differential operator of order $\leq m$ on a manifold with asymptotically cylindrical ends M. For any compact $K \subseteq M$, we have $\|Lu\|_s \leq \mathrm{const}_{L,K,s}\|u\|_{s+m}$ for $u \in C^\infty_K(M, M^{\mathrm{id}})$.*

Proof. Express the H^s-norm squared as a sum of local pieces using a partition of unity of bounded geometry (in cylindrical coordinates) as in (3.5.7). This reduces us to the local case on $\mathbb{R}^n$ (3.2.7) since L has bounded geometry in cylindrical coordinates. □

* *

The definition of ellipticity (3.3.1) makes sense as written for asymptotically cylindrical operators.

3.5.10 Exercise. Show that an asymptotically cylindrical operator L on a manifold with asymptotically cylindrical ends M is elliptic of order m iff its restriction to the interior M° is elliptic of order m and its asymptotic limit L^{id} on $\mathbb{R} \times M^{\mathrm{id}}$ is elliptic of order m.

* *

3.5.11 Definition (Reduction)**.** Let $L : E \to F$ be a cylindrical operator on $\mathbb{R} \times N$. If we restrict L to $\mathbb{R}$-invariant sections, we obtain an operator

$$L_0 : C^\infty(N, E) = C^\infty(\mathbb{R} \times N, E)^{\mathbb{R}} \to C^\infty(\mathbb{R} \times N, F)^{\mathbb{R}} = C^\infty(N, F) \tag{3.5.11.1}$$

called the *reduction* of L.

More generally, we may consider those sections on $\mathbb{R}\times N$ which transform under translation by the character e^{zt} for any complex number z. This defines operators $L_z : C^\infty(N, E) \to C^\infty(N, F)$ called *twisted reductions* of L. If $L = \sum_{i,\alpha} c_{i,\alpha}(x) D^i_t D^\alpha_x$ in coordinates $(t, x) \in \mathbb{R}\times N$, then $L_z = \sum_{i,\alpha} z^i c_{i,\alpha}(x) D^\alpha_x$ (hence this gives a bijection between cylindrical differential operators on $\mathbb{R} \times N$ and differential operators on N depending polynomially on a parameter z).

An asymptotically cylindrical operator L on M has an associated cylindrical operator L^{id} on $\mathbb{R} \times M^{\mathrm{id}}$, whose reductions L^{id}_z on M^{id} may also simply be denoted L_z.

3.5.12 Exercise. Show that any twisted reduction of an elliptic operator is elliptic.

3.5.13 Definition (Twist)**.** Given a cylindrical vector bundle V on $\mathbb{R} \times N$, its *twist* $\tau_z V$ by a complex number $z \in \mathbb{C}$ is obtained by multiplying the $\mathbb{R}$-translation action on V by e^{zt}. Multiplication by e^{zt} thus defines an isomorphism $V \to \tau_z V$. The twist of a cylindrical differential operator $L : E \to F$ is given by $\tau_z L = e^{zt} L e^{-zt} : E_z \to F_z$ (explicitly, if $L = \sum_{i,\alpha} c_{i,\alpha}(x) D_t^i D_x^\alpha$ then $\tau_z L = \sum_{i,\alpha} c_{i,\alpha}(x)(D_t - z)^i D_x^\alpha$). Twisting and reduction are compatible in the evident way: $(\tau_z L)_w = L_{w-z}$.

The twist of an asymptotically cylindrical vector bundle V on M may be defined by the property that $\tau_z V = V$ over M° and a map $V \to \tau_z V$ over M° extends smoothly to M if it is given over M° by multiplication by a function f which in cylindrical charts $(0, \infty] \times U$ has the form $f(x,t) = e^{zt}(m(x) + o(1)_{C^\infty})$ for some nonzero smooth function m. To ensure such $\tau_z V$ exists, we should note that coordinate changes between cylindrical charts (3.5.2.1) preserve the class of functions of the form $(x,t) \mapsto e^{zt}(m(x) + o(1)_{C^\infty})$ and that for any two such functions f and g, their ratio f/g extends smoothly on M. An asymptotically cylindrical operator $L : E \to F$ evidently induces a twisted operator $\tau_z L : \tau_z E \to \tau_z F$ by conjugating L by the isomorphisms $E = \tau_z E$ and $F = \tau_z F$ over M° (noting that the result of such conjugation on M° extends smoothly to M).

Beware that twisting a cylindrical vector bundle or operator by z corresponds to twisting by z at $+\infty$ and by $-z$ at $-\infty$.

⋆ **3.5.14 Definition** (Non-degenerate cylindrical elliptic operator)**.** Let L be a cylindrical elliptic operator on $\mathbb{R} \times N$ where N is compact. We say L is *non-degenerate* when its twisted reductions $L_{i\xi}$ are invertible for all $\xi \in \mathbb{R}$. An asymptotically cylindrical elliptic operator L is called non-degenerate when its asymptotic limit L^{id} is non-degenerate.

⋆ **3.5.15 Proposition.** *A non-degenerate cylindrical elliptic operator is invertible.*

3.6 Elliptic boundary conditions

3.7 Families of elliptic operators

In this section, we study how the results of the previous sections on elliptic operators (3.3)–(3.6) apply in families. Our objects of study are proper submersions $\pi : Q \to B$ equipped with vertical (i.e. fiberwise) elliptic operators L. The main result is that such a family determines a (homotopically) canonical two-term complex of vector bundles $\pi_* L$ on B whose cohomology at $b \in B$ is identified with the kernel and cokernel of L_b (**??**). In particular, the set of $b \in B$ for which L_b is surjective is open, and over this open set $\pi_* L$ is a vector bundle.

* *

Due to the categorical nature of the following definition, it makes sense in quite a number of different contexts (namely any setting with a reasonable notion of submersion and vertical differential operator).

⋆ **3.7.1 Definition** (Pushforward)**.** Let $\pi : Q \to B$ be a submersion, and let $L : E \to F$ be a vertical elliptic operator on Q. The pushforward $\pi_* L$ is the fiber product

$$\pi_* L = \underline{\mathrm{Sec}}_B(Q, E) \times_{\underline{\mathrm{Sec}}_B(Q,F)} 0, \tag{3.7.1.1}$$

where $\underline{\mathrm{Sec}}$ is the stack of sections (2.4.14). It is evident that $(\pi_* L) \times_B B' = \pi'_* L'$, where $\pi' : Q' \to B'$ and $L' : E' \to F'$ denote the pullback of (π, L) under a map $B' \to B$.

* *

We first study vertical elliptic operators on proper submersions of smooth manifolds. In this context, the most immediate interpretation of the pushforward (3.7.1) is as a smooth stack. Namely a map $Z \to \pi_* L$ from a smooth manifold Z is, by definition (3.7.1.1)(2.4.14), a pair (f, u) consisting of a map $f : Z \to B$ and a section $u : Q \times_B Z \to E$ satisfying $Lu = 0$. The following is the key analytic result from which everything else about $\pi_* L$ follows formally.

3.7.2 Proposition (Fiberwise isomorphism implies isomorphism)**.** *Let L be a vertical elliptic operator on a proper submersion $Q \to B$ of smooth manifolds.*

(3.7.2.1) *The set of $b \in B$ for which L_b is an isomorphism is open.*

(3.7.2.2) *If L_b is an isomorphism for every $b \in B$, then $L : \underline{\mathrm{Sec}}_B(Q, E) \to \underline{\mathrm{Sec}}_B(Q, F)$ is an isomorphism of smooth stacks.*

Proof. The desired assertion is local on B. By Ehresmann (2.6.30), the family $Q \to B$ is locally trivial on the base. The same argument applies moreover to the vector bundles E and F on the total space. We are therefore in the setting of a compact Hausdorff smooth manifold M and a family of elliptic operators $L_b : E \to F$ on M depending smoothly on $b \in B$. In this context, the set of $b \in B$ for which L_b is an isomorphism is open by (**??**). It thus remains to prove that if L_b is an isomorphism for every $b \in B$, then $L : \underline{\mathrm{Sec}}_B(Q, E) \to \underline{\mathrm{Sec}}_B(Q, F)$ is an isomorphism of smooth stacks.

Now a map $Z \to \underline{\mathrm{Sec}}_B(Q, E)$ is the same thing as a map $Z \to B$ and a section of E over $Z \times M$. By replacing B with Z, we reduce (3.7.2.2) to the following concrete assertion.

(3.7.2.3) If $f : B \times M \to F$ is smooth, then $L^{-1}f : B \times M \to E$ is also smooth (under the assumption that every L_b is an isomorphism, which thus defines $L^{-1}f$ fiberwise).

Let us now prove (3.7.2.3), which we note is a local assertion on B. Fix a basepoint $0 \in B$, and note that L_b^{-1} may be described in terms of L_0^{-1} by the usual series

$$L_b^{-1}f = \sum_{i=0}^{\infty} L_0^{-1}(\mathbf{1} - L_bL_0^{-1})^i f, \tag{3.7.2.4}$$

provided we can appropriately estimate its convergence. The series (3.7.2.4) converges in the fiberwise H^s-norm over a given b provided $\|\mathbf{1} - L_bL_0^{-1}\|_{(s,s)} < 1$. We have $\mathbf{1} - L_bL_0^{-1} = (L_0 - L_b)L_0^{-1}$, so this norm is bounded by $\|L_0^{-1}\|_{(s,s+m)}$ (constant) times $\|L_b - L_0\|_{(s+m,s)}$ (small provided b is sufficiently close to 0 (3.2.7)). We thus conclude that (3.7.2.4) converges exponentially in $H^s(M)$ for any fixed b, uniformly over a neighborhood of $0 \in B$ (depending on s). By Sobolev embedding (3.1.31), this implies the same for $C^k(M)$ in place of $H^s(M)$. It follows that arbitrary derivatives of $L^{-1}f$ in the M direction are continuous on $B \times M$.

To treat the derivatives in the B direction, we simply differentiate each term of the series (3.7.2.4) with respect to b. A derivative with respect to b applied to a term $(\mathbf{1} - L_bL_0^{-1})^i f$ will hit either L_b or f. Differentiating ℓ times leaves at least $i - \ell$ factors of $\mathbf{1} - L_bL_0^{-1}$, so the series remains exponentially convergent in fiberwise $C^k(M)$ for every $k < \infty$, uniformly over a neighborhood of $0 \in B$ depending on k. It follows that $L^{-1}f$ is smooth on $B \times M$.

□

We now unfold the consequences of the key analytic result (3.7.2).

3.7.3 Corollary (Fiberwise surjective implies π_*L is a vector bundle)**.** *Let L be a vertical elliptic operator on a proper submersion $\pi : Q \to B$ of smooth manifolds.*

(3.7.3.1) *The set of $b \in B$ for which L_b is surjective is open.*

(3.7.3.2) *If L_b is surjective for every $b \in B$, then π_*L is a vector bundle.*

The same holds for matrix operators as in (**??**).

Proof. Fix a basepoint $b \in B$ for which L_b is surjective, and let us show that b has a neighborhood over which every $L_{b'}$ is surjective and over which π_*L is a vector bundle.

Since L_b is elliptic and Q_b is compact, the kernel of L_b is finite-dimensional (3.3.25). Fix a map $\beta_b : C^\infty(Q_b, E_b) \to \mathbb{R}^k$ whose restriction to $\ker L_b$ is an isomorphism (equivalently, for which $L_b \oplus \beta_b$ is an isomorphism). Extend β_b to a map $\beta : \underline{\mathrm{Sec}}_B(Q, E) \to \underline{\mathbb{R}}^k_B$ over B (possibly after replacing B with a neighborhood of the point b); for example, a choice of local trivialization of $Q \to B$ induces such an extension. Now (3.7.2.1) implies that $L_{b'} \oplus \beta_{b'}$ is an isomorphism (and hence that $L_{b'}$ is surjective) for all b' in a neighborhood of b. Over such a neighborhood, the next part (3.7.2.2) says $L \oplus \beta$ is an isomorphism of stacks, which gives $\pi_*L = \underline{\mathbb{R}}^k_B$. Now π_*L has fiberwise scaling and addition maps induced by the same such maps for the stacks $\underline{\mathrm{Sec}}_B$. The entire construction respects such maps ('linear structure'), hence so does the resulting identification $\pi_*L = \underline{\mathbb{R}}^k_B$, and so π_*L is a vector bundle as desired.

The same argument applies to matrix operators. □

3.7.4 Proposition (Fiberwise isomorphism implies isomorphism)**.** *Let L be a vertical elliptic operator on a proper simply-broken submersion $Q \to B$ of log smooth manifolds.*

(3.7.4.1) *If L_b is an isomorphism and non-degenerate for some $b \in B$, then $L_{b'}$ is an isomorphism and non-degenerate for all b' in a neighborhood of b.*

(3.7.4.2) *If L_b is an isomorphism and non-degenerate for every $b \in B$, then $L : \underline{\mathrm{Sec}}_B(Q, E) \to \underline{\mathrm{Sec}}_B(Q, F)$ is an isomorphism of log smooth stacks.*

Proof. It suffices to show that if L_0 is an isomorphism for some basepoint $0 \in B$, then after replacing B with a neighborhood of said basepoint, every L_b is an isomorphism and for smooth $f : Q \to F$, the map $L^{-1}f : Q \to E$ (defined fiberwise since every L_b is an isomorphism) is also smooth. This assertion is manifestly local on B.

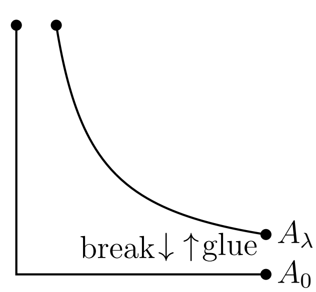
By (2.8.104), our proper simply-broken submersion $Q \to B$ is, locally on B, a pullback of a standard gluing family $M \to \mathbb{R}_{\geq 0}^{\pi_0 N/\sigma}$ (2.8.103) associated to a tuple $(M_0^{\mathrm{pre}}, N, i, \sigma)$. The same argument shows that every vector bundle on Q is the pullback of a vector bundle on M obtained via gluing from a vector bundle on M_0^{pre} which over the image of i is the pullback of a σ-equivariant vector bundle on N. We may thus identify fibers Q_b of our family $Q \to B$ with fibers $M_{\lambda(b)}$ of the standard gluing family $M \to \mathbb{R}_{\geq 0}^{\pi_0 N/\sigma}$.

Now let us consider the operator L_0 on $Q_0 = M_0$ which is an isomorphism, and let us show that the nearby operator L_b on $Q_b = M_{\lambda(b)}$ is an isomorphism provided b is sufficiently close to zero (hence $\lambda(b)$ is arbitrarily close to zero). To do this, we will construct an approximate inverse to L_b using L_0^{-1} and certain maps

$$\mathrm{break} : C^\infty(M_\lambda) \rightleftarrows C^\infty(M_0) : \mathrm{glue} \tag{3.7.4.3}$$

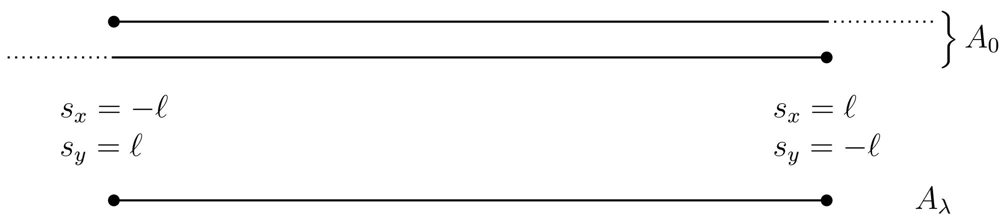
which express $C^\infty(M_\lambda)$ as a retract of $C^\infty(M_0)$ (smooth sections of E and F, though this is omitted from the notation).

We now define the gluing and breaking maps (3.7.4.3). Away from a small neighborhood of the 'singular locus' $N \times (0,0) \subseteq N \times {}'\mathbb{R}_{\geq 0}^2 \subseteq M$, the fibers M_0 and M_λ are identified by construction (2.8.103), and glue and break are both the 'identity'. Near the singular locus, the maps glue and break will depend only on the ${}'\mathbb{R}_{\geq 0}^2$ coordinate, hence the vector bundles E and F may be ignored since they are pulled back from N. Denote the fiber of the multiplication map ${}'\mathbb{R}_{\geq 0}^2 \to {}'\mathbb{R}_{\geq 0}$ over $\lambda \in {}'\mathbb{R}_{\geq 0}$ by A_λ.



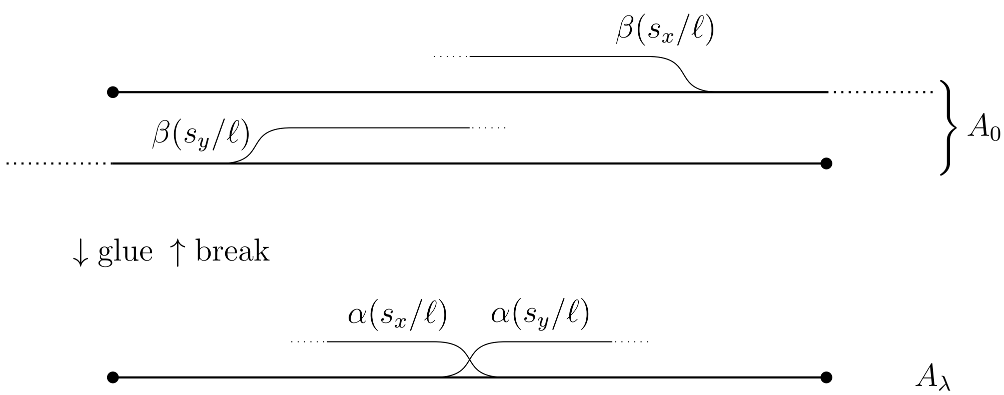
(3.7.4.4)

We will use both the natural coordinates $(x, y) \in {}'\mathbb{R}_{\geq 0}^2$ (say $x, y \leq 1$) and the shifted log (aka cylindrical) coordinates $s_x = -\ell - \log x$ and $s_y = -\ell - \log y$ (where $\ell = -\log \lambda^{1/2} > 0$). Thus the two ‘branches’ of A_0 have separate coordinates $s_x, s_y \in [-\ell, \infty)$, while on A_λ we have coordinates $s_x, s_y \in [-\ell, \ell]$ which are negatives of each other. We thus have, in some sense, a common coordinate system on A_0 and A_λ, which away from $(x, y) = (0, 0)$ is compatible with the identification of M_0 and M_λ away from the singular locus.



(3.7.4.5)

Now the gluing and breaking maps (3.7.4.3) are defined in terms of an arbitrary choice of smooth cutoff functions α and β as follows.



(3.7.4.6)

$$\operatorname{break}(f)(s_x) = \alpha(s_x/\ell) f(s_x) \tag{3.7.4.7}$$
$$\operatorname{break}(f)(s_y) = \alpha(s_y/\ell) f(s_y) \tag{3.7.4.8}$$
$$\operatorname{glue}(f) = \beta(s_x/\ell) f(s_x) + \beta(s_y/\ell) f(s_y) \tag{3.7.4.9}$$

$$\alpha : \mathbb{R} \to [0, 1] \qquad \alpha(s) = \begin{cases} 1 & s \le -\frac{1}{5} \\ 0 & s \ge \frac{1}{5} \end{cases} \qquad \alpha(s) + \alpha(-s) = 1 \tag{3.7.4.10}$$
$$\beta : \mathbb{R} \to [0, 1] \qquad \beta(s) = \begin{cases} 1 & s \le \frac{3}{5} \\ 0 & s \ge \frac{4}{5} \end{cases} \tag{3.7.4.11}$$

It is evident that $\operatorname{glue} \circ \operatorname{break} = \mathbf{1}$.

$\square$

Chapter 4

Riemann surfaces

Chapter 5

Pseudo-holomorphic maps

5.1 Symplectic and almost complex structures

We begin with the linear story.

5.1.1 Definition. Let V be a finite-dimensional real vector space.

(5.1.1.1) A *complex structure* on V is an endomorphism $J : V \to V$ satisfying $J^2 = -\mathbf{1}$ (equivalently, it is a lift of V to a complex vector space).

(5.1.1.2) A *symplectic form* on V is an anti-symmetric pairing $\omega : V \otimes V \to \mathbb{R}$ which is *non-degenerate*, meaning that the induced map $V \to V^*$ is an isomorphism.

(5.1.1.3) A *metric* on V is a symmetric pairing $g : V \otimes V \to \mathbb{R}$ which is *positive definite*, meaning that $g(v, v) > 0$ for $v \neq 0$.

(5.1.1.4) A pair (J, ω) is called *compatible* when $g(v, w) = \omega(v, Jw)$ is a metric (i.e. is symmetric and positive definite).

(5.1.1.5) A pair (J, ω) is called *tame* when $\omega(v, Jv) > 0$ for $v \neq 0$. Compatible pairs are evidently tame. A tame pair also determines a metric $g(v, w) = \omega(v, Jw) + \omega(w, Jv)$.

More generally, these notions apply (fiberwise) to any vector bundle V over a smooth manifold.

5.1.2 Exercise. Let V be a symplectic vector space. Show that if a subspace $P \subseteq V$ is symplectic, then its ω-orthogonal subspace $P^\perp$ ($v \in P^\perp$ iff $\omega(v, p) = 0$ for all $p \in P$) is also symplectic and is a complement of P. Conclude that there exists a basis $v_1, \ldots, v_n, w_1, \ldots, w_n$ of V satisfying $\omega(v_i, v_j) = \omega(w_i, w_j) = 0$ and $\omega(v_i, w_j) = \delta_{ij}$.

5.1.3 Lemma. *Let V be a finite-dimensional real vector space. The maps*

$$\begin{tikzcd} & \{(\omega, J) \text{ compatible}\} \arrow[dl] \arrow[d] \arrow[dr] & \\ \{\omega\} & & \{J\} \\ & \{(\omega, J) \text{ tame}\} \arrow[ul] \arrow[ur] & \end{tikzcd} \tag{5.1.3.1}$$

are homotopy equivalences.

Proof. For fixed J, the space of tame (resp. compatible) ω is convex and non-empty, hence contractible. It thus suffices to show that for fixed ω, the space of compatible J is contractible, which goes as follows. Fix $v \in V$ arbitrarily. The value of $J(v)$ must lie in $\{w \in V : \omega(v, w) > 0\}$, which is convex and non-empty, hence contractible. For $v, w \in V$ with $\omega(v, w) > 0$, a compatible J satisfying $J(v) = w$ stabilizes the ω-orthogonal complement of $\operatorname{span}(v, w) \subseteq V$. The space of compatible almost complex structures on this orthogonal complement is contractible by induction on the dimension of V. $\square$

5.1.4 Definition. Let V be a symplectic vector space. For a subspace $P \subseteq V$, denote by $P^\perp \subseteq V$ its *ω-orthogonal*, consisting of the vectors v for which $\omega(v, p) = 0$ for all $p \in P$.

(5.1.4.1) P is called *isotropic* when $P \subseteq P^\perp$.

(5.1.4.2) P is called *co-isotropic* when $P \supseteq P^\perp$.

(5.1.4.3) P is called *Lagrangian* when $P = P^{\perp}$.

5.1.5 Definition. Let V be a complex vector space. A subspace $P \subseteq V$ is called *totally real* when $P \cap JP = 0$ and $P + JP = V$ (equivalently, when the natural map $P \otimes_{\mathbb{R}} \mathbb{C} \to V$ is an isomorphism).

5.1.6 Exercise. Let V be a symplectic vector space equipped with a tame almost complex structure. Show that a Lagrangian subspace $P \subseteq V$ is totally real.

* *

We now continue on to the setting of manifolds.

5.1.7 Definition. Let M be a smooth manifold.

(5.1.7.1) A *metric* on M is a smooth fiberwise metric on TM.

(5.1.7.2) An *almost symplectic form* on M is a smooth fiberwise symplectic form on TM. A *symplectic form* is an almost symplectic form ω satisfying $d\omega = 0$.

(5.1.7.3) An *almost complex structure* on M is a smooth fiberwise complex structure on TM. A *complex structure* is an almost complex structure which is locally isomorphic to $(\mathbb{C}^n, J_{\mathrm{std}} = i)$; such almost complex structures are also called *integrable*.

5.1.8 Definition. Let M be a symplectic manifold. A submanifold $L \subseteq M$ is called *Lagrangian* when $TL \subseteq TM$ is Lagrangian at every point of L.

5.1.9 Definition. Let M be an almost complex manifold. A submanifold $L \subseteq M$ is called *totally real* when $TL \subseteq TM$ is totally real at every point of L.

5.2 Pseudo-holomorphicity

In this section, we recall the pseudo-holomorphic map equation and the various geometric settings in which this equation and its variants are defined.

⋆ **5.2.1 Definition** (Pseudo-holomorphic map)**.** A map $u : C \to X$ from a Riemann surface C to an almost complex manifold X is called *pseudo-holomorphic* when its differential $du : TC \to u^*TX$ is complex linear.

A pair (C, X) as above is the simplest instance of what we will call a pseudo-holomorphic map *problem*. The pseudo-holomorphic maps $u : C \to X$ are called the *solutions* of this pseudo-holomorphic map problem. We will see pseudo-holomorphic map problems which involve sections, allow domains with boundary (paired with appropriate boundary conditions), impose point constraints, and allow varying domains and targets.

5.2.2 Definition (Complex conjugate vector space)**.** For a complex vector space V, we denote by $\overline{V}$ its complex conjugate, namely its pullback under the conjugation automorphism of $\mathbb{C}$. Concretely, $\overline{V} = V$ as sets; for $v \in V$, the corresponding element of $\overline{V}$ is denoted $\overline{v}$; and the vector space structure on $\overline{V}$ is that suggested by the notation, namely $\overline{v} + \overline{w} = \overline{v + w}$ and $\lambda\overline{v} = \overline{\overline{\lambda}v}$. There is an evident identification $\overline{\overline{V}} = V$. A complex linear map $\overline{V} \to W$ is the same as a complex conjugate linear map $V \to W$.

5.2.3 Definition (Decomposition of the complexification of a complex vector space)**.** Let V be a complex vector space, and consider $V \otimes_{\mathbb{R}} \mathbb{C}$, regarded as a complex vector space via the second factor. There is a canonical map

$$V \otimes_{\mathbb{R}} \mathbb{C} \to V \oplus \overline{V}, \tag{5.2.3.1}$$

$$v \otimes \lambda \mapsto \lambda v \oplus \bar{\lambda}\bar{v}, \tag{5.2.3.2}$$

of complex vector spaces. In fact, this map is an isomorphism, with inverse given by

$$V \oplus \overline{V} \to V \otimes_{\mathbb{R}} \mathbb{C}, \tag{5.2.3.3}$$

$$v \oplus \bar{w} \mapsto \tfrac{1}{2}(v \otimes 1 - iv \otimes i) + \tfrac{1}{2}(w \otimes 1 + iw \otimes i). \tag{5.2.3.4}$$

5.2.4 Definition (Identifying the real dual and the complex dual)**.** Let V be a complex vector space. It has both a complex dual and a real dual

$$V^{*_{\mathbb{C}}} = \operatorname{Hom}_{\mathbb{C}}(V, \mathbb{C}) \quad \text{and} \quad V^{*_{\mathbb{R}}} = \operatorname{Hom}_{\mathbb{R}}(V, \mathbb{R}). \tag{5.2.4.1}$$

The complex dual $V^{*_{\mathbb{C}}}$ is evidently a complex vector space. We equip the real dual $V^{*_{\mathbb{R}}}$ with the complex structure $\xi \mapsto J^*\xi$. We identify $V^{*_{\mathbb{R}}} = V^{*_{\mathbb{C}}}$ via the inverse pair of complex linear isomorphisms given by

$$\operatorname{Hom}_{\mathbb{R}}(V, \mathbb{R}) \to \operatorname{Hom}_{\mathbb{C}}(V, \mathbb{C}), \qquad \operatorname{Hom}_{\mathbb{C}}(V, \mathbb{C}) \to \operatorname{Hom}_{\mathbb{R}}(V, \mathbb{R}), \tag{5.2.4.2}$$

$$\xi \mapsto \frac{1}{2}(\xi - iJ^*\xi), \qquad \zeta \mapsto 2\operatorname{Re}\zeta, \tag{5.2.4.3}$$

and henceforth we will simply write V^* for $V^{*_{\mathbb{R}}} = V^{*_{\mathbb{C}}}$ except when there is a need to distinguish between the two.

5.2.5 Exercise. Let V and W be complex vector spaces. Show that writing a real linear map $f : V \to W$ as the sum $f = f^{1,0} + f^{0,1}$ of the complex linear map $f^{1,0} = \frac{1}{2}(f - i \circ f \circ i)$ and the complex conjugate linear map $f^{0,1} = \frac{1}{2}(f + i \circ f \circ i)$ defines a direct sum decomposition

$$\operatorname{Hom}_{\mathbb{R}}(V, W) = \operatorname{Hom}_{\mathbb{C}}(V, W) \oplus \operatorname{Hom}_{\mathbb{C}}(\overline{V}, W). \tag{5.2.5.1}$$

Moreover, show that the following diagram of isomorphisms defined thus far

$$\begin{tikzcd}
\operatorname{Hom}_{\mathbb{R}}(V,W) \arrow[r, equal] \arrow[ddd, equal] & V^{*_{\mathbb{R}}} \otimes_{\mathbb{R}} W \arrow[d, equal] \\
& V^{*_{\mathbb{C}}} \otimes_{\mathbb{R}} W \arrow[d, equal] \\
& V^{*_{\mathbb{C}}} \otimes_{\mathbb{R}} \mathbb{C} \otimes_{\mathbb{C}} W \arrow[d, equal] \\
\operatorname{Hom}_{\mathbb{C}}(V,W) \oplus \operatorname{Hom}_{\mathbb{C}}(\overline{V},W) \arrow[r, equal] & (V^{*_{\mathbb{C}}} \oplus \overline{V^{*_{\mathbb{C}}}}) \otimes_{\mathbb{C}} W
\end{tikzcd} \tag{5.2.5.2}$$

commutes. Conclude that the equation asserting pseudo-holomorphicity of a smooth map $u : C \to X$ thus reads $(du)^{0,1} = 0$ in the space of sections of $u^*TX \otimes \overline{T^*C}$ over C.

⋆ **5.2.6 Definition** (Pseudo-holomorphic section)**.** Let $\pi : W \to C$ be an almost complex submersion over a Riemann surface C (meaning W is an almost complex manifold and $d\pi$ is complex linear). Such a map $W \to C$ is a pseudo-holomorphic map problem whose solutions are the pseudo-holomorphic sections $u : C \to W$.

5.2.7 Exercise. Show that for any smooth section $u : C \to W$ of an almost complex submersion, the anti-holomorphic derivative $(du)^{0,1}$ takes values in $u^*T_{W/C} \otimes \overline{T^*C}$.

5.2.8 Exercise (Adding an inhomogeneous term)**.** Let C be a Riemann surface and X an almost complex manifold. Consider almost complex structures on $C \times X$ for which the projection to C and the inclusions of the fibers $c \times X$ are pseudo-holomorphic. Show that such almost complex structures are in natural bijection with sections $\gamma : C \times X \to TX \otimes \overline{T^*C}$. Show that the graph of $u : C \to X$ is pseudo-holomorphic for such an almost complex structure iff $(du)^{0,1} + \gamma(u) = 0$.

5.2.9 Exercise (Linear almost complex structures are real Cauchy–Riemann operators)**.** Let $E \to C$ be a complex vector bundle over a Riemann surface. Show that real Cauchy–Riemann operators $D : C^\infty(C, E) \to C^\infty(C, E \otimes \overline{T^*C})$ (**??**) are in natural bijection with almost complex structures on E for which the vector bundle structure maps $E \to C$, $\cdot : \mathbb{C} \times E \to E$, and $+ : E \times_C E \to E$ are pseudo-holomorphic. Show that a section of E lies in the kernel of D iff it is pseudo-holomorphic.

5.2.10 Definition (Point constraint)**.** A single point constraint on a pseudo-holomorphic section problem $W \to C$ is a smooth manifold A with a map $f : A \to W$. A solution to the constrained section problem $(W \to C, A \to W)$ is a pseudo-holomorphic section $u : C \to W$ along with a point $a \in A$ such that $u(C)$ contains $f(a)$.

More generally, we can consider constraints on the derivatives of the map u. Such a constraint is a map $f : A \to J^k(W/C)$ (recall the jet bundle (**??**)), to which a solution is a pseudo-holomorphic section $u : C \to W$ along with a point $a \in A$ such that $J^k u : C \to J^k(W/C)$ passes through $f(a)$. A constraint $A \to J^k(W/C)$ is evidently 'equivalent' to the constraint $A \times_{J^k(W/C)} J^{k+1}(W/C) \to J^{k+1}(W/C)$.

Multiple simultaneous point constraints may be specified by a map $A \to J^k(W/C)^n$ whose composition to C^n lands inside the locus of n-tuples of distinct points.

Here is a slight generalization of the pseudo-holomorphic curve equation which we will be relevant later as an auxiliary tool for reasons explained in (**??**). Recall the notion of the *jet space* of a submersion (**??**).

5.2.11 Definition (Quasi-holomorphic sections)**.** Consider a triple $(W \to C, H, \varphi)$ where C is a smooth surface, $\pi : W \to C$ is a submersion, H/W is a real vector bundle, and

$$\varphi : J^1(W/C) \to H \tag{5.2.11.1}$$

is an affine linear map over W (recall that $J^1(W/C) \to W$ is a torsor for $\operatorname{Hom}(\pi^* TC, T_{W/C})$) which is 'elliptic' in the following sense (5.2.11.2).

(5.2.11.2) An affine linear map $\varphi : J^1(W/C) \to H$ is called *elliptic* when its linear part $\operatorname{Hom}(\pi^* TC, T_{W/C}) \to H$ sends nonzero elements of $\pi^* TC$ to isomorphisms $T_{W/C} \to H$.

A section $u : C \to W$ is then called *quasi-holomorphic* iff $\varphi(du) = 0$. The quasi-holomorphic sections are the solutions of the quasi-holomorphic map problem $(W \to C, H, \varphi)$.

5.2.12 Example (Pseudo-holomorphicity as quasi-holomorphicity)**.** If $W \to C$ is an almost complex fibration and we set $H = T_{W/C} \otimes \overline{T^* C}$ and $\varphi(\alpha) = \alpha^{0,1}$, then the quasi-holomorphic section equation $\varphi(du) = 0$ becomes the pseudo-holomorphic section equation $(du)^{0,1} = 0$.

The relevant generalization of the notion of a pseudo-holomorphic map to the setting of bordered Riemann surfaces is that of a pseudo-holomorphic map satisfying totally real boundary conditions.

⋆ **5.2.13 Definition** (Totally real boundary conditions)**.** Given a bordered Riemann surface $(C, \partial C)$ and an almost complex manifold X with a totally real submanifold $L \subseteq X$, we may consider pseudo-holomorphic maps $u : (C, \partial C) \to (X, L)$ (the pair notation indicating that $u : C \to X$ and $u(\partial C) \subseteq L$). More generally, given a totally real immersion $L \to X$, we may consider diagrams

$$\begin{array}{ccc} \partial C & \xrightarrow{\partial u} & L \\ \downarrow & & \downarrow \\ C & \xrightarrow{u} & X \end{array} \tag{5.2.13.1}$$

in which u is pseudo-holomorphic. Such pairs $(u, \partial u) : (C, \partial C) \to (X, L)$ are the solutions of the problem $(C, \partial C, X, L)$.

Similarly, a pseudo-holomorphic section problem over a bordered Riemann surface $(C, \partial C)$ consists of an almost complex submersion $\pi : W \to C$ and a submersion $K \to \partial C$ along with a totally real immersion $K \to \partial W = \pi^{-1}(\partial C)$ over ∂C. A solution of such a problem is then a diagram

$$\begin{array}{ccc} \partial C & \xrightarrow{\partial u} & K \\ \downarrow & & \downarrow \\ C & \xrightarrow{u} & W \end{array} \tag{5.2.13.2}$$

in which u is pseudo-holomorphic and both u and ∂u are sections.

5.2.14 Exercise. Show that, in the context of the definition of a pseudo-holomorphic section problem over a bordered Riemann surface, the immersion $K \to \partial W$ is totally real iff its fibers $K_p \to W_p$ (for $p \in \partial C$) are totally real.

* *

There is a natural notion (which we now make precise) of a *family* of pseudo-holomorphic map problems (in any of the senses considered thus far) parameterized by a smooth manifold B. Such a family $\{\mathcal{P}_b\}_{b\in B}$ is itself a pseudo-holomorphic map problem, which we call a *parameterized* pseudo-holomorphic map problem, a solution to which is a pair (b, u) consisting of a point $b \in B$ and a solution u of $\mathcal{P}_b$.

⋆ **5.2.15 Definition** (Parameterized pseudo-holomorphic map problem)**.** Let B be a smooth manifold. We introduce various sorts of pseudo-holomorphic map problems over B.

A pseudo-holomorphic section problem over by B consists of a pair of submersions $W \to C \to B$ where $C \to B$ has fiber dimension two, both $C \to B$ and $W \to B$ are equipped with relative almost complex structures (i.e. $T_{C/B}$ and $T_{W/B}$ have complex structures), and the map $W \to C$ is almost complex relative B (i.e. its derivative $T_{W/B} \to T_{C/B}$ is complex linear). A solution of the problem $W \to C \to B$ is a point $b \in B$ along with a pseudo-holomorphic section $u : C_b \to W_b$ (where $C_b = C \times_B b$ and $W_b = W \times_B b$ denote the fibers over b).

A quasi-holomorphic section problem over B is a pair of submersions $W \to C \to B$ along with a vector bundle H/W and an affine linear map $\varphi : J^1_B(W/C) \to H$ (recall the relative jet space (**??**)) which is elliptic (5.2.11.2). A solution of such a problem is a point $b \in B$ along with a quasi-holomorphic section $u : C_b \to W_b$.

Allowing domains with boundary in the parameterized context means that we allow $C \to B$ to be a submersion-with-boundary (**??**) (though $W \to C$ remains a submersion), and we impose boundary conditions taking the form of a submersion $K \to \partial C$ and an immersion $K \to W$ over C whose fibers $K_b \to W_b$ over points $b \in B$ are totally real (5.2.13) (or, in the quasi-holomorphic setting, elliptic (**??**)).

Parameterized problems in all the above senses pull back under maps $B' \to B$.

5.2.16 Example (Family of inhomogeneous terms)**.** Let C be a Riemann surface and X an almost complex manifold. We saw earlier (5.2.8) that almost complex structures on $X \times C$ for which the fiber inclusions $X = X \times c \subseteq X \times C$ and the projection $X \times C \to C$ are both almost complex are in natural bijection with sections $\gamma : C \times X \to TX \otimes \overline{T^*C}$, and that pseudo-holomorphicity of a section $(u, \mathbf{1}) : C \to X \times C$ with respect to such an almost complex structure amounts to the equation $(du)^{0,1} + \gamma(u) = 0$ for the map $u : C \to X$. Now fix a smooth manifold E and a section $\gamma : C \times X \times E \to TX \otimes \overline{T^*C}$. This gives rise to a pseudo-holomorphic section problem $C \times X \times E \to C \times E \to E$ to which a solution is a pair $(e \in E, u : C \to X)$ satisfying $(du)^{0,1} + \gamma(u, e) = 0$.

In fact, all that is really required to make sense of the various sorts of parameterized problems defined in (5.2.15) is a suitable notion of submersion (or submersion-with-boundary). Thus, the base B could in fact be a log smooth manifold (2.8), a derived smooth manifold (2.10), or an object of one of the 'hybrid categories' discussed in (2.12). It could also be any stack over these categories.

We will adopt the following definition of a point constraint for parameterized map problems. At first glance, it appears much less general than the class of point constraints considered earlier (5.2.10), but we will see that in fact it is not.

5.2.17 Definition (Parameterized point constraints)**.** A point constraint for a parameterized section problem $W \to C \to B$ is a map $f : A \to J^k_B(W/C)$ (recall the relative jet space (**??**)) whose composition $A \to C$ is a closed embedding and whose composition $A \to B$ is a proper local isomorphism (2.1.28). A solution to the constrained problem is a solution $(b, u : C_b \to W_b)$ of the unconstrained problem whose k-jet $J^k u : C_b \to J^k(W_b/C_b) = J^k_B(W/C)_b$ agrees with f under pullback to A_b.

5.2.18 Example. Consider a single point constraint in the sense of (5.2.10) for a section problem $W \to C$, namely a smooth manifold A with a map $A \to J^k(W/C)$. Such a constrained problem is 'equivalent' to the parameterized problem $W \times A \to C \times A \to A$ equipped with the single point constraint induced by the map $A \to J^k(W/C)$ regarded as a section of $J^k(W/C) \times A = J^k_A((W \times A)/(C \times A)) \to A$.

More generally, given a parameterized problem $W \to C \to B$ and a map $f : A \to J^k_B(W/C)$, we may wish to consider solutions $(b, u : C_b \to W_b)$ together with a point $a \in A$ such that the image of $J^k u$ contains $f(a)$. This is equivalent to the pullback $W \times_B A \to C \times_B A \to A$ equipped with the point constraint in the sense of (5.2.17) induced by f.

* *

The theory of pseudo-holomorphic maps becomes most interesting when we allow domains and targets with cylindrical structure (2.8.41)(3.5.2) and when we allow them to degenerate/break and glue as the base parameter $b \in B$ is varied. To describe such domains/targets and families thereof, we will use the language of log smooth manifolds developed in (2.8).

5.3 Moduli stacks

In the previous section (5.2), we introduced various sorts of pseudo-holomorphic map problems and solutions thereof (pseudo-holomorphic maps satisfying the relevant boundary conditions, point constraints, etc.). The goal of the present section is to formalize various notions (continuous, smooth, and otherwise) of *families* of solutions of such problems.

The *moduli stack* $\underline{\mathrm{Hol}}_B(C,W)$ associated to a given pseudo-holomorphic map problem $\wp = (W \to C \to B)$ associates to an object Z of the relevant geometric category (Top, Sm, Der, etc.) the collection of families of solutions of $\wp$ parameterized by Z. Being a sheaf on a topological (∞-)site (2.9) such as Top, Sm, Der, etc., the moduli stack may be regarded as a geometric object. To distinguish between the moduli stacks on different categories, we will say 'topological moduli stack' $\underline{\mathrm{Hol}}_B(C,W)_{\mathsf{Top}}$, 'smooth moduli stack' $\underline{\mathrm{Hol}}_B(C,W)_{\mathsf{Sm}}$, etc.

It will help to be familiar with mapping stacks (2.4).

* *

The definition of the moduli stack on smooth manifolds Sm is straightforward.

⋆ **5.3.1 Definition** (Smooth moduli stack)**.** Fix a quasi-holomorphic section problem ($W \to C \to B, H, \varphi$) (5.2.15). Recall that this means B is a smooth manifold, the map $C \to B$ is a submersion with two-dimensional fibers, $W \to C$ is a submersion, H/W is a vector bundle, and $\varphi : J^1(W/C) \to H$ is an affine linear map which is elliptic in the sense of (5.2.11.2); more generally, the map $C \to B$ can be a submersion-with-boundary, in which case suitable boundary conditions are imposed; we can also include point constraints (5.2.17).

The *smooth moduli stack* $\underline{\mathrm{Hol}}_B(C,W)$ associated to the problem $(W \to C \to B, H, \varphi)$ assigns to a smooth manifold Z the set of pairs (f,u) consisting of a smooth map $f : Z \to B$ and a smooth map $u : C \times_B Z \to W$ over C whose specialization $u_z : C_z \to W_z$ is quasi-holomorphic for every point $z \in Z$.

$$
\begin{tikzcd}
 & W \arrow[d] \\
C \times_B Z \arrow[ur, "u"] \arrow[r] \arrow[d] & C \arrow[d] \\
Z \arrow[r, "f"] & B
\end{tikzcd}
\tag{5.3.1.1}
$$

The map u is subject to whatever boundary conditions or point constraints exist in the input problem.

5.3.2 Example. Let C be a Riemann surface and X an almost complex manifold. A map $Z \to \underline{\mathrm{Hol}}(C,X)$ is a map $Z \times C \to X$ whose restriction to each fiber $z \times C$ is pseudo-holomorphic.

5.3.3 Exercise. Let $(W \to C \to B, H, \varphi)$ be a quasi-holomorphic section problem, and let $(W' \to C' \to B', H', \varphi')$ be its pullback under a map of smooth manifolds $B' \to B$. Define a tautological isomorphism $\underline{\mathrm{Hol}}_{B'}(C',W') = \underline{\mathrm{Hol}}_B(C,W) \times_B B'$.

The definition of the moduli stack $\underline{\mathrm{Hol}}_B(C, W)$ makes sense more generally for any smooth stack B.

* *

We now define the moduli stack on topological spaces Top. Its definition depends on the 'hybrid category' of topological-smooth spaces TopSm (2.12).

⋆ **5.3.4 Definition** (Topological moduli stack)**.** Given a quasi-holomorphic section problem $(W \to C \to B, H, \varphi)$, a map $Z \to \underline{\mathrm{Hol}}_B(C, W)$ from a topological space Z is defined by replacing the category Sm in the definition of the moduli stack on Sm (5.3.1) with the category of topological-smooth spaces TopSm. That is, a map $Z \to \underline{\mathrm{Hol}}_B(C, W)$ is a diagram (5.3.1.1) in TopSm in which the specialization of the map u to the fiber over every point $z \in Z$ is quasi-holomorphic (and satisfies the relevant boundary conditions and point constraints, if any).

In this definition, the base B does not need to be a smooth manifold, rather it can be any topological-smooth space or every topological-smooth stack (in which case the maps $W \to C \to B$ must be submersive in the relevant sense (2.12.9)). The topological stack $\underline{\mathrm{Hol}}_B(C, W)$ is evidently unchanged by pulling back the moduli problem under $(\mathsf{Top} \to \mathsf{TopSm})_!(\mathsf{Top} \to \mathsf{TopSm})^*B \to B$ (indeed, formation of the moduli stack is compatible with pullback (2.4.16.2), and the operation $\times_B(\mathsf{Top} \to \mathsf{TopSm})_!(\mathsf{Top} \to \mathsf{TopSm})^*B$ is trivial over $\mathsf{Top} \subseteq \mathsf{TopSm}$). Thus for the purpose of defining the topological moduli stack, we lose no generality by restricting consideration to bases $B \in \mathsf{Shv}(\mathsf{Top}) \subseteq \mathsf{Shv}(\mathsf{TopSm})$. We are, in fact, usually interested in the case of smooth stacks $B \in \mathsf{Shv}(\mathsf{Sm})$, which thus for the purpose of defining the topological moduli stack can be replaced by their image $|B|_! = (\mathsf{Sm} \to \mathsf{Top})_! B \in \mathsf{Shv}(\mathsf{Top})$ (recall that $(\mathsf{Top} \hookrightarrow \mathsf{TopSm})^* = (\mathsf{TopSm} \xrightarrow{|\cdot|} \mathsf{Top})_!$ (**??**)).

We will use subscripts to distinguish the moduli stacks on different categories. This notation is, in particular, essential when discussing comparison maps between them.

⋆ **5.3.5 Definition** (Comparing smooth and topological moduli stacks)**.** A map $Z \to \underline{\mathrm{Hol}}_B(C, W)_{\mathsf{Sm}}$ determines, by forgetting structure, a map $|Z| \to \underline{\mathrm{Hol}}_B(C, W)_{\mathsf{Top}}$. This defines a tautological map $\underline{\mathrm{Hol}}_B(C, W)_{\mathsf{Sm}} \to (\mathsf{Sm} \to \mathsf{Top})^* \underline{\mathrm{Hol}}_B(C, W)_{\mathsf{Top}}$ for any smooth stack B. More geometrically significant is the associated (by adjunction) 'comparison map'

$$(\mathsf{Sm} \to \mathsf{Top})_! \underline{\mathrm{Hol}}_B(C, W)_{\mathsf{Sm}} \to \underline{\mathrm{Hol}}_B(C, W)_{\mathsf{Top}}. \tag{5.3.5.1}$$

Generally speaking, an 'open substack' of a moduli stack $\underline{\mathrm{Hol}}_B(C, W)$ refers to an open substack of the *topological* moduli stack $\underline{\mathrm{Hol}}_B(C, W)_{\mathsf{Top}}$. Such an open substack determines (by pulling back under the comparison map) corresponding open substacks of all other flavors of moduli stacks we consider. These other moduli stacks may have open substacks which do not arise like this (see (2.6.7) for example), but they are not of much relevance. The terms 'open covering' and 'locally' are to be understood accordingly. Open substacks (and open coverings) of $\underline{\mathrm{Hol}}_B(C, W)$ usually arise via pullback from open substacks/coverings of $\underline{\mathrm{Sec}}_B(C, W)$.

* *

The moduli stacks presented thus far admit evident fiber product presentations. Due to the 'diagrammatic' nature of these presentations, they are valid independent of which flavor of moduli stack is being considered.

⋆ **5.3.6 Definition** (Moduli stacks as fiber products)**.** Given a quasi-holomorphic section problem $(W \to C \to B, H, \varphi)$, there is a (quite tautological) pullback square presentation of the moduli stack $\underline{\mathrm{Hol}}_B(C, W)$ in terms of the stacks of smooth sections $\underline{\mathrm{Sec}}_B(C, W)$ and $\underline{\mathrm{Sec}}_B(C, H)$.

$$\begin{tikzcd} \underline{\mathrm{Hol}}_B(C,W) \arrow[r] \arrow[d] & \underline{\mathrm{Sec}}_B(C,W) \arrow[d, "u\mapsto(u,0)"] \\ \underline{\mathrm{Sec}}_B(C,W) \arrow[r, "u\mapsto\varphi(du)"] & \underline{\mathrm{Sec}}_B(C,H) \end{tikzcd} \tag{5.3.6.1}$$

In the presence of boundary conditions $(W, K) \to (C, \partial C)$, we consider the stacks of sections $\underline{\mathrm{Sec}}_B((C, \partial C), (W, K))$ which parameterize diagrams (5.2.13.2), meaning a map $Z \to \underline{\mathrm{Sec}}_B((C, \partial C), (W, K))$ is a diagram of the following shape.

$$\begin{tikzcd} & & & K \arrow[r] \arrow[d] & W \arrow[d] \\ \partial C_Z \arrow[urrr] \arrow[r] \arrow[dr] & C_Z \arrow[urrr] \arrow[d] & & \partial C \arrow[r] \arrow[dr] & C \arrow[d] \\ & Z \arrow[rrr] & & & B \end{tikzcd} \tag{5.3.6.2}$$

For $\underline{\mathrm{Sec}}_B(C, H)$ with boundary conditions, only the map to W is lifted to H.

Point constraints (5.2.17) may also be imposed via fiber product. Namely, for a point constraint $f : A \to J^k_B(W/C)$, we have the following fiber product presentation of the moduli stack $\underline{\mathrm{Hol}}_B(C, W)_f$ of solutions to the constrained problem.

$$\begin{tikzcd} \underline{\mathrm{Hol}}_B(C,W)_f \arrow[r] \arrow[d] & B \arrow[d, "f"] \\ \underline{\mathrm{Hol}}_B(C,W) \arrow[r] & \underline{\mathrm{Sec}}_B(A, J^k_B(W/C)) \end{tikzcd} \tag{5.3.6.3}$$

The bottom map is the composition $\underline{\mathrm{Hol}}_B(C, W) \to \underline{\mathrm{Sec}}_B(C, W) \to \underline{\mathrm{Sec}}_B(C, J^k_B(W/C)) \to \underline{\mathrm{Sec}}_B(A, J^k_B(W/C))$ sending a map u to its k-jet restricted to $A \subseteq C$.

5.3.7 Corollary. *The map* $\underline{\mathrm{Hol}}_B(C, W) \to \underline{\mathrm{Sec}}_B(C, W)$ *is a closed embedding of topological stacks.*

Proof. It is a pullback (5.3.6.1) of 'zero' map $\underline{\mathrm{Sec}}_B(C, W) \to \underline{\mathrm{Sec}}_B(C, H)$, which is a closed embedding of topological stacks since $W \to H$ is a closed embedding and $C \to B$ is open (2.4.19). □

5.3.8 Corollary. *The map $\underline{\mathrm{Hol}}_B(C,W) \to B$ is separated (as a map of topological stacks) if $W \to C$ is separated.*

Proof. Since $\underline{\mathrm{Hol}}_B(C,W) \to \underline{\mathrm{Sec}}_B(C,W)$ is a closed embedding, it is separated, so it is enough to know that $\underline{\mathrm{Sec}}_B(C,W) \to B$ is separated, which is (2.4.21). $\square$

* *

We now define the moduli stack on the ∞-category of derived smooth manifolds Der. In this context, vertical quasi-holomorphicity (i.e. vanishing of $\varphi(du)$) is not a fiberwise condition (a real valued function on a derived smooth manifold may be nonzero yet vanish at every point). In fact, it is not a condition at all, rather it is the *extra data* of a path between $\varphi(du)$ and zero in the space of sections over the total space of the family. For this reason, the most transparent definition of the derived smooth moduli stack of quasi-holomorphic maps is via the fiber product presentation (5.3.6) (in other words, vertical quasi-holomorphicity is expressed *diagrammatically*).

⋆ **5.3.9 Definition** (Derived smooth moduli stack)**.** The moduli stack $\underline{\mathrm{Hol}}_B(C,W)$ on the ∞-category of derived smooth manifolds is defined as the fiber product (5.3.6) of derived smooth stacks, where the stacks of sections $\underline{\mathrm{Sec}}_B(C,W)$ have their usual categorical meaning (2.4) (which is purely diagrammatic hence applies in any ∞-category). The bottom map $u \mapsto \varphi(du)$ in (5.3.6.1) is defined using the tangent functor on derived smooth manifolds (the tangent functor gives a map of derived smooth stacks $\underline{\mathrm{Sec}}_B(C,W) \to \underline{\mathrm{Sec}}_B(C,J^1(W/C))$ sending $u \mapsto du$). Point conditions are also imposed by fiber product against the relevant evaluation map(s), i.e. diagrammatically.

5.3.10 Definition (Comparing moduli functors over Sm, Der, Top)**.** There are tautological maps

$$\underline{\mathrm{Hol}}_B(C,W)_{\mathsf{Sm}} \to \underline{\mathrm{Hol}}_B(C,W)_{\mathsf{Der}} \to \underline{\mathrm{Hol}}_B(C,W)_{\mathsf{Top}} \tag{5.3.10.1}$$

where the notion of a map from $X \in \mathsf{Shv}(\mathsf{C})$ to $Y \in \mathsf{Shv}(\mathsf{D})$ over a topological functor $f : \mathsf{C} \to \mathsf{D}$ is defined via the adjunction $(f_!, f^*)$ (2.9.35), namely it is a map $f_!X \to Y$ or equivalently a map $X \to f^*Y$. Indeed, the functors $\mathsf{Sm} \to \mathsf{Der} \to \mathsf{Top}$ induce such comparison maps on stacks of sections $\underline{\mathrm{Sec}}$, which induce the same on the stacks $\underline{\mathrm{Hol}}$ via their definition in terms of fiber products. Concretely, this just amounts to noting that the functors $\mathsf{Sm} \to \mathsf{Der} \to \mathsf{Top}$ send families of quasi-holomorphic sections to families of quasi-holomorphic sections, since the notion of such a family is defined diagrammatically and these functors are compatible with the relevant tangent functors.

5.4 Tangent complexes

Every moduli stack $\underline{\mathrm{Hol}}(C, W)$ carries a family of elliptic operators on C (called the 'deformation operators') which describes the first order deformation theory of quasi-holomorphic sections of $W \to C$. This family of operators is termed the *analytic tangent complex* of $\underline{\mathrm{Hol}}(C, W)$. The 'total space' of this complex is itself a moduli stack of quasi-holomorphic sections $\underline{\mathrm{Hol}}(C, T_{W/C})$, whose underlying quasi-holomorphic section problem $T_{W/C} \to C$ we term the *tangent moduli problem* associated to original problem $W \to C$. A point of $\underline{\mathrm{Hol}}(C, W)$ is called *regular* when its analytic tangent cohomology is concentrated in degree zero (i.e. when its deformation operator is surjective). By semi-continuity of the cohomology of elliptic operators, the regular locus is an open substack $\underline{\mathrm{Hol}}(C, W)^{\mathrm{reg}} \subseteq \underline{\mathrm{Hol}}(C, W)$.

It turns out that the setting of derived smooth stacks provides a precise sense in which the moduli stack $\underline{\mathrm{Hol}}(C, T_{W/C})$ associated to the tangent moduli problem is the tangent space of the original moduli stack $\underline{\mathrm{Hol}}(C, W)$. Indeed, every derived smooth stack has a tangent complex in the sense of left Kan extension $T_! : \mathsf{Shv}(\mathsf{Der}) \to \mathsf{Shv}(\mathsf{Der})$ along the tangent complex functor $T : \mathsf{Der} \to \mathsf{Der}$, and there is a canonical identification $T_! \underline{\mathrm{Hol}}(C, W) = \underline{\mathrm{Hol}}(C, T_{W/C})$. This has an important consequence: if $\underline{\mathrm{Hol}}(C, W)$ is representable by a derived smooth manifold, then its *geometric tangent complex* (i.e. the tangent complex of this representing object) is automatically identified with its analytic tangent complex, for essentially formal reasons.

This discussion of tangent complexes generalizes readily to the parameterized setting. Moduli stacks $\underline{\mathrm{Hol}}_B(C, W)$ carry families of elliptic operators termed the relative (or vertical) analytic tangent complex of $\underline{\mathrm{Hol}}_B(C, W) \to B$. When B is a (derived log) smooth manifold (or sufficiently nice stack), there is also a notion of absolute analytic tangent complex of $\underline{\mathrm{Hol}}_B(C, W)$, which maps to TB with fiber the relative analytic tangent complex. There are correspondingly two notions of regularity for points of $\underline{\mathrm{Hol}}_B(C, W)$ depending on which analytic tangent complex is being considered: 'regular' means the absolute analytic tangent complex vanishes in degrees > 0 (this locus is denoted $\underline{\mathrm{Hol}}_B(C, W)^{\mathrm{reg}}$), while 'regular relative B' means the same for the analytic tangent complex relative B (this locus is denoted $\underline{\mathrm{Hol}}_B(C, W)^{\mathrm{reg}/B}$). The tangent moduli problem $T\wp$ of $\wp$ is also defined in the parameterized setting, as is the canonical identification $T_! \underline{\mathrm{Hol}}(\wp) = \underline{\mathrm{Hol}}(T\wp)$ (in both absolute and relative flavors). A subtlety is that the absolute tangent moduli problem in the parameterized setting is not canonically defined, rather it depends on a choice of connection on $W \to B$. It is also this absolute tangent moduli problem which forces us into the setting of quasi-holomorphic section problems rather than just pseudo-holomorphic section problems.

This discussion would apply equally well in any other non-linear elliptic Fredholm setting.

* *

Recall the tangent functor $T : \mathsf{Shv}(\mathsf{Der}) \to \mathsf{Shv}(\mathsf{Der})$ on derived smooth stacks (2.10.32). This functor is the left Kan extension $T_!$ of the tangent functor $T : \mathsf{Der} \to \mathsf{Der}$ on derived smooth manifolds. It is also the pullback $(- \times \tau)^*$ along the multiplication by the universal tangent vector functor $\times\tau : \mathsf{Der} \to \mathsf{Der}$. The tangent functor $T : \mathsf{Shv}(\mathsf{Der}) \to \mathsf{Shv}(\mathsf{Der})$ is both continuous and cocontinuous. There is a natural identification $T\underline{\mathrm{Sec}}(C,W)=\underline{\mathrm{Sec}}(C,T_{W/C})$ and, more generally, $T(\underline{\mathrm{Sec}}_B(C,W)/B)=\underline{\mathrm{Sec}}_B(C,T_{W/C})$ (2.10.33).

5.4.1 Definition (Tangent space of $\underline{\mathrm{Hol}}(C,W)$)**.** Let $(W\to C,H,\varphi)$ be a quasi-holomorphic section problem. To understand the tangent space $T\underline{\mathrm{Hol}}(C,W)$, we apply the tangent functor T to the fiber product presentation of $\underline{\mathrm{Hol}}(C,W)$ (5.3.6), resulting in a fiber product presentation of $T\underline{\mathrm{Hol}}(C,W)$ (since T preserves fiber products).

$$\begin{tikzcd} T\underline{\mathrm{Hol}}(C,W) \arrow[r] \arrow[d] & T\underline{\mathrm{Sec}}(C,W) \arrow[d] \\ T\underline{\mathrm{Sec}}(C,W) \arrow[r, "T(u\mapsto\varphi(du))"] & T\underline{\mathrm{Sec}}(C,H) \end{tikzcd} \tag{5.4.1.1}$$

Given the natural identifications $T\underline{\mathrm{Sec}}(C,W)=\underline{\mathrm{Sec}}(C,T_{W/C})$ (2.10.33), this may be rewritten as follows.

$$\begin{tikzcd} T\underline{\mathrm{Hol}}(C,W) \arrow[r] \arrow[d] & \underline{\mathrm{Sec}}(C,T_{W/C}) \arrow[d] \\ \underline{\mathrm{Sec}}(C,T_{W/C}) \arrow[r, "T(u\mapsto\varphi(du))"] & \underline{\mathrm{Sec}}(C,T_{H/C}) \end{tikzcd} \tag{5.4.1.2}$$

Here the variation of $\varphi(du)$ induced by a section $(u,\dot u)$ of $T_{W/C}\to C$ takes the form $T\varphi(d(u,\dot u))$ for the affine linear map $T_C\varphi:T_{J^1(W/C)/C}=J^1(T_{W/C}/C)\to T_{H/C}$ over $T_{W/C}$ (the vector bundle structure on $T_{H/C}\to T_{W/C}$ comes from vertical differentation of the vector bundle $H\to W$). We have thus identified $T\underline{\mathrm{Hol}}(C,W)$ with the moduli stack $\underline{\mathrm{Hol}}(C,T_{W/C})$ of solutions to the *tangent moduli problem* $(T_{W/C}\to C,T_{H/C},T\varphi)$ obtained by simply applying the tangent functor to the input moduli problem $(W\to C,H,\varphi)$ in the appropriate way.

The tangent moduli problem $(T_{W/C}\to C,T_{H/C},T\varphi)$ is 'linear' over $(W\to C,H,\varphi)$, in the sense that it encodes, quite directly, a family of elliptic operators over $\underline{\mathrm{Hol}}(C,W)$ whose stack of solutions is $\underline{\mathrm{Hol}}(C,T_{W/C})\to\underline{\mathrm{Hol}}(C,W)$. Indeed, at any point $u\in\underline{\mathrm{Hol}}(C,W)$, there is a first order differential operator

$$D_u:C^\infty(C,u^*T_{W/C})\to C^\infty(C,u^*H) \tag{5.4.1.3}$$

which measures the variation in $\varphi(du)$ induced by variations of u. This *deformation operator* D_u can be obtained formally by linearizing the triple $(W\to C,H,\varphi)$ around u to obtain $(u^*T_{W/C}\to C,u^*H,T\varphi(u,\cdot))$. The operators D_u form a family of operators over $\underline{\mathrm{Hol}}(C,W)$, and its solution stack (3.7.1) coincides (by construction) with $\underline{\mathrm{Hol}}(C,T_{W/C})\to\underline{\mathrm{Hol}}(C,W)$. The symbol $\sigma(D_u):T^*C\to\mathrm{Hom}(u^*T_{W/C},u^*H)$ of the deformation operator D_u is simply the linear part of φ pulled back under u, hence D_u is elliptic by hypothesis on φ (5.2.11.2). The solution stack of a family of elliptic operators is a perfect complex on the base (3.7), so this upgrades $T\underline{\mathrm{Hol}}(C,W)=\underline{\mathrm{Hol}}(C,T_{W/C})\to\underline{\mathrm{Hol}}(C,W)$ to an object of $\mathsf{Perf}^{[0\ 1]}(\underline{\mathrm{Hol}}(C,W))$.

5.5 Elliptic bootstrapping

We now come to the first bit of analysis in our discussion of pseudo-holomorphic maps: *elliptic boostrapping*, which is a generalization of the linear elliptic estimates discussed in (3.3)–(3.6) to the present non-linear setting of pseudo-holomorphic (and more generally quasi-holomorphic) maps. Elliptic bootstrapping refers to estimates of the form

$$\|u\|_{s+1} \leq F_s(\|u\|_s) \tag{5.5.0.1}$$

for some functions F_s (depending on the geometry of the source and target), under the assumption that u satisfies some particular (possibly non-linear) elliptic equation. While for linear elliptic operators (with smooth coefficients) such estimates hold for all s and with F_s linear, in the present non-linear setting they only hold for sufficiently large s and with not necessarily linear F_s.

5.5.1 Exercise. Let $\Omega \subseteq \mathbb{R}^n$ be open and let $f : \Omega \to \mathbb{R}$ satisfy $\|f\|_{C^{k+1}} \leq M$. Show that for every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon, M) > 0$ such that if $\|f\|_{C^0} \leq \delta$ then $\|f\|_{C^k} \leq \varepsilon$. (Prove the case $k = 1$ directly and then use induction.)

5.6 Regularity

In this section, we show that the regular loci in smooth moduli spaces of pseudo-holomorphic curves are regular. The reasoning would apply equally well in any other non-linear elliptic Fredholm setting.

5.6.1 Regularity Theorem. *For any quasi-holomorphic section problem* $(W \to C \to B, H, \varphi)$ *over a log smooth manifold* B*, the regular locus* $\underline{\mathrm{Hol}}_B(C,W)^{\mathrm{reg}}_{\mathsf{Sm}} \subseteq \underline{\mathrm{Hol}}_B(C,W)_{\mathsf{Sm}}$ *is representable and the comparison map* $(\mathsf{Sm} \to \mathsf{Top})_! \underline{\mathrm{Hol}}_B(C,W)^{\mathrm{reg}}_{\mathsf{Sm}} \to \underline{\mathrm{Hol}}_B(C,W)^{\mathrm{reg}}_{\mathsf{Top}}$ *is an isomorphism.*

The Regularity Theorem is a non-linear generalization of the results about kernels of elliptic operators (3.3)–(3.7), and its proof follows the same outline just in a non-linear setting. For the sake of exposition, we prove the Regularity Theorem in stages of ever increasing generality (5.6.6)(5.6.8)(5.6.10).

* *

We begin our treatment of the Regularity Theorem (5.6.1) with the 'absolute' case of $\underline{\mathrm{Hol}}(C,W)$.

It is convenient to work here with *affine* vector spaces and bundles as it eliminates the distraction caused by irrelevant choice of zero. The functor ('linear part') from affine vector spaces to vector spaces is denoted with a subscript 0.

The first step is to fix local linear coordinates on the target.

5.6.2 Definition (Linear model)**.** A *linear model* quasi-holomorphic section problem is a quasi-holomorphic section problem $(W \to C, H, \varphi)$ together with the structure of an affine vector bundle on $W \to C$ and a choice of vector bundle H/C whose pullback to W is identified with H/W.

If C is compact Hausdorff, then the moduli stack $\underline{\mathrm{Hol}}(C,W)$ has an open cover by moduli stacks associated to linear models. Indeed, we saw earlier a corresponding result for the stack of all sections $\underline{\mathrm{Sec}}(C,W)$ (2.6.28). The only difference here is that we must in addition ensure that $H|_{W^\circ}$ is pulled back from C, and we leave it as an exercise to upgrade the argument of (2.6.27) to take care of this.

Thus to prove any local result about $\underline{\mathrm{Hol}}(C,W)$ for C compact Hausdorff, it suffices to prove it in the case that $(W \to C, H, \varphi)$ is a linear model.

5.6.3 Definition (Linear projection)**.** Let $(W \to C, H, \varphi)$ be a linear model quasi-holomorphic section problem. Let

$$\lambda : C^\infty(C,W) \to K \tag{5.6.3.1}$$

be a continuous affine linear map to a finite-dimensional affine vector space K (recall that such λ induces maps of topological and smooth stacks $\underline{\mathrm{Sec}}(C,W)$ (**??**)(**??**)). We will be interested in the composition

$$\underline{\mathrm{Hol}}(C,W) \to C^\infty(C,W) \xrightarrow{\lambda} K \tag{5.6.3.2}$$

which we refer to as a *linear projection* on $\underline{\mathrm{Hol}}(C,W)$.

5.6.4 Definition (λ-regular locus)**.** Let $(W \to C, H, \varphi)$ be a linear model quasi-holomorphic section problem and $\lambda : C^\infty(C, W) \to K$ a linear projection. Consider the restriction of the linear part $\lambda_0 : C^\infty(C, W_0) \to K_0$ of λ (a distribution $\lambda_0 \in C^{-\infty}(C, \Omega_C \otimes \operatorname{Hom}(W_0, K_0))$) to $\ker D$.

$$
\begin{tikzcd}
\ker D \arrow[r] \arrow[d] & C^\infty(C, T_{W/C}) \arrow[r] \arrow[d] & C^\infty(C, W_0) \arrow[r, "\lambda_0"] & K_0 \\
\operatorname{Hol}(C, W) \arrow[r] & C^\infty(C, W)
\end{tikzcd}
\tag{5.6.4.1}
$$

The upper middle horizontal map comes from the identification $T_{W/C} = W \times W_0$ coming from the affine vector bundle structure on $W \to C$. Over the regular locus $\operatorname{Hol}(C, W)^{\mathrm{reg}} \subseteq \operatorname{Hol}(C, W)$, where $\ker D$ is a vector bundle (3.7), we have defined a map of vector bundles $\ker D \to K_0$. The *λ-regular locus* $\operatorname{Hol}(C, W)^{\lambda\text{-reg}} \subseteq \operatorname{Hol}(C, W)^{\mathrm{reg}}$ is the locus where this map $\ker D \to K_0$ is an isomorphism.

5.6.5 Exercise. Show that $\underline{\operatorname{Hol}}(C, W)^{\mathrm{reg}} = \bigcup_\lambda \underline{\operatorname{Hol}}(C, W)^{\lambda\text{-reg}}$.

5.6.6 Proposition. *For any quasi-holomorphic section problem $(W \to C, H, \varphi)$ with C compact Hausdorff, the regular locus $\underline{\operatorname{Hol}}(C, W)^{\mathrm{reg}}_{\mathsf{Sm}} \subseteq \underline{\operatorname{Hol}}(C, W)_{\mathsf{Sm}}$ is representable and the comparison map $(\mathsf{Sm} \to \mathsf{Top})_! \underline{\operatorname{Hol}}(C, W)^{\mathrm{reg}}_{\mathsf{Sm}} \to \underline{\operatorname{Hol}}(C, W)^{\mathrm{reg}}_{\mathsf{Top}}$ is an isomorphism.*

Proof. The desired assertion is local, so we may assume wlog that the section problem in question $(W \to C, H, \varphi)$ is a linear model (5.6.2). Since the regular locus is covered by the λ-regular loci for linear projections $\lambda : C^\infty(C, W) \to K$ (5.6.5), it suffices to show that the restriction $\lambda|_{\underline{\operatorname{Hol}}(C,W)^{\lambda\text{-reg}}}$ is a local isomorphism of both topological and smooth stacks.

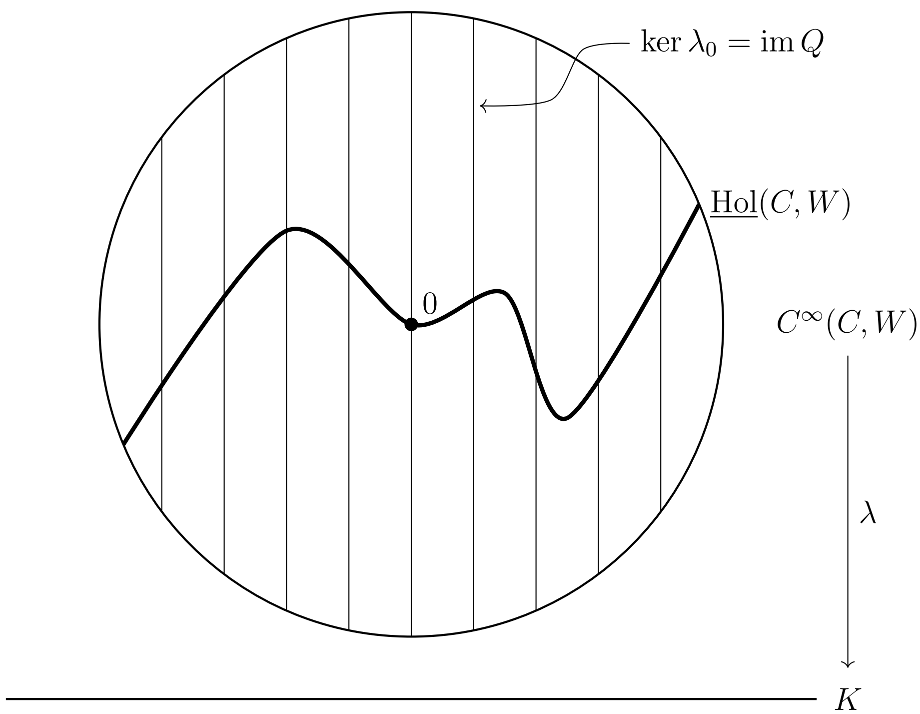
Fix a basepoint in $\operatorname{Hol}(C, W)^{\lambda\text{-reg}}$ around which we shall show the restriction of λ is a local isomorphism. Declare this basepoint to be the zero section of $W \to C$.

Fix the notation

$$\mathbf{D} : \underline{\operatorname{Sec}}(C, W) \to \underline{\operatorname{Sec}}(C, H) \tag{5.6.6.1}$$
$$\xi \mapsto \varphi(d\xi) \tag{5.6.6.2}$$

which is a map of topological stacks. We have $\underline{\operatorname{Hol}}(C, W) = \mathbf{D}^{-1}(0) = \underline{\operatorname{Sec}}(C, W) \times_{\underline{\operatorname{Sec}}(C,H)} 0$ by definition.

Denote by $D : C^\infty(C, W) \to C^\infty(C, H)$ the linearization of $\mathbf{D}$ at zero, which is a linear elliptic operator (hence also a map of topological stacks). Since the zero section is λ-regular, the map $D \oplus \lambda_0 : C^\infty(C, W) \to C^\infty(C, H) \oplus K_0$ is an isomorphism of topological vector spaces (hence also of topological and smooth stacks). Its linear inverse $Q \oplus \alpha$ enjoys the bound $\|Q\|_{(s,s+1)} < \infty$ for sufficiently large s (depending on λ) by elliptic regularity (**??**).



(5.6.6.3)

We now consider the endomorphism

$$R : C^\infty(C,W) \to C^\infty(C,W) \tag{5.6.6.4}$$
$$\xi \mapsto \xi - Q\mathbf{D}\xi \tag{5.6.6.5}$$

which is a map of topological stacks (the point of this map is that $R(\xi)$ is 'closer' to lying in $\mathbf{D}^{-1}(0)$ than ξ is since Q is right inverse to the linearization of $\mathbf{D}$). Since $\operatorname{im} Q \subseteq \ker \lambda_0$, the map R restricts to an endomorphism of each fiber $\lambda^{-1}(k)$. The identity $\mathbf{D}\xi = DQ\mathbf{D}\xi = D(\xi - R(\xi))$ implies the estimate

$$\|\mathbf{D}\xi\|_{s-1} \le \|D\|_{(s,s-1)}\|\xi - R\xi\|_s. \tag{5.6.6.6}$$

In particular, a point $\xi \in C^\infty(C,W)$ lies in $\mathrm{Hol}(C,W)$ iff $R(\xi) = \xi$.

We wish to show that in a neighborhood of zero, the restriction of R to any fiber $\lambda^{-1}(k)$ is a contraction mapping in a certain precise sense. To prove this, we should bound, for $\xi - \zeta \in \ker \lambda_0$, the quantity

$$R\xi - R\zeta = (\xi - \zeta) - (Q\mathbf{D}\xi - Q\mathbf{D}\zeta) \tag{5.6.6.7}$$
$$= -Q(\mathbf{D}\xi - \mathbf{D}\zeta - D(\xi - \zeta)) \tag{5.6.6.8}$$

(note the use of $\xi - \zeta \in \ker \lambda_0 = \operatorname{im} Q$ to write $\xi - \zeta = QD(\xi - \zeta)$). The expression $A(\xi,\zeta) = \mathbf{D}\xi - \mathbf{D}\zeta - D(\xi - \zeta)$ is a non-linear first order differential operator applied to (ξ,ζ). That is, we have $A(\xi,\zeta) = B(J^1\xi, J^1\zeta)$ for some smooth function B over C. Since

$A(\xi, \xi) = 0$, it follows that the function B vanishes along the diagonal. Since $A(\xi, \zeta)$ vanishes to second order at $\xi = \zeta = 0$ (because D is the derivative of $\mathbf{D}$ at zero), it follows that B vanishes to second order along the zero section. Together, these imply (3.1.37) the estimate

$$\|B(\alpha, \beta)\|_s \leq \text{const}_s \|\alpha - \beta\|_s (\|\alpha\|_s + \|\beta\|_s) \tag{5.6.6.9}$$

provided $H^s \subseteq C^0$ and $\|\alpha\|_{C^0}, \|\beta\|_{C^0} \leq 1$. We thus have the following *quadratic estimate*

$$\begin{aligned} \|\mathbf{D}\xi - \mathbf{D}\zeta - D(\xi - \zeta)\|_{s-1} &= \|B(J^1\xi, J^1\zeta)\|_{s-1} \\ &\leq \text{const}_s \|\xi - \zeta\|_s (\|\xi\|_s + \|\zeta\|_s) \end{aligned} \tag{5.6.6.10}$$

provided $H^{s-1} \subseteq C^0$ and $\|\xi\|_s, \|\zeta\|_s \leq 1$. This implies our desired contraction estimate

$$\begin{aligned} \|R\xi - R\zeta\|_s &\leq \|Q\|_{(s-1,s)} \|\mathbf{D}\xi - \mathbf{D}\zeta - D(\xi - \zeta)\|_{s-1} \\ &\leq \text{const}_s \|\xi - \zeta\|_s (\|\xi\|_s + \|\zeta\|_s) \end{aligned} \tag{5.6.6.11}$$

provided we also have $\|Q\|_{(s-1,s)} < \infty$ (which hold for sufficiently large s). Note that this is a nontrivial H^s-bound only over a neighborhood of zero depending on s.

We now define a local inverse to $\lambda|_{\underline{\mathrm{Hol}}(C,W)}$ near zero by iterating the map R. Let $N_0 : K \to C^\infty(C, W)$ be any linear section of λ, and inductively define $N_i := R(N_{i-1})$ for $i \geq 1$. Since R is a map of topological stacks, so is each N_i. We have the contraction estimate $\|N_{i+1}(k) - N_i(k)\|_s \leq \frac{1}{2}\|N_i(k) - N_{i-1}(k)\|_s$ by (5.6.6.11) provided $\|N_i(k)\|_s$ and $\|N_{i-1}(k)\|_s$ are sufficiently small, hence we have by induction that

$$\|N_{i+1}(k) - N_i(k)\|_s \leq \text{const}\, 2^{-i} \tag{5.6.6.12}$$

over a neighborhood of zero in K (depending on s); note this implies $\|\mathbf{D}N_i(k)\|_s \leq \text{const}\, 2^{-i}$ as well by (5.6.6.6). By completeness of H^s, we conclude that the limit $N_\infty : K \to H^s(C, W)$ exists in a neighborhood of zero (depending on s) and satisfies $\mathbf{D}N_\infty = 0$. Now elliptic boostrapping for quasi-holomorphic sections implies that each $N_\infty(k)$ is in fact smooth (**??**) and hence that $N_\infty : K \to C^\infty(C, W)$ is continuous (5.5.1). Now fix s so that $N_\infty : K \to C^\infty(C, W)$ exists and is continuous in a fixed neighborhood of zero. The composition $\lambda \circ N_\infty : K \to K$ is the identity map by construction (indeed $\lambda \circ N_i$ is the identity for every $i \geq 0$). The other composition $N_\infty \circ \lambda|_{\underline{\mathrm{Hol}}(C,W)} : \underline{\mathrm{Hol}}(C, W) \to \underline{\mathrm{Hol}}(C, W)$ is the identity map in an H^s-neighborhood of the origin by the contraction estimate (5.6.6.11) which implies that $N_\infty(k)$ is the unique zero of $\mathbf{D}$ in an H^s-neighborhood of zero lying over k.

We have shown that $\lambda|_{\underline{\mathrm{Hol}}(C,W)^{\lambda\text{-reg}}}$ is a local isomorphism of topological stacks. It remains to show that it is a local isomorphism of smooth stacks. Concretely, this means we should show that the local inverse $\lambda|^{-1}_{\underline{\mathrm{Hol}}(C,W)^{\lambda\text{-reg}}}$, regarded as a map $K \times C \to W$, is smooth.

We will make an inductive argument based on the tangent moduli problem $T\underline{\mathrm{Hol}}(C, W) = \underline{\mathrm{Hol}}(C, T_{W/C})$ (5.4). The local linear projection on $\underline{\mathrm{Hol}}(C, W)$ naturally induces one on $\underline{\mathrm{Hol}}(C, T_{W/C})$, in the sense that the affine linear map $\lambda : C^\infty(C, W) \to K$ has a derivative $T\lambda : C^\infty(C, T_{W/C}) \to TK$ giving a linear projection $T\lambda|_{\underline{\mathrm{Hol}}(C,T_{W/C})}$. If $u \in \underline{\mathrm{Hol}}(C, W)$ is λ-regular, then $(u, 0) \in \underline{\mathrm{Hol}}(C, T_{W/C})$ is $T\lambda$-regular (the derivative of $T\lambda$ is an extension of two copies of the derivative of λ). Thus $T\lambda|_{\underline{\mathrm{Hol}}(C,T_{W/C})}$ is a local isomorphism of topological stacks at $(u,0)$, hence has a local inverse $T\lambda|^{-1}_{\underline{\mathrm{Hol}}(C,T_{W/C})}$. Now the key is to prove (at λ-regular points) that $\lambda|^{-1}_{\underline{\mathrm{Hol}}(C,W)}$ is differentiable and that its derivative is given by

$$T(\lambda|^{-1}_{\underline{\mathrm{Hol}}(C,W)}) = (T\lambda)|^{-1}_{\underline{\mathrm{Hol}}(C,T_{W/C})}. \tag{5.6.6.13}$$

This implies smoothness of $\lambda|^{-1}_{\underline{\mathrm{Hol}}(C,W)}$ by induction (if $\lambda|^{-1}_{\underline{\mathrm{Hol}}(C,W)}$ is always of class C^k, then in particular $(T\lambda)|^{-1}_{\underline{\mathrm{Hol}}(C,T_{W/C})}$ is of class C^k, so the identity (5.6.6.13) implies that $T(\lambda|^{-1}_{\underline{\mathrm{Hol}}(C,W)})$ is of class C^k and hence that $\lambda|^{-1}_{\underline{\mathrm{Hol}}(C,W)}$ is always of class C^{k+1}).

It thus remains to prove the tangent identity (5.6.6.13), which will see is a consequence of the quadratic estimate (5.6.6.10). It suffices to verify it at a single point, which we may assume wlog is the zero section of $W \to C$ (and mapped to the zero of K by λ). It then amounts to proving that

$$\|\lambda|^{-1}_{\underline{\mathrm{Hol}}(C,W)}(k) - \lambda|^{-1}_{\ker D}(k)\|_s = o(|k|) \quad \text{as } k \to 0 \tag{5.6.6.14}$$

for all $s < \infty$. Rewriting this in terms of $\xi = \lambda|^{-1}_{\underline{\mathrm{Hol}}(C,W)}(k)$, we should show that

$$\|\xi - \alpha(\lambda(\xi))\|_s = o(|\lambda(\xi)|) \quad \text{as } \xi \to 0 \text{ and } \mathbf{D}(\xi) = 0, \tag{5.6.6.15}$$

where we recall that $Q \oplus \alpha : C^\infty(C,H) \oplus K \to C^\infty(C,W)$ is the inverse of $D \oplus \lambda : C^\infty(C,W) \to C^\infty(C,H) \oplus K$. Now the left hand side is commensurate with the H^{s-1}-norm of $(D\oplus\lambda)(\xi - \alpha(\lambda(\xi))) = D(\xi)$ since $D\oplus\lambda$ and $Q\oplus\alpha$ are bounded $H^s \to H^{s-1}$ and $H^{s-1} \to H^s$. Similarly, the quantity $|\lambda(\xi)|$ is commensurate with the H^s-norm of $\alpha(\lambda(\xi)) = (1-QD)\lambda$. It is thus equivalent to show that

$$\|D(\xi)\|_{s-1} = o(\|(1-QD)\xi\|_s) \quad \text{as } \xi \to 0 \text{ and } \mathbf{D}(\xi) = 0. \tag{5.6.6.16}$$

Now the quadratic estimate (5.6.6.10) implies $\|D(\xi)\|_{s-1} \le \mathrm{const}_s \|\xi\|_s^2$ for $\mathbf{D}(\xi) = 0$ and $\|\xi\|_s \le 1$, from which the estimate above follows immediately. $\square$

* *

We now treat the Regularity Theorem (5.6.1) for parameterized moduli problems over a smooth manifold. Compared with the absolute case discussed just above, not much more is needed other than upgrading the notation.

5.6.7 Definition (Linear model over a smooth base)**.** A *linear model* quasi-holomorphic section problem over a smooth manifold is one of the form $(W \to C, H/C) \times B$ where $W \to C$ is an affine vector bundle and B is an affine vector space. Note that $\varphi : J^1(W/C) \times B \to H$ is not assumed independent of the B-coordinate.

If $C \to B$ is proper, then the moduli stack $\underline{\mathrm{Hol}}_B(C,W)$ has an open cover by moduli stacks associated to linear models. Indeed, we saw earlier a corresponding result for the stack of all sections $\underline{\mathrm{Sec}}_B(C,W)$ (2.6.32), and we leave it as an exercise to upgrade the argument to ensure H is pulled back from C.

5.6.8 Proposition. *The Regularity Theorem* (5.6.1) *holds for B a smooth manifold.*

Proof. We follow the same basic argument used above for the case $B = *$ (5.6.6).

It suffices to consider the case of a linear model quasi-holomorphic section problem (5.6.7). A linear projection in this case is the restriction to $\underline{\mathrm{Hol}}_B(C, W) \subseteq C^\infty(C, W) \times B$ of a continuous affine linear map

$$\lambda : C^\infty(C, W) \times B \to K. \tag{5.6.8.1}$$

The λ-*regular locus* of $\underline{\mathrm{Hol}}_B(C, W)$ is the subset of the regular locus where λ defines an isomorphism from the (absolute) analytic tangent space to K_0; equivalently, this is where the linearization

$$C^\infty(C, W_0) \oplus B_0 = C^\infty(C, u^* T_{W/C}) \oplus T_b B \to C^\infty(C, H) \oplus K_0 \tag{5.6.8.2}$$

is an isomorphism. It suffices to show that $\lambda|_{\underline{\mathrm{Hol}}_B(C,W)^{\lambda\text{-reg}}}$ is a local isomorphism of topological and smooth stacks.

Fix a basepoint in $\underline{\mathrm{Hol}}_B(C, W)^{\lambda\text{-reg}}$ around which to show $\lambda|_{\underline{\mathrm{Hol}}_B(C,W)^{\lambda\text{-reg}}}$ is a local isomorphism, and declare this point to be the zero section of $W \to C$ and the zero of B. We consider the map $\mathbf{D} : \underline{\mathrm{Sec}}_B(C, W) = \underline{\mathrm{Sec}}(C, W) \times B \to \underline{\mathrm{Sec}}(C, H)$ given by $\mathbf{D}(\xi, b) = \varphi_b(d\xi)$. The derivative of $\mathbf{D}$ at the origin is denoted D; this operator is surjective, and the restriction of λ_0 to its kernel is an isomorphism. We let $Q : C^\infty(C, H) \to C^\infty(C, W) \oplus B$ denote the right inverse to D with image $\ker \lambda_0$.

We again consider the endomorphism $R(\xi) = \xi - Q\mathbf{D}\xi$ of $C^\infty(C, W) \times B$. To prove the contraction estimate for R, we should estimate the quantity $\mathbf{D}\xi - \mathbf{D}\zeta - D(\xi - \zeta)$ for $\xi, \zeta \in C^\infty(C, W) \times B$. More precisely, writing instead (ξ, b) and (ζ, c), this is the quantity

$$\varphi_b(d\xi) - \varphi_c(d\zeta) - D(\xi - \zeta) - \dot{\varphi}_{b-c}(0). \tag{5.6.8.3}$$

The bound (5.6.6.11) applies to $\varphi_0(d\xi) - \varphi_0(d\zeta) - D(\xi - \zeta)$, so the remaining quantity we need to bound here is

$$(\varphi_b - \varphi_0)(d\xi) - (\varphi_c - \varphi_0)(d\zeta) - \dot{\varphi}_{b-c}(0) \tag{5.6.8.4}$$

which equals $(\varphi_b - \varphi_0)(d\xi - d\zeta) + (\varphi_b - \varphi_c)(d\zeta - 0) + (\varphi_b - \varphi_c)(0) - \dot{\varphi}_{b-c}(0)$. The first two terms are both bounded quadratically as desired. The remainder $(\varphi_b - \varphi_c)(0) - \dot{\varphi}_{b-c}(0)$ is simply a smooth function in (b, c) which vanishes along the diagonal $b = c$ and to second order at $(0, 0)$, hence is also bounded as desired.

The rest of the proof is the same. □

* *

We now treat the Regularity Theorem (5.6.1) for parameterized moduli problems over a log smooth manifold (2.8). Recall this means $C \to B$ is a proper simply-broken submersion and $W \to C$ is a strict submersion.

5.6.9 Definition (Linear model over a log smooth base)**.** A *linear model* quasi-holomorphic section problem over a log smooth manifold is one of the form $W_B \to C_B \to B$ where:

(5.6.9.1) $B = V \times X_P$ where V is an affine vector space and P is sharp.

(5.6.9.2) $(W_B \to C_B \to B, H/W_B)$ is the pullback of $(W \to C \to {}'\mathbb{R}^n_{\geq 0}, H/C)$ under a monomial map $X_P \to {}'\mathbb{R}^n_{\geq 0}$.

(5.6.9.3) $(W \to C \to {}'\mathbb{R}^n_{\geq 0}, H/C)$ is a standard gluing family (2.8.106) associated to a two-dimensional log smooth manifold of depth one C_0^{pre} carrying an affine vector bundle $W_0^{\mathrm{pre}} \to C_0^{\mathrm{pre}}$ and a vector bundle $H_0^{\mathrm{pre}} \to C_0^{\mathrm{pre}}$, together with a collar and involution of the ideal locus (thus $W \to C$ is an affine vector bundle).

Every quasi-holomorphic section problem over a log smooth manifold has an open cover by linear models. Indeed, after fixing local coordinates $B = V \times X_P$, every proper simply-broken submersion over B is pulled back under a monomial map $X_P \to {}'\mathbb{R}^n_{\geq 0}$ of a standard gluing family (2.8.105), and every section of a strict submersion over such a family is covered by the desired vector bundle coordinates (**??**) (so every moduli stack $\underline{\mathrm{Sec}}_B(C, W)$ has an open cover by moduli stacks associated to linear model section problems).

Note that this argument proves a stronger result, namely that every point of $\underline{\mathrm{Sec}}_B(C, W)$ is covered by a linear model $\underline{\mathrm{Sec}}_{B^\circ}(C^\circ, W^\circ)$ in which its image in X_P is the cone point. For such points (i.e. lying over the cone point of X_P), regularity is equivalent to regularity relative X_P (**??**), so we conclude that $\underline{\mathrm{Hol}}_B(C, W)^{\mathrm{reg}}$ is the union of $\underline{\mathrm{Hol}}_{B^\circ}(C^\circ, W^\circ)^{\mathrm{reg}/X_P}$ where $W^\circ \to C^\circ \to B^\circ$ ranges over local linear models for $W \to C \to B$.

5.6.10 Proposition. *The Regularity Theorem* (5.6.1) *holds for B a log smooth manifold.*

Proof. We modify the arguments of (5.6.6) and (5.6.8) as appropriate.

It suffices to consider linear models and to treat the regular locus relative X_P (5.6.9) (we shall show that $\underline{\mathrm{Hol}}_B(C, W)^{\mathrm{reg}/X_P}$ is represented by a strict submersion over X_P).

Let us describe the relevant notion of linear projection. Begin with a continuous affine linear map

$$\lambda_0^{\mathrm{pre}} : C^\infty(C_0^{\mathrm{pre}}, W_0^{\mathrm{pre}}) \times V \to K \tag{5.6.10.1}$$

whose linear part, regarded as a distribution on C_0^{pre} valued in $K_0 \otimes (W_0^{\mathrm{pre}})^*$, is supported away from the ideal locus. Thus λ_0^{pre} depends only on the restriction of the map $C_0^{\mathrm{pre}} \to W_0^{\mathrm{pre}}$ to some fixed compact subset of the interior of C_0^{pre}, so it determines a map $\lambda : \underline{\mathrm{Sec}}_{{}'\mathbb{R}^N_{\geq 0}}(C, W) \times V \to K$, whose pullback to B we denote by

$$\lambda_B : \underline{\mathrm{Sec}}_B(C_B, W_B) = \underline{\mathrm{Sec}}_{X_P}(C, W) \times V \to \underline{\mathrm{Sec}}_{{}'\mathbb{R}^N_{\geq 0}}(C, W) \times V \to K. \tag{5.6.10.2}$$

The restriction of λ_B to $\underline{\mathrm{Hol}}_B(C_B, W_B)$ is our linear projection.

□

5.7 Derived Regularity

Here we prove the Derived Regularity Theorem (0.0.1) (see (5.7.10) below), which states that every quasi-holomorphic section problem is 'derived regular' in the following sense.

5.7.1 Definition (Derived Regular)**.** A quasi-holomorphic section problem $\wp = (W \to C \to B)$ over a derived smooth stack B is called *derived regular* when the morphism $\underline{\mathrm{Hol}}(\wp)_{\mathsf{Der}} \to B$ is representable and the comparison map $(\mathsf{Der} \to \mathsf{Top})_!\underline{\mathrm{Hol}}(\wp)_{\mathsf{Der}} \to \underline{\mathrm{Hol}}(\wp)_{\mathsf{Top}}$ is an isomorphism.

5.7.2 Exercise (Locality of derived regularity)**.** Show that derived regularity is a *local property* in the sense that if $\wp$ is derived regular over each of a collection of open substacks $U_i \subseteq \underline{\mathrm{Hol}}(\wp)_{\mathsf{Top}}$, then it is derived regular over their union $\bigcup_i U_i \subseteq \underline{\mathrm{Hol}}(\wp)_{\mathsf{Top}}$.

In brief, the proof we are about to give of the Derived Regularity Theorem proceeds as follows. Recall the Regularity Theorem (5.6.1), which asserts that $\underline{\mathrm{Hol}}(\wp)^{\mathrm{reg}}_{\mathsf{Sm}}$ is representable and that $(\mathsf{Sm} \to \mathsf{Top})_!\underline{\mathrm{Hol}}(\wp)^{\mathrm{reg}}_{\mathsf{Sm}} \to \underline{\mathrm{Hol}}(\wp)^{\mathrm{reg}}_{\mathsf{Top}}$ is an isomorphism. The Derived Regularity Theorem is stronger in two respects: it concerns the entire moduli stack rather than just the regular locus, and it concerns the derived smooth moduli stack rather than just the smooth moduli stack. The first difference is easily dealt with: since derived regularity is preserved under pullback, a standard thickening argument (5.7.6) shows that it is enough to prove the Derived Regularity Theorem over the regular locus. To prove the Derived Regularity Theorem over the regular locus (which is the main difficulty), it suffices (given the Regularity Theorem) to show that the comparison map $(\mathsf{Sm} \to \mathsf{Der})_!\underline{\mathrm{Hol}}(\wp)^{\mathrm{reg}}_{\mathsf{Sm}} \to \underline{\mathrm{Hol}}(\wp)^{\mathrm{reg}}_{\mathsf{Der}}$ is an isomorphism (5.7.9). The analogous comparison map for the stack $\underline{\mathrm{Sec}}_B(C, W)$ of all sections is an isomorphism by (2.10.35)(**??**), and the moduli stack $\underline{\mathrm{Hol}}_B(C, W)$ is a fiber product of these. Now left Kan extension $(\mathsf{Sm} \to \mathsf{Der})_!$ does not preserve all pullbacks, but it does preserve submersive pullbacks, and the Regularity Theorem implies that the relevant pullback is submersive over $\underline{\mathrm{Hol}}_B(C, W)^{\mathrm{reg}}$, so we are done.

It is remarkable that this argument reveals the Derived Regularity Theorem to be a *formal (yet nontrivial) consequence* of the Regularity Theorem! At no point in the argument do we need to contemplate the meaning of, or do any *hard analysis* (such as invoking Sobolev spaces or elliptic regularity) with, a family of quasi-holomorphic sections parameterized by a derived smooth manifold. This was quite a welcome surprise to the present author.

* *

We begin our treatment of the Derived Regularity Theorem with some initial reductions.

5.7.3 Lemma (Pullback and descent for derived regularity)**.** *Let $\wp$ be a quasi-holomorphic section problem over a derived smooth stack $B \in \mathsf{Shv}(\mathsf{Der})$.*

(5.7.3.1) *If $\wp$ is derived regular, then so is its pullback $\wp' = \wp \times_B B'$ under any map of derived smooth stacks $B' \to B$.*

(5.7.3.2) *If the pullback $\wp'$ is derived regular for every map $B' \to B$ from a derived smooth manifold B', then $\wp$ is derived regular.*

Proof. It follows directly from the definition that $\underline{\mathrm{Hol}}(\wp') = \underline{\mathrm{Hol}}(\wp) \times_B B'$.

$$\begin{tikzcd} \underline{\mathrm{Hol}}(\wp') \arrow[r] \arrow[d] & \underline{\mathrm{Hol}}(\wp) \arrow[d] \\ B' \arrow[r] & B \end{tikzcd} \tag{5.7.3.3}$$

Thus representability of $\underline{\mathrm{Hol}}(\wp) \to B$ implies representability of the pullback $\underline{\mathrm{Hol}}(\wp') \to B'$, and representability of the pullback $\underline{\mathrm{Hol}}(\wp') \to B'$ for all maps from derived smooth manifolds $B' \to B$ implies representability of $\underline{\mathrm{Hol}}(\wp) \to B$.

Now let us compare the comparison maps (between derived smooth and topological moduli stacks) associated to $\wp$ and its pullback $\wp'$. Let us abbreviate $|\cdot|_! = (\mathsf{Der} \to \mathsf{Top})_!$, and recall that $\underline{\mathrm{Hol}}(\wp)_{\mathsf{Top}} = \underline{\mathrm{Hol}}(\wp \times_B |B|_!)$, so the comparison map for $\wp$ takes the form $|\underline{\mathrm{Hol}}(\wp)|_! \to \underline{\mathrm{Hol}}(\wp \times_B |B|_!)$. Now the comparison maps for $\wp$ and $\wp'$ fit into a commuting diagram of the following shape.

$$\begin{tikzcd} {|\underline{\mathrm{Hol}}(\wp')|_!} \arrow[r] \arrow[d] & {|\underline{\mathrm{Hol}}(\wp)|_!} \arrow[d] \\ \underline{\mathrm{Hol}}(\wp' \times_{B'} |B'|_!) \arrow[r] \arrow[d] & \underline{\mathrm{Hol}}(\wp \times_B |B|_!) \arrow[d] \\ {|B'|_!} \arrow[r] & {|B|_!} \end{tikzcd} \tag{5.7.3.4}$$

The bottom square is a fiber square since the formation of $\underline{\mathrm{Hol}}$ is compatible with pullback (by inspection). The composite square is a pullback provided $\underline{\mathrm{Hol}}(\wp) \to B$ is representable, since left Kan extension $(\mathsf{Der} \to \mathsf{Top})_!$ preserves pullbacks of representable morphisms (2.9.42). It thus follows from cancellation (1.1.52) that the top square is a pullback (when $\underline{\mathrm{Hol}}(\wp) \to B$ is representable).

Now if $\underline{\mathrm{Hol}}(\wp) \to B$ is representable and the comparison map for $\wp$ is an isomorphism, it follows that the comparison map for $\wp'$ is an isomorphism (since the top square in (5.7.3.4) above is a pullback). Conversely, suppose $\underline{\mathrm{Hol}}(\wp') \to B'$ is representable and the comparison map of $\wp'$ is an isomorphism, for all maps from derived smooth manifolds $B' \to B$. We already saw that this means $\underline{\mathrm{Hol}}(\wp) \to B$ is representable, so the top square in (5.7.3.4) is a pullback for all $B' \to B$. To check that the comparison map for $\wp$ is an isomorphism, it suffices to check that it pulls back to an isomorphism under any map from a topological space $Z \to |B|_!$ (1.1.95). By the definition of left Kan extension $|\cdot|_! = (\mathsf{Der} \to \mathsf{Top})_! : \mathsf{Shv}(\mathsf{Der}) \to \mathsf{Shv}(\mathsf{Top})$, such a map locally factors through the map $|B'|_! \to |B|_!$ associated to some map from a derived smooth manifold $B' \to B$. Now the comparison map of $\wp$ pulls back to an isomorphism under any such map by assumption. □

5.7.4 Lemma (Reduction to smooth bases)**.** *If every quasi-holomorphic section problem over a smooth manifold is derived regular, then every quasi-holomorphic section problem over a derived smooth manifold is derived regular.*

Proof. Let $W \to C \to B$ be a quasi-holomorphic section problem over a derived smooth manifold B. The construction of an open cover of $\underline{\mathrm{Hol}}_B(C, W)$ by linear models (5.6.2)(5.6.7)(5.6.9) given earlier in the case B is a smooth manifold applies without change in the case B is a derived smooth manifold. Since derived regularity is local on $\underline{\mathrm{Hol}}_B(C, W)$, we may assume wlog that our quasi-holomorphic section problem $W \to C \to B$ is a linear model.

Since derived regularity is preserved under pullback (5.7.3.1), it suffices to show that every linear model quasi-holomorphic section problem over a derived smooth manifold B is, locally on B, a pullback of a quasi-holomorphic section problem over a smooth manifold. In fact, something much stronger is true, namely that the stack of linear model quasi-holomorphic section problems over derived smooth manifolds B is left Kan extended from smooth manifolds (**??**). □

5.7.5 Exercise (Derived regularity and point conditions)**.** Fix a morphism $A \to S$ in $(\mathsf{Shv}(\mathsf{Der}) \downarrow \mathsf{Shv}(\mathsf{Top}))$ (meaning $A = (A_{\mathsf{Der}}, A_{\mathsf{Top}})$ consists of a pair of derived smooth and topological stacks together with a comparison map $(\mathsf{Der} \to \mathsf{Top})_! A_{\mathsf{Der}} \to A_{\mathsf{Top}}$, etc.). Consider a pullback diagram of the following shape.

$$\begin{tikzcd} \underline{\mathrm{Hol}}_B(C, W)' \arrow[r] \arrow[d] & A \arrow[d] \\ \underline{\mathrm{Hol}}_B(C, W) \arrow[r] & S \end{tikzcd} \tag{5.7.5.1}$$

Recall that left Kan extension $\mathsf{Shv}(\mathsf{Der}) \to \mathsf{Shv}(\mathsf{Top})$ preserves pullbacks of representable morphisms (2.9.42), and conclude that if the comparison maps of A and S are isomorphisms and $A_{\mathsf{Der}} \to S_{\mathsf{Der}}$ (hence also $A_{\mathsf{Top}} \to S_{\mathsf{Top}}$) is representable, then derived regularity of $\underline{\mathrm{Hol}}_B(C, W)$ implies the same for $\underline{\mathrm{Hol}}_B(C, W)'$.

* *

The following (trivial) result is quite useful for proving results about moduli stacks of pseudo-holomorphic maps/sections.

⋆ **5.7.6 Proposition** (Reduction to the regular locus)**.** *A condition on solutions to quasi-holomorphic section problems over log smooth manifolds which is preserved under pullback and holds over the regular locus holds everywhere.*

Proof. It suffices (in fact, is equivalent) to show that for any quasi-holomorphic moduli problem $\wp$ over a log smooth manifold B, we have

$$\underline{\mathrm{Hol}}(\wp) = \bigcup_{\wp = \tilde{\wp} \times_{\tilde{B}} B} \underline{\mathrm{Hol}}(\tilde{\wp})^{\mathrm{reg}} \times_{\tilde{B}} B \tag{5.7.6.1}$$

where the union (of open substacks of $\underline{\mathrm{Hol}}(\wp)$) is over all maps of log smooth manifolds $B \to \tilde{B}$, all quasi-holomorphic moduli problems $\tilde{\wp}$ over $\tilde{B}$, and all isomorphisms $\wp = \tilde{\wp} \times_{\tilde{B}} B$.

Let $\wp = (W \to C \to B, H, \varphi)$, and fix a point $(b \in B, u : C_b \to W_b) \in \underline{\mathrm{Hol}}(\wp)$ which we would like to show lies in the union (5.7.6.1) above. Let $\tilde{B} = B \times \mathbb{R}^k \supseteq B \times 0 = B$. Define $\tilde{\wp}/\tilde{B}$ as the pullback of $\wp$ under the projection $\tilde{B} \to B$, except that instead of taking

$$\tilde{\varphi} : J^1_{\tilde{B}}(\tilde{W}/\tilde{C}) = J^1_B(W/C) \times \mathbb{R}^k \to H \tag{5.7.6.2}$$

to simply equal φ, we add to it some linear map $\alpha : \mathbb{R}^k \to H$ over W (evidently $\tilde{\wp} \times_{\tilde{B}} B = \wp$). The point $(b, u) \in \underline{\mathrm{Hol}}(\wp) = \underline{\mathrm{Hol}}(\tilde{\wp}) \times_{\tilde{B}} B$ will lie in $\underline{\mathrm{Hol}}(\tilde{\wp})^{\mathrm{reg}} \times_{\tilde{B}} B$ iff the composition

$$\mathbb{R}^k \xrightarrow{\alpha} C^\infty(W, H) \xrightarrow{u_b^*} C^\infty(C_b, u^*H) \twoheadrightarrow T^1_u \underline{\mathrm{Hol}}(C_b, W_b) \twoheadrightarrow T^1_{(u,b)} \underline{\mathrm{Hol}}_B(C, W) \tag{5.7.6.3}$$

is surjective. In fact, we can choose α so that *a fortiori* the composition $\mathbb{R}^k \to T^1_u \underline{\mathrm{Hol}}(C_b, W_b)$ is surjective, since $C^\infty(C_b, u^*H) \twoheadrightarrow T^1_u \underline{\mathrm{Hol}}(C_b, W_b)$ is the cokernel of an elliptic operator on C_b (5.4), hence is finite-dimensional since C_b is compact (3.3)–(3.5)–(3.6). $\square$

* *

We now come to the main point in the proof of the Derived Regularity Theorem, namely the proof that the comparison map $(\mathsf{Sm} \to \mathsf{Der})_! \underline{\mathrm{Hol}}(\wp)^{\mathrm{reg}}_{\mathsf{Sm}} \to \underline{\mathrm{Hol}}(\wp)^{\mathrm{reg}}_{\mathsf{Der}}$ is an isomorphism for any quasi-holomorphic section problem $\wp$ over a smooth manifold B (5.7.9). In other words, families of regular quasi-holomorphic sections parameterized by derived smooth manifolds are completely classified by such families over smooth manifolds. It is quite surprising that this turns out to be a formal consequence of the fact that $\underline{\mathrm{Hol}}(\wp')^{\mathrm{reg}}_{\mathsf{Sm}}$ is representable (5.6.1) for all moduli problems $\wp'$ over smooth manifolds. The key inputs are the fact that $\mathsf{Sm} \to \mathsf{Der}$ preserves submersive pullbacks, hence so does $(\mathsf{Sm} \to \mathsf{Der})_! : \mathsf{Shv}(\mathsf{Sm}) \to \mathsf{Shv}(\mathsf{Der})$ (2.9.42), and the fact that the comparison map for stacks of all sections $(\mathsf{Sm} \to \mathsf{Der})_! \underline{\mathrm{Sec}}_B(C, W)_{\mathsf{Sm}} \to \underline{\mathrm{Sec}}_B(C, W)_{\mathsf{Der}}$ is an isomorphism (**??**).

To make the argument, we need a technical fact, namely we need to realize $\underline{\mathrm{Hol}}_B(C, W)$ as a fiber of a map of stacks of smooth sections (note that the most apparent fiber product presentation of $\underline{\mathrm{Hol}}_B(C, W)$ (5.3.6) is not of this form).

5.7.7 Lemma. *Let $(W \to C \to B, H, \varphi)$ be quasi-holomorphic section problem. If H/W is the pullback of H_0/C, then there is a fiber diagram*

$$\begin{tikzcd} \underline{\mathrm{Hol}}_B(C, W) \arrow[r] \arrow[d] & \underline{\mathrm{Sec}}_B(C, W) \arrow[d] \\ B \arrow[r] & \underline{\mathrm{Sec}}_B(C, H_0) \end{tikzcd} \tag{5.7.7.1}$$

for all flavors of moduli stacks.

Proof. We have $H = W \times_C H_0$ as a fiber product in Sm (and also in Der since $\mathsf{Sm} \to \mathsf{Der}$ preserves transverse fiber products (2.10.2.2)(2.10.17)). We thus have $\underline{\mathrm{Sec}}_B(C, H) = \underline{\mathrm{Sec}}_B(C, W) \times_B \underline{\mathrm{Sec}}_B(C, H_0)$ (this is a purely categorical consequence of $H = W \times_C H_0$). Now $\underline{\mathrm{Hol}}_B(C, W) \to \underline{\mathrm{Sec}}_B(C, W)$ is by definition a pullback of the zero section $\underline{\mathrm{Sec}}_B(C, W) \to \underline{\mathrm{Sec}}_B(C, H) = \underline{\mathrm{Sec}}_B(C, W) \times_B \underline{\mathrm{Sec}}_B(C, H_0)$, which is a pullback of $B \to \underline{\mathrm{Sec}}_B(C, H_0)$. $\square$

The presentation (5.7.7) is ‘better’ than the fiber product presentation (5.3.6) in that one of the factors is B. We need to improve it further to make this factor as small as possible.

5.7.8 Lemma. *In the setup of* (5.7.7)*, if $H_0 \to C \to B$ is the pullback of $H'_0 \to C'_0 \to B'$ under a map $B \to B'$, then there is a fiber diagram*

$$\begin{tikzcd} \underline{\mathrm{Hol}}_B(C,W) \arrow[r] \arrow[d] & \underline{\mathrm{Sec}}_B(C,W) \arrow[d] \\ B' \arrow[r] & \underline{\mathrm{Sec}}_{B'}(C',H'_0) \end{tikzcd} \tag{5.7.8.1}$$

for all flavors of moduli stacks.

Proof. We have (by cancellation (1.1.52)) a pair of fiber squares.

$$\begin{tikzcd} B \arrow[r] \arrow[d] & \underline{\mathrm{Sec}}_B(C,H_0) \arrow[r] \arrow[d] & B \arrow[d] \\ B' \arrow[r] & \underline{\mathrm{Sec}}_{B'}(C',H'_0) \arrow[r] & B' \end{tikzcd} \tag{5.7.8.2}$$

Stacking the left square with the square (5.7.7) gives the desired result. ☐

5.7.9 Proposition. *For any quasi-holomorphic section problem $\wp$ over a smooth manifold, the comparison map $(\mathsf{Sm} \to \mathsf{Der})_!\underline{\mathrm{Hol}}(\wp)^{\mathrm{reg}}_{\mathsf{Sm}} \to \underline{\mathrm{Hol}}(\wp)^{\mathrm{reg}}_{\mathsf{Der}}$ is an isomorphism.*

Proof. The desired assertion is local, so we may fix a basepoint of $\underline{\mathrm{Hol}}(\wp)^{\mathrm{reg}}$ and prove it just in a neighborhood of this point. There exists a local linear model in which this point is regular relative X_P (5.6.9). We are thus reduced to considering the comparison map for $\underline{\mathrm{Hol}}_B(C_B,W_B)^{\mathrm{reg}/X_P}$ for a linear model.

By (5.7.8) we have a fiber product presentation of the following form.

$$\begin{tikzcd} \underline{\mathrm{Hol}}_B(C_B,W_B) \arrow[r] \arrow[d] & \underline{\mathrm{Sec}}_B(C_B,W_B) \arrow[d] \\ X_P \arrow[r] & \underline{\mathrm{Sec}}_{X_P}(C,H) \end{tikzcd} \tag{5.7.9.1}$$

We now consider the $(\mathsf{Sm} \to \mathsf{Der})_!$ comparison cube of this fiber square. The comparison maps for the parameterized section functors $\underline{\mathrm{Sec}}$ on the right are isomorphisms (**??**), as is the comparison map for X_P. Thus to show that the comparison map for $\underline{\mathrm{Hol}}_B(C_B,W_B)^{\mathrm{reg}/X_P}$ is an isomorphism, it suffices to show that $(\mathsf{Sm} \to \mathsf{Der})_!$(5.7.9.1) is a fiber square (over the regular locus relative X_P).

To show that $(\mathsf{Sm} \to \mathsf{Der})_!$(5.7.9.1) is a fiber square over the regular locus relative X_P, recall that $\mathsf{Sm} \to \mathsf{Der}$ preserves submersive pullbacks (2.10.17), hence so does left Kan extension $(\mathsf{Sm} \to \mathsf{Der})_!$ (2.9.42). It thus suffices to show that the right vertical map

$\underline{\mathrm{Sec}}_B(C_B, W_B) \to \underline{\mathrm{Sec}}_{X_P}(C, H)$ is submersive over the relative regular locus. For a map $Z \to \underline{\mathrm{Sec}}_{X_P}(C, H)$ from a log smooth manifold Z, the pullback

$$\begin{tikzcd} \underline{\mathrm{Hol}}_{B\times_{X_P}Z}(C_{B\times_{X_P}Z}, W_{B\times_{X_P}Z}) \arrow[r] \arrow[d] & \underline{\mathrm{Sec}}_B(C_B, W_B) \arrow[d] \\ Z \arrow[r] & \underline{\mathrm{Sec}}_{X_P}(C, H) \end{tikzcd} \tag{5.7.9.2}$$

is itself a moduli stack of quasi-holomorphic sections over the parameter space $B \times_{X_P} Z = V \times Z$, hence its relative regular locus is submersive over Z by the Regularity Theorem (5.6.1). Thus $\underline{\mathrm{Sec}}_B(C_B, W_B) \to \underline{\mathrm{Sec}}_{X_P}(C, H)$ is submersive over the relative regular locus, as desired. ☐

* *

We may now conclude with the proof of the Derived Regularity Theorem (0.0.1).

5.7.10 Theorem (Derived Regularity Theorem (0.0.1))**.** *Every quasi-holomorphic section problem over a derived smooth stack is derived regular.*

Proof. By our initial reductions (5.7.3.2)(5.7.4), it suffices to consider quasi-holomorphic section problems over smooth manifolds. Derived regularity is a local property on $\underline{\mathrm{Hol}}(\wp)$, so we may consider the maximal open subset $\underline{\mathrm{Hol}}(\wp)^{\mathrm{dreg}} \subseteq \underline{\mathrm{Hol}}(\wp)$ which is derived regular. Derived regularity is preserved under pullback (5.7.3.1), so by reduction to the regular locus (5.7.6), to show that $\underline{\mathrm{Hol}}(\wp)^{\mathrm{dreg}} = \underline{\mathrm{Hol}}(\wp)$ it suffices to show that $\underline{\mathrm{Hol}}(\wp)^{\mathrm{dreg}}$ contains $\underline{\mathrm{Hol}}(\wp)^{\mathrm{reg}}$, which follows from the Regularity Theorem (5.6.1) and the fact that the comparison map $(\mathsf{Sm} \to \mathsf{Der})_! \underline{\mathrm{Hol}}_B(C, W)^{\mathrm{reg}}_{\mathsf{Sm}} \to \underline{\mathrm{Hol}}_B(C, W)^{\mathrm{reg}}_{\mathsf{Der}}$ is an isomorphism (5.7.9). ☐

5.8 A priori estimates

We now provide a treatment of the standard *a priori* estimates on pseudo-holomorphic maps.

It is difficult to trace the origin of the results in this section. Many appear in some form in Gromov [26], where they were considered too trivial to require anything more than a very brief justification. Subsequent work of many authors has supplied various different ways turning Gromov's brief hints into complete proofs.

⋆ **5.8.1 Definition** (Bound on geometry)**.** Let (X, g) be a Riemannian manifold. A *bound on the geometry* of (X, g) at a point $p \in X$ is a collection of real numbers $M_0, M_1, \ldots < \infty$ such that there exists a smooth map $\Phi : (B(1), 0) \to (X, p)$ with $\Phi^* g \geq M_0^{-1} g_{\mathrm{std}}$ and $\|\Phi^* g\|_{C^k} \leq M_k$ for all k. A bound on the geometry of (X, g) (resp. over a subset $A \subseteq X$) is a collection $(M_0, M_1, \ldots)$ which bound the geometry of X at every point (resp. of A). A bound on the geometry *and injectivity radius* means that in addition Φ is required to be injective (beware that in standard terminology, a 'bound on the geometry' is usually taken to mean what we have decided to call a 'bound on the geometry and injectivity radius').

A bound on the geometry of (X, g, τ) for some additional structure τ (e.g. a symplectic form, almost complex structure, or any combination thereof) means that $\|\Phi^* \tau\|_{C^k} \leq M_k$ as well. When the data τ itself determines a Riemannian metric g_τ (e.g. a tame pair (J, ω) determining the metric $\omega(v, Jw) + \omega(w, Jv)$), we may simply say (X, τ) has bounded geometry to mean that (X, g_τ, τ) has bounded geometry.

We say that a constant 'depends on the geometry of X (resp. over a $A \subseteq X$)' to mean that said constant may be bounded in terms of a bound on the geometry of X (resp. over A).

⋆ **5.8.2 Definition** (Energy)**.** Let (X, g) be a Riemannian manifold, and let C be a Riemann surface. The *energy* of a map $u : C \to (X, g)$ is the integral

$$E(u) = \int_C \tfrac{1}{2}|du|^2 \tag{5.8.2.1}$$

where the integrand $\frac{1}{2}|du|^2$ is by definition $\frac{1}{2}(|u_x|^2 + |u_y|^2)\, dx\, dy = g(u_z, u_{\bar{z}})\, i\, dz\, d\bar{z}$ in local holomorphic coordinates.

5.8.3 Exercise. Show that if $u : C \to (X, J)$ is pseudo-holomorphic and J is compatible with ω, then $u^*\omega = \frac{1}{2}|du|^2$, so we have $E(u) = \int_C u^*\omega$.

* *

⋆ **5.8.4 Proposition** (Gradient bounds imply C^∞ bounds)**.** *Let $u : D^2 \to (X, J, g)$ be a pseudo-holomorphic map. If* $\sup |du| \leq M$*, then*

$$|D^k u(0)| \leq \mathrm{const} \cdot E(u)^{1/2} \tag{5.8.4.1}$$

for some const $< \infty$ *depending on $k < \infty$, $M < \infty$, and the geometry of (X, J, g) over the image of u.*

Proof. Choose local linear coordinates on the target X. Write the pseudo-holomorphic map equation in local coordinates as $u_x + J(u)u_y = 0$. Applying $\frac{d}{dx} - \frac{d}{dy}J(u)$ to this equation yields the higher order equation

$$u_{xx} + u_{yy} = \dot{J}(u, u_y)u_x - \dot{J}(u, u_x)u_y, \tag{5.8.4.2}$$

which has the virtue that its leading order terms have constant coefficients.

We now bound the $W^{k,2}$-norm of u (over any compact subset of $(D^2)^\circ$) using two successive bootstrapping arguments based on (5.8.4.2). The first bounds $\|u\|_{k,2}$ by some (unspecified) function of M. The second bounds $\|u\|_{k,2}$ by $E(u)^{1/2}$ times some (unspecified) function of M (which is enough by Sobolev embedding (3.1.31)).

For the first bootstrap, we note that L^2-norm of the right side of (5.8.4.2) is bounded in terms of M, so by elliptic regularity (3.3) we have a bound on the $W^{2,2}$-norm of u in terms of M. This implies a $W^{1,2}$-bound on the right side of (5.8.4.2) (inspect its derivative) in terms of M, hence by elliptic regularity we have a bound on the $W^{3,2}$-norm of u in terms of M. Now we claim that for $k \geq 3$, a bound on the $W^{k,2}$-norm of u implies a bound on the $W^{k+1,2}$-norm of u. Indeed, the right side of (5.8.4.2) is a smooth function vanishing at zero applied to (u, Du), hence since $W^{k-1,2} \subseteq C^0$ for $k \geq 3$ (3.1.31), the $W^{k-1,2}$-norm of the right side is bounded in terms of the $W^{k-1,2}$-norm of (u, Du) (3.1.34), thus in terms of the $W^{k,2}$-norm of u. By induction, the $W^{k,2}$-norm of u is bounded in terms of M for all $k < \infty$.

Now we do the second bootstrap. We know from the first bootstrap that the derivatives of u are bounded in terms of M. In particular, the factors $\dot{J}(u, u_x)$ and $\dot{J}(u, u_y)$ are bounded in C^∞ in terms of M. It follows (3.2.7) that the $W^{k-1,2}$-norm of the right side of (5.8.4.2) is bounded linearly in terms of the $W^{k,2}$-norm of u. Applying elliptic regularity (3.3), we conclude that the $W^{k+1,2}$-norm of u is bounded linearly in terms of the $W^{k,2}$-norm of u. In the base case $k = 1$, the $W^{k-1,2}$-norm of the right side is (by inspection) bounded linearly in terms of $E(u)^{1/2}$. By induction, the $W^{k,2}$-norm of u is bounded linearly in terms of $E(u)^{1/2}$ for all $k < \infty$. □

5.8.5 Exercise. Use a rescaling argument to deduce from (5.8.4) that $|D^k u(p)| \leq \text{const} \cdot d(p, \partial D^2)^{-(k-1)} \cdot E(u)^{1/2}$ under the same hypotheses.

⋆ **5.8.6 Hofer's Lemma** ([32, Lemma 3.3])**.** *Let (X, d) be a complete metric space, let $f : X \to \mathbb{R}_{\geq 0}$ be locally bounded, and let $M < \infty$. For every $p_0 \in X$, there exists $p \in X$ with $f(p) \geq f(p_0)$ and $d(p, p_0) \leq 2M \cdot f(p_0)^{-1}$ such that $d(x, p) \leq M \cdot f(p)^{-1} \implies f(x) \leq 2f(p)$.*

Proof. If p_0 does not satisfy the desired property, then there exists a violation point p_1, i.e. $d(p_0, p_1) \leq M \cdot f(p_0)^{-1}$ and $f(p_1) \geq 2f(p_0)$. If p_1 does not satisfy the desired property, there is a subsequent violation point p_2. We have $f(p_i) \geq 2^i f(p_0)$, hence $d(p_i, p_{i+1}) \leq 2^{-i} M \cdot f(p_0)^{-1}$, so $d(p_0, p_i) \leq 2M \cdot f(p_0)^{-1}$. This process $p_0, p_1, \ldots$ will eventually terminate at a suitable point p, since otherwise it would converge (since X is complete) to a point p_∞ near which f is not locally bounded. □

⋆ **5.8.7 Proposition** (Small energy bounds imply gradient bounds)**.** *Let $u : D^2 \to (X, J, g)$ be a pseudo-holomorphic map. If $\int_{D^2} |du|^2 < \varepsilon$ then*

$$|du(0)| \leq 5 \tag{5.8.7.1}$$

for some $\varepsilon > 0$ depending on the geometry of (X, J, g) over the image of u.

Proof. If $|du(0)| \geq 5$, then we can use Hofer's Lemma (5.8.6) to find a point $p \in D^2$ at distance at most $\frac{2}{5}$ from the origin such that $|du| \leq 2|du(p)|$ over the disk of radius $|du(p)|^{-1}$ around p (which is entirely contained in D^2 since $\frac{2}{5} + \frac{1}{5} < 1$). Now consider the map $\tilde{u} : D^2 \to (X, J, g)$ obtained from u by rescaling the disk of radius $|du(p)|^{-1}$ around p to the disk D^2. We have $\sup |d\tilde{u}| \leq 2$ by construction, hence we have C^∞ bounds (5.8.4) on $\tilde{u}$ over D^2. We also have $|d\tilde{u}(0)| = 1$ by construction, which combined with C^∞ bounds on $\tilde{u}$ implies a lower bound on the energy of $\tilde{u}$, hence also on the energy of u. Now take $\varepsilon > 0$ to be smaller than this lower bound. □

5.8.8 Proposition (Removable singularity)**.** *A pseudo-holomorphic map $u : D^2 \setminus 0 \to (X, J)$ extends smoothly to D^2 iff its image is relatively compact in X and it has finite energy.*

Bibliography

[1] *Théorie des topos et cohomologie étale des schémas. Tome 1: Théorie des topos*. Lecture Notes in Mathematics, Vol. 269. Springer-Verlag, Berlin-New York, 1972. Séminaire de Géométrie Algébrique du Bois-Marie 1963–1964 (SGA 4), Dirigé par M. Artin, A. Grothendieck, et J. L. Verdier. Avec la collaboration de N. Bourbaki, P. Deligne et B. Saint-Donat.

[2] Robert A. Adams. *Sobolev spaces*. Academic Press [A subsidiary of Harcourt Brace Jovanovich, Publishers], New York-London, 1975. Pure and Applied Mathematics, Vol. 65.

[3] Jarod Alper, Jack Hall, and David Rydh. A Luna étale slice theorem for algebraic stacks. *Ann. of Math. (2)*, 191(3):675–738, 2020.

[4] M. Artin. Versal deformations and algebraic stacks. *Invent. Math.*, 27:165–189, 1974.

[5] M. F. Atiyah and R. Bott. A Lefschetz fixed point formula for elliptic complexes. I. *Ann. of Math. (2)*, 86:374–407, 1967.

[6] M. F. Atiyah, N. J. Hitchin, and I. M. Singer. Self-duality in four-dimensional Riemannian geometry. *Proc. Roy. Soc. London Ser. A*, 362(1711):425–461, 1978.

[7] K. Behrend. Introduction to algebraic stacks. In *Moduli spaces*, volume 411 of *London Math. Soc. Lecture Note Ser.*, pages 1–131. Cambridge Univ. Press, Cambridge, 2014.

[8] J. M. Boardman and R. M. Vogt. *Homotopy invariant algebraic structures on topological spaces*. Lecture Notes in Mathematics, Vol. 347. Springer-Verlag, Berlin-New York, 1973.

[9] Jean-Marc Cordier. Sur la notion de diagramme homotopiquement cohérent. *Cahiers Topologie Géom. Différentielle*, 23(1):93–112, 1982. Third Colloquium on Categories, Part VI (Amiens, 1980).

[10] Marius Crainic and Ivan Struchiner. On the linearization theorem for proper Lie groupoids. *Ann. Sci. Éc. Norm. Supér. (4)*, 46(5):723–746, 2013.

[11] Matias del Hoyo and Rui Loja Fernandes. Riemannian metrics on Lie groupoids. *J. Reine Angew. Math.*, 735:143–173, 2018.

[12] P. Deligne and D. Mumford. The irreducibility of the space of curves of given genus. *Inst. Hautes Études Sci. Publ. Math.*, (36):75–109, 1969.

[13] Jean Dieudonné. Une généralisation des espaces compacts. *J. Math. Pures Appl. (9)*, 23:65–76, 1944.

[14] Albrecht Dold. Homology of symmetric products and other functors of complexes. *Ann. of Math. (2)*, 68:54–80, 1958.

[15] Albrecht Dold. Partitions of unity in the theory of fibrations. *Ann. of Math. (2)*, 78:223–255, 1963.

[16] Eduardo J. Dubuc. C^∞-schemes. *Amer. J. Math.*, 103(4):683–690, 1981.

[17] Yuri V. Egorov and Bert-Wolfgang Schulze. *Pseudo-differential operators, singularities, applications*, volume 93 of *Operator Theory: Advances and Applications*. Birkhäuser Verlag, Basel, 1997.

[18] Charles Ehresmann. Sur les espaces fibrés différentiables. *C. R. Acad. Sci. Paris*, 224:1611–1612, 1947.

[19] Charles Ehresmann. Les connexions infinitésimales dans un espace fibré différentiable. In *Colloque de topologie (espaces fibrés), Bruxelles, 1950*, pages 29–55. Georges Thone, Liège; Masson et Cie., Paris, 1951.

[20] Charles Ehresmann. Catégories topologiques et catégories différentiables. In *Colloque Géom. Diff. Globale (Bruxelles, 1958)*, pages 137–150. Centre Belge Rech. Math., Louvain, 1959.

[21] Samuel Eilenberg and J. A. Zilber. Semi-simplicial complexes and singular homology. *Ann. of Math. (2)*, 51:499–513, 1950.

[22] Kenji Fukaya, Yong-Geun Oh, Hiroshi Ohta, and Kaoru Ono. *Lagrangian intersection Floer theory: anomaly and obstruction. Part I*, volume 46 of *AMS/IP Studies in Advanced Mathematics*. American Mathematical Society, Providence, RI; International Press, Somerville, MA, 2009.

[23] Kenji Fukaya, Yong-Geun Oh, Hiroshi Ohta, and Kaoru Ono. *Lagrangian intersection Floer theory: anomaly and obstruction. Part II*, volume 46 of *AMS/IP Studies in Advanced Mathematics*. American Mathematical Society, Providence, RI; International Press, Somerville, MA, 2009.

[24] Kenji Fukaya and Kaoru Ono. Arnold conjecture and Gromov-Witten invariant. *Topology*, 38(5):933–1048, 1999.

[25] Jean Giraud. *Cohomologie non abélienne*. Die Grundlehren der mathematischen Wissenschaften, Band 179. Springer-Verlag, Berlin-New York, 1971.

[26] M. Gromov. Pseudo holomorphic curves in symplectic manifolds. *Invent. Math.*, 82(2):307–347, 1985.

[27] André Haefliger. Groupoïdes d'holonomie et classifiants. Number 116, pages 70–97. 1984. Transversal structure of foliations (Toulouse, 1982).

[28] André Haefliger. Orbi-espaces. In *Sur les groupes hyperboliques d'après Mikhael Gromov (Bern, 1988)*, volume 83 of *Progr. Math.*, pages 203–213. Birkhäuser Boston, Boston, MA, 1990.

[29] André Haefliger. Complexes of groups and orbihedra. In *Group theory from a geometrical viewpoint (Trieste, 1990)*, pages 504–540. World Sci. Publ., River Edge, NJ, 1991.

[30] J. Heinloth. Notes on differentiable stacks. In *Mathematisches Institut, Georg-August-Universität Göttingen: Seminars Winter Term 2004/2005*, pages 1–32. Universitätsdrucke Göttingen, Göttingen, 2005.

[31] V. A. Hinich and V. V. Schechtman. On homotopy limit of homotopy algebras. In *K-theory, arithmetic and geometry (Moscow, 1984–1986)*, volume 1289 of *Lecture Notes in Math.*, pages 240–264. Springer, Berlin, 1987.

[32] H. Hofer and C. Viterbo. The Weinstein conjecture in the presence of holomorphic spheres. *Comm. Pure Appl. Math.*, 45(5):583–622, 1992.

[33] Lars Hörmander. *The analysis of linear partial differential operators. I*, volume 256 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1983. Distribution theory and Fourier analysis.

[34] Luc Illusie, Kazuya Kato, and Chikara Nakayama. Quasi-unipotent logarithmic Riemann-Hilbert correspondences. *J. Math. Sci. Univ. Tokyo*, 12(1):1–66, 2005.

[35] Niles Johnson and Donald Yau. *2-dimensional categories*. Oxford University Press, Oxford, 2021.

[36] A. Joyal. Quasi-categories and Kan complexes. *J. Pure Appl. Algebra*, 175(1-3):207–222, 2002. Special volume celebrating the 70th birthday of Professor Max Kelly.

[37] André Joyal. Notes on quasi-categories. 2004, 2007, 2008.

[38] André Joyal. The Theory of Quasi-Categories and its Applications. 2008.

[39] Dominic Joyce. An introduction to d-manifolds and derived differential geometry. In *Moduli spaces*, volume 411 of *London Math. Soc. Lecture Note Ser.*, pages 230–281. Cambridge Univ. Press, Cambridge, 2014.

[40] Dominic Joyce. A new definition of Kuranishi space. *arxiv:math/1409.6908v3*, pages 1–193, 2015.

[41] Dominic Joyce. Manifolds with analytic corners. *arXiv:1605.05913*, 2016.

[42] Dominic Joyce. Algebraic geometry over C^∞-rings. *Mem. Amer. Math. Soc.*, 260(1256):v+139, 2019.

[43] Dominic Joyce. Kuranishi spaces as a 2-category. In *Virtual fundamental cycles in symplectic topology*, volume 237 of *Math. Surveys Monogr.*, pages 253–298. Amer. Math. Soc., Providence, RI, 2019.

[44] Daniel M. Kan. Abstract homotopy. III. *Proc. Nat. Acad. Sci. U.S.A.*, 42:419–421, 1956.

[45] Daniel M. Kan. On c. s. s. complexes. *Amer. J. Math.*, 79:449–476, 1957.

[46] Daniel M. Kan. Functors involving c.s.s. complexes. *Trans. Amer. Math. Soc.*, 87:330–346, 1958.

[47] Kazuya Kato. Logarithmic structures of Fontaine-Illusie. In *Algebraic analysis, geometry, and number theory (Baltimore, MD, 1988)*, pages 191–224. Johns Hopkins Univ. Press, Baltimore, MD, 1989.

[48] W. Kondrachov. Sur certaines propriétés des fonctions dans l'espace. *C. R. (Doklady) Acad. Sci. URSS (N.S.)*, 48:535–538, 1945.

[49] M. Kuranishi. New proof for the existence of locally complete families of complex structures. In *Proc. Conf. Complex Analysis (Minneapolis, 1964)*, pages 142–154. Springer, Berlin, 1965.

[50] Markus Land. *Introduction to infinity-categories*. Compact Textbooks in Mathematics. Birkhäuser/Springer, Cham, [2021] ©2021.

[51] Gérard Laumon and Laurent Moret-Bailly. *Champs algébriques*, volume 39 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, 2000.

[52] H. Blaine Lawson, Jr. and Marie-Louise Michelsohn. *Spin geometry*, volume 38 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 1989.

[53] Tom Leinster. *Basic category theory*, volume 143 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2014.

[54] Jean Leray. L'anneau d'homologie d'une représentation. *C. R. Acad. Sci. Paris*, 222:1366–1368, 1946.

[55] Robert B. Lockhart and Robert C. McOwen. Elliptic differential operators on noncompact manifolds. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 12(3):409–447, 1985.

[56] Jacob Lurie. *Derived algebraic geometry*. ProQuest LLC, Ann Arbor, MI, 2004. Thesis (Ph.D.)–Massachusetts Institute of Technology.

[57] Jacob Lurie. *Derived Algebraic Geometry V: Structured Spaces*. 2009.

[58] Jacob Lurie. *Higher topos theory*, volume 170 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2009.

[59] Jacob Lurie. *Higher algebra*. Book manuscript, 2017.

[60] Saunders Mac Lane. *Categories for the working mathematician*, volume 5 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1998.

[61] Richard B. Melrose. Pseudodifferential operators, corners and singular limits. In *Proceedings of the International Congress of Mathematicians, Vol. I, II (Kyoto, 1990)*, pages 217–234. Math. Soc. Japan, Tokyo, 1991.

[62] Richard B. Melrose. Calculus of conormal distributions on manifolds with corners. *Internat. Math. Res. Notices*, (3):51–61, 1992.

[63] Richard B. Melrose. *The Atiyah-Patodi-Singer index theorem*, volume 4 of *Research Notes in Mathematics*. A K Peters, Ltd., Wellesley, MA, 1993.

[64] Haynes Miller. Leray in Oflag XVIIA: the origins of sheaf theory, sheaf cohomology, and spectral sequences. Number 84, suppl., pages 17–34. 2000. Jean Leray (1906–1998).

[65] Ieke Moerdijk and Gonzalo E. Reyes. *Models for smooth infinitesimal analysis*. Springer-Verlag, New York, 1991.

[66] J. C. Moore. Homotopie des complexes monoïdaux, I. *Séminaire Henri Cartan*, 7(2), 1954-1955. talk:18.

[67] Jürgen Moser. A rapidly convergent iteration method and non-linear partial differential equations. I. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (3)*, 20:265–315, 1966.

[68] James R. Munkres. *Topology*. Prentice Hall, Inc., Upper Saddle River, NJ, 2000. Second edition of [MR0464128].

[69] Jun-iti Nagata. On a necessary and sufficient condition of metrizability. *J. Inst. Polytech. Osaka City Univ. Ser. A*, 1:93–100, 1950.

[70] Chikara Nakayama and Arthur Ogus. Relative rounding in toric and logarithmic geometry. *Geom. Topol.*, 14(4):2189–2241, 2010.

[71] Behrang Noohi. Foundations of topological stacks I. *arXiv:math/0503247*, 2005.

[72] John Pardon. Representability in non-linear elliptic Fredholm analysis. *Submitted to the Proceedings of the International Congress of Basic Science.*, 2023.

[73] Brett Parker. Exploded manifolds. *Adv. Math.*, 229(6):3256–3319, 2012.

[74] The Univalent Foundations Program. *Homotopy type theory—univalent foundations of mathematics.* The Univalent Foundations Program, Princeton, NJ; Institute for Advanced Study (IAS), Princeton, NJ, 2013.

[75] Michael Reed and Barry Simon. *Methods of modern mathematical physics. I.* Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York, second edition, 1980. Functional analysis.

[76] C. L. Reedy. Homotopy theory of model categories. 1974.

[77] F. Rellich. Ein Satz über mittlere Konvergenz. *Nachr. Ges. Wiss. Göttingen, Math.-Phys. Kl.*, 1930:30–35, 1930.

[78] Emily Riehl and Dominic Verity. *Elements of ∞-category theory*, volume 194 of *Cambridge Studies in Advanced Mathematics.* Cambridge University Press, Cambridge, 2022.

[79] J. Rosický. On homotopy varieties. *Adv. Math.*, 214(2):525–550, 2007.

[80] I. Satake. On a generalization of the notion of manifold. *Proc. Nat. Acad. Sci. U.S.A.*, 42:359–363, 1956.

[81] Jean-Pierre Serre. *Algèbre locale. Multiplicités*, volume 11 of *Lecture Notes in Mathematics.* Springer-Verlag, Berlin-New York, 1965. Cours au Collège de France, 1957–1958, rédigé par Pierre Gabriel, Seconde édition, 1965.

[82] Yu. Smirnov. A necessary and sufficient condition for metrizability of a topological space. *Doklady Akad. Nauk SSSR (N.S.)*, 77:197–200, 1951.

[83] David I. Spivak. *Quasi-smooth derived manifolds.* ProQuest LLC, Ann Arbor, MI, 2007. Thesis (Ph.D.)–University of California, Berkeley.

[84] David I. Spivak. Derived smooth manifolds. *Duke Math. J.*, 153(1):55–128, 2010.

[85] Pelle Steffens. Derived C^∞-Geometry I: Foundations. *arxiv:2304.08671v2*, pages 1–205, 2023.

[86] William P. Thurston. *Three-dimensional geometry and topology. Vol. 1*, volume 35 of *Princeton Mathematical Series.* Princeton University Press, Princeton, NJ, 1997. Edited by Silvio Levy.

[87] Bertrand Toën and Gabriele Vezzosi. Homotopical algebraic geometry. I. Topos theory. *Adv. Math.*, 193(2):257–372, 2005.

[88] Bertrand Toën and Gabriele Vezzosi. Homotopical algebraic geometry. II. Geometric stacks and applications. *Mem. Amer. Math. Soc.*, 193(902):x+224, 2008.

[89] Alan Weinstein. Linearization problems for Lie algebroids and Lie groupoids. volume 52, pages 93–102. 2000. Conference Moshé Flato 1999 (Dijon).

[90] Alan Weinstein. Linearization of regular proper groupoids. *J. Inst. Math. Jussieu*, 1(3):493–511, 2002.

[91] R. O. Wells, Jr. *Differential analysis on complex manifolds*. Prentice-Hall Series in Modern Analysis. Prentice-Hall, Inc., Englewood Cliffs, N.J., 1973.

[92] Nguyen Tien Zung. Proper groupoids and momentum maps: linearization, affinity, and convexity. *Ann. Sci. École Norm. Sup. (4)*, 39(5):841–869, 2006.

Index

∞-category, 43
∞-groupoid, 49
$\triangleleft$, 48
$\triangleright$, 48
2-category, 29
2-out-of-3 property, 17

Ab, 11
C^∞, 167
C^∞_{loc}, 167
C^k, 167
C^k_{loc}, 167
Grp, 10
hCat_∞, 53
$\underline{\text{Hom}}$, 80
H^s, 169
hSpc, 41
LogSm, 113
LogTop, 103
Ω_M, 168
$\mathsf{P}_\Lambda(\mathsf{C})$, 60
$\underline{\text{Sec}}$, 81
Set, 10
$\mathsf{Sif}(\mathsf{C})$, 63
Sm, 85
Top, 10, 66
TopSm, 163
X_P, 107

almost complex structure, 198
almost symplectic form, 198
altas, 78
amplitude, 153
arbitrarily small, 66
arrow category, 17
Artin morphism, 78
asymptotically cylindrical, 108, 114, 186
atlas, 96, 161

bound on geometry, 224
bound on injectivity radius, 224
boundary, 38
bounded geometry, 187
bump function, 87

cancellation, 20
cancellation for fiber products, 18
cancellative, 106
cardinal, 11
cardinality, 11
cardinality of a simplicial set, 32
cartesian functor, 56
Cartier divisor, 104
categorical equivalence, 53
categorical fiber, 53
category of categories, 16
category of open subsets, 72
Cauchy swarm, 166
Čech descent, 72
Čech nerve, 74
chain rule, 113
closed, 67
closed embedding, 67
closed subset, 66
closed under composition, 17
co-isotropic, 197

coarse local isomorphism, 134
cocartesian functor, 56
cocontinuous functor, 21
coend, 21
colimit, 54
compact, 69
compact-open topology, 81
compatible, 197
complete simplex, 31
complete topological vector space, 166
completed swarm, 69
complex, 27
complex conjugate, 199
complex structure, 197, 198
complexification, 199
composition, 43
condition on morphisms, 80
cone, 48, 106
continuous, 66
continuous functor, 21
contractible, 41
contractible Kan complex, 41
convolution, 169
core, 11, 49
corporeal, 141
cosheaf, 73
cosifted, 62
cosimplicial object, 31
cotangent cone, 111
covering sieve, 73
cylinder, 186
cylindrical, 108, 114, 186

deformation to the tangent bundle, 93
degenerate simplex, 32
delta function, 168
density, 168
depth, 111, 120
derived smooth manifold, 145, 146
descent, 72, 73
dg-category, 42, 45
diagonal, 19
diagram, 18, 45
differential graded category, 42, 45
differential graded nerve, 45
differential operator, 175
Dold–Kan Correspondence, 35
dual, 11

Ehresmann's Theorem, 91
elementary derived open embedding, 151
elliptic operator, 178
embedding, 67, 105
end, 21
energy, 224
enough open embeddings, 128
enriched category, 27
epic, 16
epimorphism, 16
equivalence of ∞-categories, 53
equivalence of categories, 13
essential image, 12
essentially small category, 15
essentially surjective, 12, 47
étale space, 74
exact, 119
extension property, 38

face, 107
fibrant simplicial category, 44
final, 21, 55
final object, 17, 53
finite presheaf, 60
finite subcover property, 69
formal adjoint, 175
Fourier inversion, 169
Fourier transform, 169
free, 12
full subcategory, 11, 44
fully faithful, 12
functor, 12, 45
functor category, 12, 45
fundamental groupoid, 11

ghost sheaf, 105
gluing coordinates, 123
groupoid, 11

Hadamard's Lemma, 93
Hofer's Lemma, 225
homotopy category, 46
homotopy category of ∞-categories, 53
homotopy category of Kan complexes, 41
homotopy equivalence, 41
horn, 38

idempotent, 22
idempotent complete, 22
idempotent completion, 22
immersion, 85
immersive, 85
induced property, 25
infinity-category, 43
infinity-groupoid, 49
inhomogeneous term, 200
initial, 21, 55
initial object, 17, 53
inner horn, 43
integrable, 198
interpolation norm, 170
inverse function theorem, 85, 116
isofibration, 53
isomorphism, 11, 47
isotropic, 197
iterated diagonal, 19

join, 48
join of pairs, 48

Kan complex, 39
Kan extension, 57, 59, 136
Kan fibration, 39

Lagrangian, 197, 198
large category, 15
latching map, 33
latching object, 33
leaf structure, 163
left cone, 48
left fibration, 48
left horn, 43
left lifting property, 38
levelwise property, 31
Lie group, 89
lift, 17, 38
lifting property, 38
limit, 54, 69
limit pointed topological space, 69
linear model, 211, 215
linear projection, 211
local, 119
local isomorphism, 67, 85, 134
local object, 23
local on the source, 68
local on the target, 67
local presheaf, 60
local property, 67, 68
local sections, 67, 76
locally closed embedding, 67
locally compact, 66
locally connected, 163
locally exact, 119
locally trivial, 67
log (co)tangent short exact sequence, 111, 113
log chain rule, 113
log coordinates, 108
log differentiability, 112
log inverse function theorem, 116
log smooth, 112
log smooth manifold, 113
log structure, 103
log tangent bundle, 111
log topological space, 103

manifold, 85
manifold with (asymptotically) cylindrical ends, 186
mapping space, 52
marked horn, 50
marked simplicial set, 50
matching map, 33
matching object, 33
metric, 197, 198
minimal, 150

minimal atlas, 161
minimal submersion, 97
Mittag-Leffler condition, 28
moduli stack, 204, 205, 207
monic, 16
monoidal category, 26
monomial, 107
monomorphism, 16
Morita equivalence, 138
morphism, 43
multi-index, 167

natural transformation, 12
neighborhood, 66
neighborhood base, 66
nerve, 43–45, 74
non-degenerate simplex, 32
norm, 166
normalized fiber, 122

object, 43
open, 67
open cover, 76
open covering, 128
open embedding, 67, 85, 128
open subset, 66
opposite, 11, 44
order of an operator, 179
orientation, 27
orientation line, 27
orthogonal, 197
outer horn, 43

pair, 38
paracompact, 87
parameterized problem, 202
parametrix, 179
partition of unity, 87
perfect topological site, 139
perfection, 140
point constraint, 200, 203
point of a topological stack, 76
polyhedral cone, 106
poset, 14
pre-log structure, 103
precosheaf, 72
preservation of (co)limits, 21
preserve, 17
preserved under pullback, 19
preserved under pushout, 19
presheaf, 22, 72
principle of equivalence, 14
problem, 199
proper, 70, 77
proper atlas, 78, 97, 161
property of morphisms, 17, 47
pseudo-differential operator, 179
pseudo-holomorphic, 199, 200
pseudo-holomorphic map problem, 199
pullback of a morphism, 19
pullback of log structures, 105
pushout of a morphism, 19

quadratic estimate, 214
quasi-category, 43
quasi-holomorphic, 201
quasi-integral, 106

real affine toric variety, 107
reduction, 188
reduction to the regular locus, 220
Reedy property, 34
reflect, 17
reflective subcategory, 22
regular locus, 212
Regularity Theorem, 211
relative colimit, 57
relative depth, 120
relative diagonal, 19
relative functor category, 57
relative limit, 57
relative swarm, 69
relative tangent complex, 155
relative tangent space, 95
representable, 22
representable morphism, 25
retract, 16

retraction, 16
right cone, 48
right fibration, 48
right horn, 43
right lifting property, 38

section, 16
semi-local operator, 178
semi-norm, 166
separated, 70, 77
sharp, 109
sheaf, 73
sieve, 73
sieve descent, 73
sifted, 62
simplex category, 31
simplicial category, 44
simplicial nerve, 44
simplicial object, 31
simplicial set, 31
simply-broken submersion, 123
skeleton, 33
slice category, 48
small category, 15
small model, 15
smash product, 39
smooth, 167
smooth manifold, 85
smoothing operator, 179
Sobolev space, 169
solution, 199
source-local property, 68
split epimorphism, 16, 47
split monomorphism, 16, 47
stability structure, 84
stable, 83
strata functor, 113
strict log map, 105
strict Morita equivalence, 138
strict topological functor, 131
subcanonical, 130
submersion, 85, 153, 164
submersion of log smooth manifolds, 117
submersive, 85, 153, 164
submersive atlas, 96, 161
subswarm lifting property, 69
support of an operator, 178
swarm, 69
symbol, 175, 179
symplectic form, 197, 198

tame, 197
tangent bundle, 111
tangent complex, 96, 154, 155
tangent functor, 87
target-local property, 67
topological functor, 131
topological group, 88
topological preservation of limits, 146
topological site, 128
topological space, 66
topological stack, 75
topological stack of categories, 84
topological vector space, 166
topological-smooth space, 163
topologically fully faithful, 139
topology, 66
totally real, 198
totally real boundary conditions, 201
transfinite composition, 38
transverse diagram, 86, 147, 148
transverse limit, 86, 147, 148
trivial Kan complex, 41
trivial Kan fibration, 41
trivial log structure, 104
truncated simplicial object, 33
twist, 189
twisted arrow category, 20
twisted reduction, 188

universal property of
  local presheaves, 62
  presheaves on a topological ∞-site, 135
  reflective subcategory of presheaves, 25
  sheaves on a topological site, 135
  topological stacks, 75

universally closed, 68, 69, 77
vector space topology, 166
weak Kan complex, 43
weak Kan extension, 57